ANALYSIS OF A MODIFIED SECOND-ORDER MIXED HYBRID BDM_1 FINITE ELEMENT METHOD FOR TRANSPORT PROBLEMS IN DIVERGENCE FORM

FABIAN BRUNNER, JULIAN FISCHER, AND PETER KNABNER

ABSTRACT. We prove optimal second order convergence of a modified BDM_1 mixed finite element scheme for advection-diffusion problems in divergence form. If advection is present, it is known that the total flux is approximated only with first order accuracy by the classical BDM_1 mixed method, which is suboptimal since the same order of convergence is obtained if the computationally less expensive RT_0 element is used. The modification that was first proposed in [7] is based on the hybrid problem formulation and consists in using the Lagrange multipliers for the discretization of the advective term instead of the cellwise constant approximation of the scalar unknown.

1. INTRODUCTION

In the present work, we are concerned with mixed finite element approximations of the advection-diffusion problem

$$\partial_t u = \operatorname{div}(a\nabla u - u\mathbf{b}) + f \quad \text{in } \Omega \times (0,T), \tag{1a}$$

$$u = u_0 \quad \text{on } \Omega \times \{0\}, \tag{1b}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1c}$$

where Ω denotes a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$, and $T > 0$ is the final time. This equation describes the evolution of a conserved quantity u subject to diffusion and advection. Throughout this work, we assume the diffusion coefficient a to be essentially bounded and uniformly elliptic on $\Omega \times [0,T]$. The advection velocity field $\mathbf{b}$ is required to belong to $L^\infty([0,T];W^{2,\infty}(\Omega))$, and the source term f is required to belong to $L^2(\Omega \times [0,T])$. Further regularity assumptions on a, $\mathbf{b}$ and f are only introduced implicitly by the definition of our numerical scheme requiring that they are well-defined at each discrete time t_n, and through the regularity assumptions on the exact solution u and the total flux $\mathbf{q}$.

Mixed finite element methods for the equation (1) are based on the mixed reformulation of the problem: In this reformulation, the total flux $\mathbf{q}$ is introduced as an additional explicit variable and the equation (1a) translates into the system

$$\partial_t u = -\operatorname{div}\mathbf{q} + f \quad \text{in } \Omega \times (0,T), \tag{2a}$$

$$\mathbf{q} = -a\nabla u + u\mathbf{b} \quad \text{in } \Omega \times (0,T). \tag{2b}$$

A *mixed finite element method* consists in approximating the quantities $\mathbf{q}$ and u simultaneously in appropriate function spaces by discretizing the system (2). Mixed methods are well suited for solving elliptic and parabolic problems which arise in many fields of applications. In particular, they are locally conservative, provide continuous fluxes across element boundaries and can handle distorted and unstructured grids well. For a general overview on mixed finite element methods and their applications, we refer the reader to the books [3, 4, 6], the review articles [13, 15], and the references therein.

Mixed methods for advection-diffusion problems based on the Raviart-Thomas elements have been analyzed in [11] and [12]. Using these finite element spaces, the flux variable and the scalar variable are approximated with the same order of convergence. For problems where the main interest lies in the flux variable, the BDM spaces were introduced [5], which are able to approximate the flux to one order higher than the scalar variable. For example, if the BDM_1 element is employed for the discretization of an elliptic pure diffusion equation, the approximation of the flux variable is of second order accuracy in the mesh size h, while the approximation of the scalar variable is of first order accuracy. Error estimates in L^2 and L^∞ for general second order elliptic problems with nonvanishing advection using the BDM family of elements were derived in [9] and [10], respectively. It was shown that in case of additional advective transport — more precisely, when an advective term in divergence form is present and the flux variable of the mixed method is defined to be the total flux consisting of advective and diffusive transport — , the order of convergence in the flux variable in the L^2 norm drops to one. The order of convergence is then limited by the first order accuracy of the approximation for the transported quantity itself. This phenomenon of suboptimal convergence may occur whenever the mixed finite element spaces employed use polynomials of higher degree for the approximation of the flux variable than for the approximation of the scalar variable.

In the paper [7], a scheme for advection-diffusion-reaction equations has been introduced which represents a modification of the classical BDM_1 scheme based on the hybrid problem formulation. More precisely, in the usual system of linear equations for the mixed hybrid BDM_1 scheme, the cellwise constant approximation of u in the definition of the advective flux is replaced by a reconstruction based on the interelement Lagrange multipliers (which are introduced during hybridization to relax the continuity constraint of the flux across the element boundaries). The original purpose of this modification was to improve stability properties of the scheme for moderate Péclet numbers. However, it turns out that this modification additionally improves the order of convergence of the scheme and restores optimal second-order convergence for the fluxes even in the presence of a nonvanishing advective transport term.

It is a classical result dating back to the original works of Arnold and Brezzi [1] and Brezzi, Douglas, and Marini [5] that the Lagrange multipliers from the hybridization process carry higher order information about the scalar variable u: The Lagrange multipliers may be used to reconstruct a higher order nonconforming approximation of u in the space of Crouzeix-Raviart elements. The precise modification in [7] provides a higher order approximation of the advective flux $u\mathbf{b}$ using the Lagrange multipliers.

In the present work, we give an interpretation of the modification to the mixed hybrid BDM_1 scheme introduced in [7] on the finite element level. Based on this interpretation, we establish optimal order error estimates for the modified BDM_1 scheme from [7]. Furthermore, we present an extension of the modification — which has been introduced in [7] in the framework of planar problems — to the case of three space dimensions. Although our presentation is restricted to simplicial meshes consisting of triangles or tetrahedra, the same ideas can be easily transferred to quadrilateral (respectively hexahedral) mesh elements.

For the RT_0 element, the modification from [7] has been analyzed in [8] along with other upwind-mixed hybrid schemes. In [14], the idea of using the Lagrange multiplier in the advective term was employed to derive a posteriori error estimates for lowest order mixed finite element approximations of advection-diffusion-reaction equations.

This article is organized as follows. In the next section, we recall the classical BDM_1 method and the modified method of [7]; we furthermore introduce a reconstruction operator which provides a description of the modification of [7] on the finite element level. Moreover, we recall the analytical properties of the finite element spaces. Section 3 contains the statement of our main results, the proofs of which are given in Sections 4 and 5. Finally, in Section 6, numerical results are presented to illustrate the analytical results.

Notation. Throughout the paper, we use standard notation from numerical analysis. By ∇ and div we denote the distributional gradient and divergence, respectively. The expression D^2 refers to the second (distributional) derivative. For a quantity defined only on a face, we denote its distributional derivative (with the face considered as a manifold) nevertheless by ∇. By $W^{k,p}(\Omega)$ we denote the space of functions in $L^p(\Omega)$ whose kth distributional derivative also belongs to $L^p(\Omega)$. The space $W^{k,2}(\Omega)$ will also be denoted as $H^k(\Omega)$. By $H^k_0(\Omega)$ we denote the closure in $H^k(\Omega)$ of the set of compactly supported smooth functions in Ω. The space $\mathbf{H}^{\mathrm{div}}(\Omega)$ consists of the vector fields in $L^2(\Omega)$ whose distributional divergence also belongs to $L^2(\Omega)$.

By $\mathcal{P}^k(K)$ we denote the space of polynomials up to degree k on some set $K \subset \mathbb{R}^d$. By $\mathcal{T}_h$ we denote the set of simplices of some triangulation of our domain $\Omega \subset \mathbb{R}^d$; the symbol $\mathcal{F}_h$ refers to the set of all faces of the simplices $K \in \mathcal{T}_h$. By $L^2(F)$ for some $F \in \mathcal{F}_h$ we denote the L^2-space with respect to the surface measure on F. The notation $L^2(\mathcal{F}_h)$ refers to the L^2-space with respect to the surface measure on the union of all faces.

When evaluating the trace of some vector field $\mathbf{v}_h$ on a face F across which $\mathbf{v}_h$ may have a jump, we shall write $\mathbf{v}_h|_K$ to make clear from which side the trace is to be understood. In case no ambiguity may arise, we shall omit the $|_K$.

2. The modified BDM_1 mixed hybrid finite element scheme

Discretizing equation (2) in space and time, we obtain a fully discrete mixed finite element scheme for the advection-diffusion problem (1). We use an implicit Euler scheme for time discretization and denote the time elapsed at the n-th timestep by t_n, $0 \leq n \leq N$. For the space discretization with the BDM_1 mixed finite element, let $(\mathcal{T}_h)_{h>0}$ denote a family of regular triangulations of Ω consisting of closed d-simplices $K \in \mathcal{T}_h$. In this work, all elements are assumed to have flat faces, and the set of all faces associated with $\mathcal{T}_h$ is denoted by $\mathcal{F}_h$. The case of boundary elements having curved faces could be handled with minor modifications, cf. [5]. For the discretization using the BDM_1 mixed finite element method, we define the function spaces

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}^{\mathrm{div}}(\Omega) : \ \mathbf{v}_h|_K \in [\mathcal{P}_1(K)]^d \quad \forall K \in \mathcal{T}_h\}, \tag{3}$$

$$W_h := \{w_h \in L^2(\Omega) : \ w_h|_K \in \mathcal{P}_0(K) \quad \forall K \in \mathcal{T}_h\}. \tag{4}$$

Then, for given initial data $u_h^0 \in W_h$, for each time step $n \in \{1, \dots, N\}$ we seek $(\mathbf{q}_h^n, u_h^n) \in \mathbf{V}_h \times W_h$ satisfying

$$\int_\Omega \frac{u_h^n - u_h^{n-1}}{t_n - t_{n-1}} w_h \, dx = -\int_\Omega \operatorname{div} \mathbf{q}_h^n \ w_h \, dx + \int_\Omega f(\cdot, t_n) w_h \, dx, \tag{5a}$$

$$\int_\Omega a^{-1}(\cdot, t_n)\mathbf{q}_h^n \cdot \mathbf{v}_h \, dx = \int_\Omega u_h^n \operatorname{div} \mathbf{v}_h \, dx + \int_\Omega u_h^n a^{-1}(\cdot, t_n)\mathbf{b}(\cdot, t_n) \cdot \mathbf{v}_h \, dx \tag{5b}$$

for all $(\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h$.

In the mixed hybrid formulation of the problem, additionally the continuity constraint of the normal component of the fluxes at the faces of the triangulation is relaxed: We introduce the spaces

$$\hat{\mathbf{V}}_h := \{\mathbf{v}_h \in [L^2(\Omega)]^d : \ \mathbf{v}_h|_K \in [\mathcal{P}_1(K)]^d \quad \forall K \in \mathcal{T}_h\}, \tag{6}$$

$$\Lambda_{h,0} := \{\mu_h \in L^2(\mathcal{F}_h) : \ \mu_h|_F \in \mathcal{P}_1(F) \quad \forall F \in \mathcal{F}_h, \quad \mu_h|_{\partial\Omega} \equiv 0\}, \tag{7}$$

and look for solutions $(\mathbf{q}_h^n, u_h^n, \lambda_h^n) \in \hat{\mathbf{V}}_h \times W_h \times \Lambda_{h,0}$ satisfying

(8a)

$$\int_\Omega \frac{u_h^n - u_h^{n-1}}{t_n - t_{n-1}} w_h \, dx = -\sum_{K \in \mathcal{T}_h} \int_K \operatorname{div} \mathbf{q}_h^n \ w_h \, dx + \int_\Omega f(\cdot, t_n) w_h \, dx,$$

(8b)

$$\begin{aligned} \int_\Omega a^{-1}(\cdot, t_n)\mathbf{q}_h^n \cdot \mathbf{v}_h \, dx = \sum_{K \in \mathcal{T}_h} \int_K u_h^n \operatorname{div} \mathbf{v}_h \, dx + \int_\Omega u_h^n a^{-1}(\cdot, t_n)\mathbf{b}(\cdot, t_n) \cdot \mathbf{v}_h \, dx \\ - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda_h^n \, \mathbf{v}_h|_K \cdot \mathbf{n}_{\partial K} \, dS, \end{aligned}$$

(8c)

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu_h \, \mathbf{q}_h^n|_K \cdot \mathbf{n}_{\partial K} \, dS = 0$$

for all $(\mathbf{v}_h, w_h, \mu_h) \in \hat{\mathbf{V}}_h \times W_h \times \Lambda_{h,0}$. Here, $\boldsymbol{n}_{\partial K}$ is the outer unit normal to ∂K and the λ_h^n denote the Lagrange multipliers introduced to relax the continuity constraint on the flux $\mathbf{q}_h^n$ at the faces of the triangulation (which is no longer incorporated in the new ansatz space $\hat{\mathbf{V}}_h$ for $\mathbf{q}_h^n$, but instead explicitly enforced by equation (8c)). Note that the solution to the hybrid system coincides with the solution from the non-hybridized system. Moreover, the flux $\mathbf{q}_h^n$ and the scalar unknown u_h^n can be eliminated at the level of the linear system – a process called static condensation – , and a system for the Lagrange multipliers remains to be solved after this elimination process. This linear system consists of fewer variables than the linear system resulting from the nonhybrid formulation and it does no longer have the structure of a saddle point problem; therefore, standard iterative linear solvers can be employed to solve it numerically. The quantities of interest $\mathbf{q}_h^n$ and u_h^n may subsequently be reconstructed on every simplex from the Lagrange multipliers on the faces of the simplex, which is computationally cheap as it is a local procedure.

We shall use the standard basis for hybridized BDM_1 finite elements. The space W_h being the space of functions which are constant on every $K \in \mathcal{T}_h$, a basis of W_h is given by the characteristic functions of the elements $K \in \mathcal{T}_h$.

The space $\Lambda_{h,0}$ for the Lagrange multipliers consists of functions which are defined on the faces of the triangulation and which are linear on every face (e.g. in two spatial dimensions $d = 2$, the functions in $\Lambda_{h,0}$ are defined to be linear polynomials on every edge of our triangulation and may be discontinuous at the vertices). In two space dimensions, a basis of this space is given by functions vanishing everywhere except for one interior edge and taking the value 0 at one endpoint and the value 1 at the other endpoint of this edge. More precisely, for an interior edge F with endpoints x_F^1 and x_F^2, we denote by μ_F^i the function in $\Lambda_{h,0}$ which takes the value 1 at x_F^i and the value 0 at the other endpoint of F and vanishes on all other edges.

For $\hat{\mathbf{V}}_h$, in the case $d = 2$ we define our basis functions to be the vector fields that vanish on every triangle except for one triangle K, on which the normal flux is prescribed to be zero at every edge except for one edge F; on this edge, we require the (outward) normal flux to be $2/|F|$ at the point $\frac{1}{3}(2x_F^1 + x_F^2)$ and to be 0 at the point $\frac{1}{3}(x_F^1 + 2x_F^2)$ (or the other way around). The corresponding basis function is denoted by $\mathbf{v}_{KF}^1$ (respectively $\mathbf{v}_{KF}^2$).

In three space dimensions, there are three basis functions of $\Lambda_{h,0}$ associated with each interior face, and we denote by μ_F^i the basis function vanishing on all faces but F and taking the value 1 in the midpoint m_F^i of the i-th edge of F and the value 0 in the midpoints of the other edges. Accordingly, we define the basis function $\mathbf{v}_{KF}^i$ of $\hat{\mathbf{V}}_h$ associated with the face F of K to have zero normal flux on all faces but F, where it is required to take the value $3/|F|$ in m_F^i and the value 0 in the other midpoints; outside of K, the function $\mathbf{v}_{KF}^i$ is prescribed to be zero.

Note that our basis functions satisfy the following properties (both in the case $d = 2$ and in the case $d = 3$):

(9a)
$$\int_K \operatorname{div} \mathbf{v}_{KF}^i \, dx = 1 \qquad \forall F \in \mathcal{F}_h,\ F \subset \partial K,\ i = 1, \dots, d,$$
(9b)
$$\int_F \mathbf{v}_{KF'}^i|_K \cdot \boldsymbol{n}_{\partial K} \, dS = \delta_{FF'} \qquad \forall F, F' \in \mathcal{F}_h,\ F, F' \subset \partial K,\ i = 1, \dots, d,$$
(9c)
$$\int_F \mu_F^i \mathbf{v}_{KF'}^j|_K \cdot \boldsymbol{n}_{\partial K} \, dS = \delta_{FF'}\delta_{ij} \quad \forall F, F' \in \mathcal{F}_h,\ F, F' \subset \partial K,\ i, j = 1, \dots, d.$$

We then may define numbers $q_{KFi}^n, u_K^n, \lambda_{Fi}^n \in \mathbb{R}$ by expanding $\mathbf{q}_h^n$, u_h^n, and λ_h^n as

$$\mathbf{q}_h^n = \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_h, F \subset \partial K} \sum_{i=1}^{d} q_{KFi}^n \mathbf{v}_{KF}^i, \tag{10a}$$

$$u_h^n = \sum_{K \in \mathcal{T}_h} u_K^n \chi_K, \tag{10b}$$

$$\lambda_h^n = \sum_{F\in\mathcal{F}_h}\sum_{i=1}^{d}\lambda_{Fi}^n\mu_F^i. \tag{10c}$$

We introduce the abbreviations

$$B_K^{Fi\tilde{F}j,n} := \int_K ((a(\cdot,t_n)^{-1}\mathbf{v}_{KF}^i)\cdot\mathbf{v}_{K\tilde{F}}^j\;dx, \tag{11a}$$

$$F_K^n := \int_K f(\cdot,t_n)\;dx, \tag{11b}$$

and expand the advection velocity field (with Π_h^1 as introduced below) as

(12)

$$\mathbf{b}(\cdot,t_n) = \Pi_h^1\mathbf{b}(\cdot,t_n) + \mathbf{b}_r^n = \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h,F\subset\partial K}\sum_{i=1}^{d} b_{KFi}^n\mathbf{v}_{KF}^i + \mathbf{b}_r^n =: \mathbf{b}_h^n + \mathbf{b}_r^n.$$

The property (20b) below entails

$$||\mathbf{b}_r^n||_{L^\infty(\Omega)} \le Ch^2||\mathbf{b}(\cdot,t_n)||_{W^{2,\infty}(\Omega)}. \tag{13}$$

Using the properties (9) of the basis functions, we may approximate the standard mixed hybrid BDM_1 scheme by the system of linear equations (the approximation just consisting of replacing $\mathbf{b}(\cdot,t_n)$ by $\mathbf{b}_h^n$)

(14a)

$$|K|\frac{u_K^n - u_K^{n-1}}{t_n - t_{n-1}} + \sum_{F\in\mathcal{F}_h,F\subset\partial K}\sum_{i=1}^{d} q_{KFi}^n = F_K^n \quad \forall K\in\mathcal{T}_h,$$

(14b)

$$\sum_{F\in\mathcal{F}_h,F\subset\partial K}\sum_{i=1}^{d} B_K^{Fi\tilde{F}j,n}q_{KFi}^n - u_K^n - \sum_{F\in\mathcal{F}_h,F\subset\partial K}\sum_{i=1}^{d} b_{KFi}^n u_K^n B_K^{Fi\tilde{F}j,n} = -\lambda_{\tilde{F}j}^n$$
$$\forall K\in\mathcal{T}_h, \tilde{F}\in\mathcal{F}_h \text{ with } \tilde{F}\subset\partial K, 1\le j\le d,$$

(14c)

$$\sum_{K\in\mathcal{T}_h,F\subset\partial K} q_{KFi}^n = 0 \quad \forall F\in\mathcal{F}_h,\; 1\le i\le d.$$

In their modification, the authors of [7] replace the (only first-order accurate) term u_h^n in the advective flux term by an – as it will turn out higher order – reconstruction making use of the Lagrange multipliers. Their modification was limited to the two-dimensional case. The resulting system of linear equations reads

(15a)

$$|K|\frac{u_K^n - u_K^{n-1}}{t_n - t_{n-1}} + \sum_{F\in\mathcal{F}_h,F\subset\partial K}\sum_{i=1}^{2} q_{KFi}^n = F_K^n \quad \forall K\in\mathcal{T}_h,$$

(15b)

$$\sum_{F\in\mathcal{F}_h,F\subset\partial K}\sum_{i=1}^{2} B_K^{Fi\tilde{F}j,n}q_{KFi}^n - u_K^n - \sum_{F\in\mathcal{F}_h,F\subset\partial K}\sum_{i=1}^{2} b_{KFi}^n\frac{\lambda_{F1}^n + \lambda_{F2}^n + \lambda_{Fi}^n}{3}B_K^{Fi\tilde{F}j,n}$$
$$= -\lambda_{\tilde{F}j}^n \qquad \forall K\in\mathcal{T}_h, \tilde{F}\in\mathcal{F}_h \text{ with } \tilde{F}\subset\partial K,\; 1\le j\le 2,$$

(15c)

$$\sum_{K \in \mathcal{T}_h, F \subset \partial K} q^n_{KFi} = 0 \quad \forall F \in \mathcal{F}_h,\ 1 \leq i \leq 2.$$

As a first step towards the numerical analysis of this scheme, let us provide an interpretation of the modification at the finite element level. For this purpose, introduce an operator $\mathcal{B} : \Lambda_{h,0} \to \mathbf{V}_h$ which – given a piecewise affine approximation for the solution u on the faces of the triangulation – reconstructs an $\mathbf{H}^{\mathrm{div}}$-conforming approximation for the advective flux in the space $\mathbf{V}_h$. The operator $\mathcal{B}$ is defined to act on $\Lambda_{h,0}$ as

$$\text{(16a)} \quad \mathcal{B}[\lambda^n_h] := \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_h, F \subset \partial K} \sum_{i=1}^{2} \frac{\lambda^n_{Fi} + \lambda^n_{F1} + \lambda^n_{F2}}{3} b^n_{KFi} \mathbf{v}^i_{KF} \quad \text{in case } d = 2,$$

i.e. the normal advective flux $\mathbf{n}_F \cdot \mathcal{B}[\lambda^n_h]$ is prescribed at the two points of an edge F dividing the edge into three segments of equal size to match the product of the normal component of $\mathbf{b}^n_h$ and the Lagrange multiplier λ^n_h at these points. As we shall see below, this precise structure of the operator $\mathcal{B}$ is crucial for its higher order approximation property.

In the present work, we shall also propose a natural extension of the modified BDM_1 scheme of [7] to the case of three spatial dimensions. To this aim, let us define

$$\text{(16b)} \qquad \mathcal{B}[\lambda^n_h] := \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_h, F \subset \partial K} \sum_{i=1}^{3} \lambda^n_{Fi} b^n_{KFi} \mathbf{v}^i_{KF} \quad \text{in case } d = 3,$$

which leads to the linear system

(17a)

$$|K| \frac{u^n_K - u^{n-1}_K}{t_n - t_{n-1}} + \sum_{F \in \mathcal{F}_h, F \subset \partial K} \sum_{i=1}^{3} q^n_{KFi} = F^n_K \quad \forall K \in \mathcal{T}_h,$$

(17b)

$$\sum_{F \in \mathcal{F}_h, F \subset \partial K} \sum_{i=1}^{3} B^{Fi\tilde{F}j,n}_K q^n_{KFi} - u^n_K - \sum_{F \in \mathcal{F}_h, F \subset \partial K} \sum_{i=1}^{3} b^n_{KFi} \lambda^n_{Fi} B^{Fi\tilde{F}j,n}_K = -\lambda^n_{\tilde{F}j}$$
$$\forall K \in \mathcal{T}_h, \tilde{F} \in \mathcal{F}_h \text{ with } \tilde{F} \subset \partial K, 1 \leq j \leq 3,$$

(17c)

$$\sum_{K \in \mathcal{T}_h, F \subset \partial K} q^n_{KFi} = 0 \quad \forall F \in \mathcal{F}_h,\ 1 \leq i \leq 3.$$

Thus, in the case of three space dimensions, $\mathcal{B}[\lambda^n_h]$ is defined by prescribing the normal advective flux at the midpoints of the edges of the faces (to match again the product of the normal component of $\mathbf{b}^n_h$ and the Lagrange multiplier λ^n_h at these points). As we shall show below, this definition entails the desired higher order approximation property.

Using the operator $\mathcal{B}$, the modified mixed hybrid scheme is equivalent to requiring

$$\int_\Omega \frac{u_h^n - u_h^{n-1}}{t_n - t_{n-1}} w_h \, dx = -\sum_{K\in\mathcal{T}_h} \int_K \operatorname{div} \mathbf{q}_h^n \, w_h \, dx + \int_\Omega f(\cdot, t_n) w_h \, dx, \tag{18a}$$

$$\int_\Omega a^{-1}(\cdot,t_n)\mathbf{q}_h^n \cdot \mathbf{v}_h \, dx = \sum_{K\in\mathcal{T}_h} \int_K u_h^n \operatorname{div} \mathbf{v}_h \, dx + \int_\Omega a^{-1}(\cdot,t_n)\mathcal{B}[\lambda_h^n]\cdot \mathbf{v}_h \, dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \lambda_h^n \mathbf{v}_h|_K \cdot \mathbf{n}_{\partial K} \, dS, \tag{18b}$$

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu_h \mathbf{q}_h^n|_K \cdot \mathbf{n}_{\partial K} \, dS = 0 \tag{18c}$$

for any $(\mathbf{v}_h, w_h, \mu_h) \in \hat{\mathbf{V}}_h \times W_h \times \Lambda_{h,0}$.

2.1. **Interpolation operators.** In this section, we recall the projection operators that are used in our error analysis. For some fixed $s > 2$, let $\boldsymbol{V} := \mathbf{H}^{\mathrm{div}}(\Omega) \cap [L^s(\Omega)]^d$ and let $\Pi_h^1 : \boldsymbol{V} \to \mathbf{V}_h$ denote the usual BDM_1 projection operator (which is defined by the condition that the normal component of $\Pi_h^1 \boldsymbol{v}$ on F must match the $L^2(F)$-orthogonal projection of the normal component of $\boldsymbol{v}$ to the space of linear polynomials on F for every face $F \in \mathcal{F}_h$); cf. [3, 5]. Moreover, we define $P_h^0 : L^2(\Omega) \to W_h$ to be the $L^2(\Omega)$-orthogonal projection onto W_h. Assuming that $(\mathcal{T}_h)_{h>0}$ is a regular family of triangulations, i.e. that the ratios

$$\sigma_K := \frac{h_K}{\rho_K} \tag{19}$$

with ρ_K denoting the diameter of the largest inscribed ball in K and h_K denoting the diameter of the element K are uniformly bounded by a constant $\sigma_{\max}$ independent of h, the following approximation properties are known to hold for the projectors:

$$\|w - P_h^0 w\|_{L^2(\Omega)} \le Ch \|\nabla w\|_{L^2(\Omega)} \quad \forall w \in H^1(\Omega), \tag{20a}$$

$$\|\boldsymbol{v} - \Pi_h^1 \boldsymbol{v}\|_{L^p(\Omega)} \le Ch^2 \|D^2 \boldsymbol{v}\|_{L^p(\Omega)} \quad \forall \boldsymbol{v} \in W^{2,p}(\Omega), \tag{20b}$$

where $p \in [1, \infty]$ and where the constants C depend only on $\sigma_{\max}$ and (in the second formula) the exponent p. Since the projectors are defined locally, the same estimates hold on each element $K \in \mathcal{T}_h$ of the triangulation with h replaced by h_K. We will also make use of the commuting diagram property

$$\operatorname{div}(\Pi_h^1) = P_h^0(\operatorname{div}), \tag{20c}$$

which implies that for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in L^2(\Omega)$

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_h^1 \boldsymbol{v}) w_h \, dx = 0 \quad \forall w_h \in W_h, \tag{20d}$$

$$\int_\Omega \operatorname{div}(\boldsymbol{v}_h)(w - P_h^0 w) \, dx = 0 \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h. \tag{20e}$$

Finally, we denote by Q_h^k the orthogonal projection in $L^2(\mathcal{F}_h)$ onto the space of piecewise polynomials of degree smaller than or equal to k defined on the faces of our mesh.

3. Main Results

The main result of our present work, the higher order convergence property of the modified mixed hybrid BDM_1 scheme for our problem (1), reads as follows.

Theorem 1. *Let $a \in L^\infty(\Omega\times[0,T];\mathbb{R}^{d\times d})$ be uniformly elliptic with ellipticity constant $\lambda > 0$ and upper bound $\Lambda > 0$, let $f \in L^2(\Omega\times[0,T])$, $\mathbf{b} \in L^\infty([0,T];W^{2,\infty}(\Omega))$, and let Ω be a bounded Lipschitz domain. Moreover, let $u \in L^2([0,T];H^1_0(\Omega))$ denote a weak solution to the problem* (1) *and let $\mathbf{q}$ be defined by* (2b). *Let $(\mathbf{q}_h^n, u_h^n, \lambda_h^n) \in \hat{\mathbf{V}}_h \times W_h \times \Lambda_{h,0}$ be a solution to the numerical scheme (18) for a given sequence of discrete times $0 = t_0 < t_1 < \ldots < t_N = T$ and let $\sigma_{\max}$ be defined as in* (19). *Set*

$$\tau := \max_{1\le n\le N}(t_n - t_{n-1}).$$

Then there exists some $h_{\max} = h_{\max}(\Omega, \sigma_{\max}, \lambda, \Lambda, ||\mathbf{b}||_{L^\infty([0,T],W^{1,\infty}(\Omega))})$ and some maximal time step size $\tau_{\max} = \tau_{\max}(\Omega, \sigma_{\max}, \lambda, \Lambda, ||\mathbf{b}||_{L^\infty(\Omega\times[0,T])})$ such that the following holds: Provided that the smallness conditions $\tau < \tau_{\max}$ and $h < h_{\max}$ are satisfied, the a priori error estimate

$$\begin{aligned}
&\sup_{1\le n\le N} ||P_h^0 u(\cdot,t_n) - u_h^n||^2_{L^2(\Omega)} + \sum_{n=1}^{N}(t_n - t_{n-1})||\mathbf{q}(\cdot,t_n) - \mathbf{q}_h^n||^2_{L^2(\Omega)} \\
&\le C(T,\sigma_{\max},\lambda,\Lambda,\Omega,||\mathbf{b}||_{L^\infty([0,T];W^{2,\infty}(\Omega))})\Big(||P_h^0 u_0 - u_h^0||^2_{L^2(\Omega)} \\
&\qquad + ||\partial_{tt}u||^2_{L^\infty(\Omega\times[0,T])}\tau^2 + ||\mathbf{q}||^2_{L^\infty([0,T];H^2(\Omega))}h^4 + ||u||^2_{L^\infty([0,T];H^2(\Omega))}h^4\Big)
\end{aligned}$$

holds.

As in the usual case for BDM_1 finite elements, one may obtain a higher order reconstruction of the solution u itself by postprocessing applied to the Lagrange multipliers λ_h^n. This is the statement of the next corollary.

Corollary 2. *Let $a \in L^\infty(\Omega\times[0,T];\mathbb{R}^{d\times d})$ be uniformly elliptic with ellipticity constant $\lambda > 0$ and upper bound $\Lambda > 0$, let $f \in L^2(\Omega\times[0,T])$, $\mathbf{b} \in L^\infty([0,T];W^{2,\infty}(\Omega))$, and let Ω be a bounded Lipschitz domain. Moreover, let $u \in L^2([0,T];H^1_0(\Omega))$ denote a weak solution to the problem* (1) *and let $\mathbf{q}$ be defined by* (2b). *Let $(\mathbf{q}_h^n, u_h^n, \lambda_h^n) \in \hat{\mathbf{V}}_h \times W_h \times \Lambda_{h,0}$ be a solution to the numerical scheme (18) for a given sequence of discrete times $0 = t_0 < t_1 < \ldots < t_N = T$ and let $\sigma_{\max}$ be defined as in* (19). *Set*

$$\tau := \max_{1\le n\le N}(t_n - t_{n-1}).$$

Let the postprocessed variable $\tilde{u}_h^n$ be defined locally on each $K \in \mathcal{T}_h$ by

$$\begin{aligned}
&\tilde{u}_h^n|_K \in \mathcal{P}_1(K),\\
&\int_F \tilde{u}_h^n|_K - \lambda_h^n \, dS = 0 \quad \forall F \in \mathcal{F}_h, F \subset \partial K,
\end{aligned}$$

i.e. in particular $\tilde{u}_h^n$ belongs to the space of lowest order Crouzeix-Raviart elements. Then there exists some $h_{\max} = h_{\max}(\Omega, \sigma_{\max}, \lambda, \Lambda, ||\mathbf{b}||_{L^\infty([0,T],W^{1,\infty}(\Omega))})$ and some maximal time step size $\tau_{\max} = \tau_{\max}(\Omega, \sigma_{\max}, \lambda, \Lambda, ||\mathbf{b}||_{L^\infty(\Omega\times[0,T])})$ such that the *following holds: Provided that the smallness conditions $\tau < \tau_{\max}$ and $h < h_{\max}$ are satisfied, the a priori error estimate*

$$\sum_{n=1}^{N}(t_n - t_{n-1})||u(\cdot,t_n) - \tilde{u}_h^n||_{L^2(\Omega)}^2$$
$$\leq C(T,\sigma_{\max},\lambda,\Lambda,\Omega,||\mathbf{b}||_{L^\infty([0,T];W^{2,\infty}(\Omega))})\Big(||P_h^0 u_0 - u_h^0||_{L^2(\Omega)}^2$$
$$+ ||\partial_{tt}u||_{L^\infty(\Omega\times[0,T])}^2\tau^2 + ||\mathbf{q}||_{L^\infty([0,T];H^2(\Omega))}^2 h^4 + ||u||_{L^\infty([0,T];H^2(\Omega))}^2 h^4\Big)$$

holds.

4. Derivation of the Error Estimates

4.1. **Error estimation is equivalent to proving approximation properties for the operator $\mathcal{B}$.** Let us now proceed to the proof of our main result. We first reduce our main theorem to an improved estimate for the approximation quality of the advective flux reconstruction obtained by the operator $\mathcal{B}$; this improved estimate will then be the content of Lemma 3 below.

Proof of Theorem 1. In the following, we use the abbreviations $u^n := u(\cdot,t_n)$, $\mathbf{b}^n := \mathbf{b}(\cdot,t_n)$, $f^n := f(\cdot,t_n)$, and $a^n := a(\cdot,t_n)$. We then have by (1)

$$\frac{u^n - u^{n-1}}{t_n - t_{n-1}} = -\operatorname{div}\mathbf{q}^n + f^n + \frac{u^n - u^{n-1}}{t_n - t_{n-1}} - \partial_t u(\cdot,t_n), \tag{21a}$$

$$\mathbf{q}^n = -a^n\nabla u^n + u^n\mathbf{b}^n \tag{21b}$$

for $\mathbf{q}^n$ defined by the second equation, which implies

$$\int_\Omega \frac{u^n - u^{n-1}}{t_n - t_{n-1}} w\ dx = -\int_\Omega \operatorname{div}\mathbf{q}^n w\ dx + \int_\Omega f^n w\ dx + \int_\Omega \left(\frac{u^n - u^{n-1}}{t_n - t_{n-1}} - \partial_t u(\cdot,t_n)\right) w\ dx, \tag{22a}$$

$$\int_\Omega (a^n)^{-1}\mathbf{q}^n\cdot\mathbf{v}\ dx = \sum_{K\in\mathcal{T}_h}\int_K u^n\operatorname{div}\mathbf{v}\ dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K} u^n\mathbf{v}|_K\cdot\mathbf{n}_{\partial K}\ dS + \int_\Omega u^n(a^n)^{-1}\mathbf{b}^n\cdot\mathbf{v}\ dx \tag{22b}$$

for any $w \in L^2(\Omega)$ and any $\mathbf{v} \in \bigoplus_{K\in\mathcal{T}_h}\mathbf{H}^{\mathrm{div}}(K)$.

Arguing analogously to [5] and setting

$$\mathbf{d}_h^n := \mathbf{q}^n - \mathbf{q}_h^n, \tag{23a}$$

$$z_h^n := P_h^0 u^n - u_h^n, \tag{23b}$$

$$\mathbf{e}_h^n := \Pi_h^1\mathbf{q}^n - \mathbf{q}_h^n, \tag{23c}$$

it follows from (18) (with (18c) implying that $\mathbf{q}_h^n \in \mathbf{V}_h$, i.e. we may replace the broken integral $\sum_K\int_K$ in (18a) by the integral over the full domain $\int_\Omega$), (22), and the definition of P_h^0 (which implies that $\int_\Omega(u^n - P_h^0u^n)w_h\ dx = 0$ for any $w_h \in W_h$ and any $0 \leq n \leq N$; due to $\operatorname{div}\mathbf{v}_h \in W_h$ for $\mathbf{v}_h \in \hat{\mathbf{V}}_h$, this in particular entails $\int_\Omega(u^n - u_h^n)\operatorname{div}\mathbf{v}_h\ dx = \int_\Omega z_h^n\operatorname{div}\mathbf{v}_h\ dx$) that

(24a)

$$\int_\Omega \frac{z_h^n - z_h^{n-1}}{t_n - t_{n-1}} w_h \; dx = -\int_\Omega \operatorname{div} \mathbf{d}_h^n \; w_h \; dx + \int_\Omega \left(\frac{u^n - u^{n-1}}{t_n - t_{n-1}} - \partial_t u(\cdot, t_n) \right) \; w_h \; dx,$$

(24b)

$$\int_\Omega (a^n)^{-1} \mathbf{d}_h^n \cdot \mathbf{v}_h \; dx = \sum_{K \in \mathcal{T}_h} \int_K z_h^n \operatorname{div} \mathbf{v}_h \; dx + \int_\Omega (a^n)^{-1} (u^n \mathbf{b}^n - \mathcal{B}[\lambda_h^n]) \cdot \mathbf{v}_h \; dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (u^n - \lambda_h^n) \mathbf{v}_h|_K \cdot \mathbf{n}_{\partial K} \; dS,$$

(24c)

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu_h \mathbf{q}_h^n|_K \cdot \mathbf{n}_{\partial K} \; dS = 0$$

for any $(\mathbf{v}_h, w_h, \mu_h) \in \hat{\mathbf{V}}_h \times W_h \times \Lambda_{h,0}$.

Testing the error equation (24b) with $\mathbf{e}_h^n \in \mathbf{V}_h$ (to see the latter inclusion, note that $\mathbf{q}_h^n \in \mathbf{V}_h$ holds by (24c); due to $\mathbf{e}_h^n \in \mathbf{V}_h$, we may in particular replace the broken integral $\sum_K \int_K$ in (24b) by $\int_\Omega$) and taking into account $\mathbf{d}_h^n = \mathbf{e}_h^n + \mathbf{q}^n - \Pi_h^1 \mathbf{q}^n$, we infer (note that the integrals over the faces cancel due to $\mathbf{v}_h := \mathbf{e}_h^n$ having continuous normal component across the faces and due to the $\mathbf{n}_{\partial K}$ having opposite signs for the two simplices K adjacent to an interior face)

$$\int_\Omega (a^n)^{-1} \mathbf{e}_h^n \cdot \mathbf{e}_h^n \; dx = \int_\Omega \operatorname{div} \mathbf{e}_h^n \; z_h^n \; dx - \int_\Omega (a^n)^{-1} (\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n) \cdot \mathbf{e}_h^n \; dx + \int_\Omega (a^n)^{-1} (u^n \mathbf{b}^n - \mathcal{B}[\lambda_h^n]) \cdot \mathbf{e}_h^n \; dx.$$

Taking into account the fact that

$$\begin{aligned} &\int_\Omega \operatorname{div} \mathbf{e}_h^n \; w_h \; dx = \int_\Omega \operatorname{div} \mathbf{d}_h^n \; w_h \; dx \\ &= -\int_\Omega \frac{z_h^n - z_h^{n-1}}{t_n - t_{n-1}} w_h \; dx + \int_\Omega \left(\frac{u^n - u^{n-1}}{t_n - t_{n-1}} - \partial_t u(\cdot, t_n) \right) w_h \; dx \end{aligned}$$

holds for any $w_h \in W_h$ by (20d) and (24a) (i.e. in particular for $w_h := z_h^n$), we deduce

$$\begin{aligned} &\int_\Omega (z_h^n - z_h^{n-1}) z_h^n \; dx + (t_n - t_{n-1}) \int_\Omega (a^n)^{-1} \mathbf{e}_h^n \cdot \mathbf{e}_h^n \; dx \\ &= \int_\Omega \big(u^n - u^{n-1} - (t_n - t_{n-1}) \partial_t u(\cdot, t_n) \big) z_h^n \; dx \\ &\quad - (t_n - t_{n-1}) \int_\Omega (a^n)^{-1} (\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n) \cdot \mathbf{e}_h^n \; dx \\ &\quad + (t_n - t_{n-1}) \int_\Omega (a^n)^{-1} (u^n \mathbf{b}^n - \mathcal{B}[\lambda_h^n]) \cdot \mathbf{e}_h^n \; dx. \end{aligned}$$

Noting that $(z_h^n - z_h^{n-1}) z_h^n = \frac{1}{2}|z_h^n|^2 - \frac{1}{2}|z_h^{n-1}|^2 + \frac{1}{2}|z_h^n - z_h^{n-1}|^2$, it follows by Young's inequality that

$$\begin{aligned}
&||z_h^n||^2_{L^2(\Omega)} + ||z_h^n - z_h^{n-1}||^2_{L^2(\Omega)} + (t_n - t_{n-1})c||\mathbf{e}_h^n||^2_{L^2(\Omega)} \\
&\le ||z_h^{n-1}||^2_{L^2(\Omega)} + (t_n - t_{n-1})C||\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})C||u^n \mathbf{b}^n - \mathcal{B}[\lambda_h^n]||^2_{L^2(\Omega)} \\
&\quad + 2\left|\left|u^n - u^{n-1} - (t_n - t_{n-1})\partial_t u(\cdot, t_n)\right|\right|_{L^2(\Omega)} ||z_h^n||_{L^2(\Omega)} \\
&\le ||z_h^{n-1}||^2_{L^2(\Omega)} + (t_n - t_{n-1})C||\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})C||u^n \mathbf{b}^n - \mathcal{B}[\lambda_h^n]||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})^2 C||\partial_{tt} u||_{L^\infty(\Omega\times[0,T])}||z_h^n||_{L^2(\Omega)} \\
&\le ||z_h^{n-1}||^2_{L^2(\Omega)} + (t_n - t_{n-1})C||\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})C||u^n \mathbf{b}^n - \mathcal{B}[\lambda_h^n]||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})||z_h^n||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})^3 C||\partial_{tt} u||^2_{L^\infty(\Omega\times[0,T])}.
\end{aligned}$$

Furthermore, we have

$$||\mathbf{d}_h^n||_{L^2(\Omega)} \le ||\mathbf{e}_h^n||_{L^2(\Omega)} + ||\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n||_{L^2(\Omega)}.$$

Summing up, the estimate (25) below then implies in connection with the two previous estimates in case $h \le h_{\max}$ (recall (23a) and (23b))

$$\begin{aligned}
&||z_h^n||^2_{L^2(\Omega)} + ||z_h^n - z_h^{n-1}||^2_{L^2(\Omega)} + (t_n - t_{n-1})(c - Ch^2||\mathbf{b}^n||^2_{W^{1,\infty}(\Omega)})||\mathbf{d}_h^n||^2_{L^2(\Omega)} \\
&\le ||z_h^{n-1}||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})C||\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})C(1 + ||\mathbf{b}^n||^2_{L^\infty(\Omega)})||z_h^n||^2_{L^2(\Omega)} \\
&\quad + (t_n - t_{n-1})Ch^4||\mathbf{b}^n||^2_{W^{2,\infty}(\Omega)}||u^n||^2_{H^2(\Omega)} \\
&\quad + (t_n - t_{n-1})^3 C||\partial_{tt} u||^2_{L^\infty(\Omega\times[0,T])}.
\end{aligned}$$

Possibly decreasing $h_{\max}$, we see that for the prefactor of the third term on the left-hand side, we may assume that $(c - Ch^2||\mathbf{b}^n||^2_{W^{1,\infty}(\Omega)}) \ge c$.

Set $\eta_n := \prod_{k=1}^n (1 - C \cdot (1 + ||\mathbf{b}||^2_{L^\infty(\Omega\times[0,T])})(t_k - t_{k-1}))$, with C denoting the constant in the third line of the right-hand side in the previous estimate. We then get by multiplying the previous inequality by η_{n-1} and taking the sum with respect to n (recall that $\tau = \max_{1\le n\le N}(t_n - t_{n-1})$)

$$\begin{aligned}
&\eta_N ||P_h^0 u^N - u_h^N||^2_{L^2(\Omega)} + \sum_{n=1}^N \eta_{n-1}||z_h^n - z_h^{n-1}||^2_{L^2(\Omega)} \\
&+ \sum_{n=1}^N \eta_{n-1}(t_n - t_{n-1})c||\mathbf{q}^n - \mathbf{q}_h^n||^2_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\leq ||P_h^0 u^0 - u_h^0||^2_{L^2(\Omega)} \\
&\quad + \sum_{n=1}^{N} \eta_{n-1}(t_n - t_{n-1}) C ||\mathbf{q}^n - \Pi_h^1 \mathbf{q}^n||^2_{L^2(\Omega)} \\
&\quad + \sum_{n=1}^{N} \eta_{n-1}(t_n - t_{n-1}) C h^4 ||\mathbf{b}^n||^2_{W^{2,\infty}(\Omega)} ||u^n||^2_{H^2(\Omega)} \\
&\quad + \sum_{n=1}^{N} \eta_{n-1}(t_n - t_{n-1}) C \tau^2 ||\partial_{tt} u||^2_{L^\infty(\Omega\times[0,T])},
\end{aligned}
$$

which gives the desired error estimate after an application of (20b) (note that η_n is bounded from below by $\exp(-C(1+||\mathbf{b}||^2_{L^\infty(\Omega\times[0,T])})T)$, provided that $\tau \leq \tau_{\max}$; note also that $\eta_n \leq 1$ under the same condition). □

4.2. **Approximation properties of the reconstruction operator $\mathcal{B}$.** It remains to establish the following lemma, which quantifies the approximation properties of our operator $\mathcal{B}$ as defined in (16).

Lemma 3. *Let $(\mathbf{q}_h^n, u_h^n, \lambda_h^n) \in \mathbf{V}_h \times W_h \times \Lambda_{h,0}$ be a solution to the numerical scheme (18) for a given $n \in \mathbb{N}$. There exists a constant $c = c(\Omega, \sigma_{\max}, \lambda) > 0$ such that the following holds: Provided that the smallness condition $h^2||\mathbf{b}^n||^2_{L^\infty(\Omega)} + h^4||\nabla \mathbf{b}^n||^2_{L^\infty(\Omega)} \leq c$ is satisfied, the operator $\mathcal{B}$ admits the estimate*

$$
\begin{aligned}
||u^n\mathbf{b}^n - \mathcal{B}[\lambda_h^n]||^2_{L^2(\Omega)} \leq & C h^2 ||\mathbf{b}^n||^2_{W^{1,\infty}(\Omega)} ||\mathbf{q}^n - \mathbf{q}_h^n||^2_{L^2(\Omega)} \\
& + C ||\mathbf{b}^n||^2_{L^\infty(\Omega)} ||P_h^0 u^n - u_h^n||^2_{L^2(\Omega)} \\
& + C h^4 ||\mathbf{b}^n||^2_{W^{2,\infty}(\Omega)} ||u^n||^2_{H^2(\Omega)}.
\end{aligned} \tag{25}
$$

Proof. First, we observe that it is enough to establish the estimate

$$
\begin{aligned}
&||u^n\mathbf{b}^n - \mathcal{B}[\lambda_h^n]||^2_{L^2(\Omega)} \\
&\leq C h^2 \big(||\mathbf{b}^n||^2_{L^\infty(\Omega)} + h^2 ||\nabla \mathbf{b}^n||^2_{L^\infty(\Omega)}\big) ||\mathbf{q}^n - \mathbf{q}_h^n||^2_{L^2(\Omega)} \\
&\quad + C ||\mathbf{b}^n||^2_{L^\infty(\Omega)} ||P_h^0 u^n - u_h^n||^2_{L^2(\Omega)} \\
&\quad + C h^2 \big(||\mathbf{b}^n||^2_{L^\infty(\Omega)} + h^2 ||\nabla \mathbf{b}^n||^2_{L^\infty(\Omega)}\big) ||\mathcal{B}[\lambda_h^n] - u^n \mathbf{b}^n||^2_{L^2(\Omega)} \\
&\quad + C h^4 ||\nabla \mathbf{b}^n||^2_{L^\infty(\Omega)} ||\nabla u^n||^2_{L^2(\Omega)} \\
&\quad + C h^6 ||\nabla \mathbf{b}^n||^2_{L^\infty(\Omega)} ||D^2 u^n||^2_{L^2(\Omega)} \\
&\quad + C h^4 ||D^2 \mathbf{b}^n||^2_{L^\infty(\Omega)} ||u^n||^2_{L^2(\Omega)} \\
&\quad + C h^6 ||D^2 \mathbf{b}^n||^2_{L^\infty(\Omega)} ||\nabla u^n||^2_{L^2(\Omega)} \\
&\quad + C h^4 ||D^2(\mathbf{b}^n u^n)||^2_{L^2(\Omega)}.
\end{aligned} \tag{26}
$$

Indeed, by an absorption argument applied to the third term on the right-hand side, the bound (26) entails our lemma.

To prove (26), we first show that the Lagrange multipliers represent a better approximation for the scalar unknown on the edges. Following along the lines of the proof of Theorem 4.3 in [5], we first test (18b) with a test function $\mathbf{v}_h \in \hat{\mathbf{V}}_h$ supported in some $K \in \mathcal{T}_h$ and satisfying $\mathbf{v}_h \cdot \mathbf{n}_{\partial K} = \lambda_h^n - Q_h^1 u^n$ on F and $\mathbf{v}_h \cdot \mathbf{n}_{\partial K} =$

0 on $\partial K \setminus F$ (where $F \in \mathcal{F}_h, F \subset \partial K$); note that we have

$$||\mathbf{v}_h||_{L^2(K)} + h_K ||\mathbf{v}_h||_{H^1(K)} \le C h_K^{\frac{1}{2}} ||\lambda_h^n - Q_h^1 u^n||_{L^2(F)}.$$

This yields

$$\int_K (a^n)^{-1} \mathbf{q}_h^n \cdot \mathbf{v}_h \ dx - \int_K u_h^n \operatorname{div} \mathbf{v}_h \ dx - \int_K (a^n)^{-1} \mathcal{B}[\lambda_h^n] \cdot \mathbf{v}_h \ dx$$
$$+ \int_F \lambda_h^n (\lambda_h^n - Q_h^1 u^n) \ dS = 0.$$

On the other hand, from (22b), we have

$$\int_K (a^n)^{-1} \mathbf{q}^n \cdot \mathbf{v}_h \ dx - \int_K u^n \operatorname{div} \mathbf{v}_h \ dx - \int_K (a^n)^{-1} u^n \mathbf{b}^n \cdot \mathbf{v}_h \ dx$$
$$+ \int_F u^n (\lambda_h^n - Q_h^1 u^n) \ dS = 0.$$

Subtracting the last two equations from each other and using that $\operatorname{div} \mathbf{v}_h \in \mathcal{P}_0(K)$, we obtain

$$\int_K (a^n)^{-1} (\mathbf{q}_h^n - \mathbf{q}^n) \cdot \mathbf{v}_h \ dx - \int_K (u_h^n - P_h^0 u^n) \operatorname{div} \mathbf{v}_h \ dx$$
$$- \int_K (a^n)^{-1} (\mathcal{B}[\lambda_h^n] - u^n \mathbf{b}^n) \cdot \mathbf{v}_h \ dx + \int_F |\lambda_h^n - Q_h^1 u^n|^2 \ dS = 0.$$

This gives

$$||\lambda_h^n - Q_h^1 u^n||_{L^2(F)} \le C h_K^{\frac{1}{2}} ||\mathbf{q}_h^n - \mathbf{q}^n||_{L^2(K)} + C h_K^{-\frac{1}{2}} ||u_h^n - P_h^0 u^n||_{L^2(K)}$$
$$+ C h_K^{\frac{1}{2}} ||\mathcal{B}[\lambda_h^n] - u^n \mathbf{b}^n||_{L^2(K)}. \tag{27}$$

Therefore, λ_h^n is a better approximation for u^n on the edges, provided that we can bound the right-hand side of this estimate appropriately.

By Proposition 4 below, this estimate entails an improved bound for the approximation quality of $\mathbf{n} \cdot \mathcal{B}[\lambda_h^n]$ for the advective normal flux $\mathbf{n} \cdot u\mathbf{b}$ on the edges. More precisely, from the previous formula, we deduce the estimate

$$\begin{aligned}
&||\mathbf{n} \cdot \mathcal{B}[\lambda_h^n] - \mathbf{n} \cdot \mathbf{b}^n Q_h^1 u^n||_{L^2(F)}^2 \\
&\le C||\mathbf{n} \cdot \Pi_h^1 \mathbf{b}^n \ \lambda_h^n - \mathbf{n} \cdot \Pi_h^1 \mathbf{b}^n Q_h^1 u^n||_{L^2(F)}^2 \\
&\quad + C||\mathbf{n} \cdot \Pi_h^1 \mathbf{b}^n Q_h^1 u^n - \mathbf{n} \cdot \mathbf{b}^n Q_h^1 u^n||_{L^2(F)}^2 \\
&\quad + C||\mathbf{n} \cdot \mathcal{B}[\lambda_h^n] - \mathbf{n} \cdot \Pi_h^1 \mathbf{b}^n \lambda_h^n||_{L^2(F)}^2 \\
&\le C h_K ||\mathbf{b}^n||_{L^\infty(\Omega)}^2 ||\mathbf{q}_h^n - \mathbf{q}^n||_{L^2(K)}^2 + C h_K^{-1} ||\mathbf{b}^n||_{L^\infty(\Omega)}^2 ||u_h^n - P_h^0 u^n||_{L^2(K)}^2 \\
&\quad + C h_K ||\mathbf{b}^n||_{L^\infty(\Omega)}^2 ||\mathcal{B}[\lambda_h^n] - u^n \mathbf{b}^n||_{L^2(K)}^2 + C h_K^4 ||D^2 \mathbf{b}^n||_{L^\infty(\Omega)}^2 ||u^n||_{L^2(F)}^2 \\
&\quad + C h_K^4 ||\nabla \Pi_h^1 \mathbf{b}^n||_{L^\infty(\cup_{K \in \mathcal{T}_h} K)}^2 ||\nabla \lambda_h^n||_{L^2(F)}^2,
\end{aligned} \tag{28}$$

where the estimate for the term in the fourth line is a consequence of the rescaled version of Proposition 4 below and the definition (16) of the operator $\mathcal{B}$: By (16), the quantity $\mathbf{n} \cdot \mathcal{B}[\lambda_h^n]$ (which by definition is a linear polynomial on any face F of the triangulation) is prescribed to match the product of the normal component of $\Pi_h^1 \mathbf{b}^n = \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_h, F \subset \partial K} \sum_{i=1}^d b_{KFi}^n \mathbf{v}_{KF}^i$ and λ_h^n at the two points which divide F into three segments of equal size (case $d = 2$) respectively at the midpoints of the edges of the face (case $d = 3$). For this approximation of the product of the linear polynomials $\mathbf{n}\cdot\Pi_h^1\mathbf{b}^n$ and λ_h^n on the face F, Proposition 4 after a change of variables precisely provides the higher-order approximation property.

To obtain a bound on the derivative of the Lagrange multiplier λ_h^n, we test equation (24b) with a suitable test function $\mathbf{v}_h \in \hat{\mathbf{V}}_h$ satisfying $\operatorname{supp}\mathbf{v}_h \subset K$, $\int_K \operatorname{div}\mathbf{v}_h \, dx = 0$, $\mathbf{n}_{\partial K}\cdot\mathbf{v}_h = \lambda_h^n - Q_h^0\lambda_h^n$ on F and $\mathbf{n}_{\partial K}\cdot\mathbf{v}_h = 0$ on $\partial K \setminus F$ (note that we have $\int_F u^n(\lambda_h^n - Q_h^0\lambda_h^n)\, dS = \int_F (u^n - Q_h^0 u^n)(\lambda_h^n - Q_h^0\lambda_h^n)\, dS$ and $\int_F \lambda_h^n(\lambda_h^n - Q_h^0\lambda_h^n)\, dS = \int_F |\lambda_h^n - Q_h^0\lambda_h^n|^2\, dS$ by the properties of the projection operator Q_h^0). This yields

$$\begin{aligned}||\lambda_h^n - Q_h^0\lambda_h^n||^2_{L^2(F)} \leq \int_K (a^n)^{-1}\mathbf{d}_h^n\cdot\mathbf{v}_h\, dx - \int_K (a^n)^{-1}(u^n\mathbf{b}^n - \mathcal{B}[\lambda_h^n])\cdot\mathbf{v}_h\, dx \\ + ||u^n - Q_h^0 u^n||_{L^2(F)}||\lambda_h^n - Q_h^0\lambda_h^n||_{L^2(F)},\end{aligned}$$

which gives using the estimate $||\mathbf{v}_h||_{L^2(K)} \leq Ch_K^{1/2}||\lambda_h^n - Q_h^0\lambda_h^n||_{L^2(F)}$ (this estimate holds on the reference simplex and for a general simplex $K \in \mathcal{T}_h$ by transformation)

$$\begin{aligned}&||\nabla\lambda_h^n||_{L^2(F)} \leq Ch_K^{-1}||\lambda_h^n - Q_h^0\lambda_h^n||_{L^2(F)} \\ &\leq Ch_K^{-1/2}||\mathbf{q}_h^n - \mathbf{q}^n||_{L^2(K)} + Ch_K^{-1/2}||u^n\mathbf{b}^n - \mathcal{B}[\lambda_h^n]||_{L^2(K)} + C||\nabla u^n||_{L^2(F)}.\end{aligned} \tag{29}$$

Finally, using the definition of the interpolation operator Π_h^1 and its approximation properties as well as the fact that $\mathcal{B}$ takes values in $\mathbf{V}_h$ (note that elements $\mathbf{v}_h \in \mathbf{V}_h$ satisfy the bound $||\mathbf{v}_h||^2_{L^2(K)} \leq Ch_K||\mathbf{n}_{\partial K}\cdot\mathbf{v}_h||^2_{L^2(\partial K)}$), we obtain from (28) (applied to the different faces of K) and the previous estimate

$$\begin{aligned}&||\mathcal{B}[\lambda_h^n] - u^n\mathbf{b}^n||^2_{L^2(K)} \\
&\overset{(20b)}{\leq} 2||\mathcal{B}[\lambda_h^n] - \Pi_h^1(\mathbf{b}^n u^n)||^2_{L^2(K)} + Ch_K^4||D^2(\mathbf{b}^n u^n)||^2_{L^2(K)} \\
&\leq Ch_K||\mathbf{n}_{\partial K}\cdot(\mathcal{B}[\lambda_h^n] - Q_h^1(\mathbf{b}^n u^n))||^2_{L^2(\partial K)} + Ch_K^4||D^2(\mathbf{b}^n u^n)||^2_{L^2(K)} \\
&\leq Ch_K||\mathbf{n}_{\partial K}\cdot(\mathcal{B}[\lambda_h^n] - \mathbf{b}^n Q_h^1 u^n)||^2_{L^2(\partial K)} + Ch_K||Q_h^1(\mathbf{b}^n u^n) - \mathbf{b}^n Q_h^1 u^n||^2_{L^2(\partial K)} \\
&\quad + Ch_K^4||D^2(\mathbf{b}^n u^n)||^2_{L^2(K)} \\
&\overset{(28,30)}{\leq} Ch_K^2||\mathbf{b}^n||^2_{L^\infty(\Omega)}||\mathbf{q}_h^n - \mathbf{q}^n||^2_{L^2(K)} + C||\mathbf{b}^n||^2_{L^\infty(\Omega)}||u_h^n - P_h^0 u^n||^2_{L^2(K)} \\
&\quad + Ch_K^2||\mathbf{b}^n||^2_{L^\infty(\Omega)}||\mathcal{B}[\lambda_h^n] - u^n\mathbf{b}^n||^2_{L^2(K)} + Ch_K^5||D^2\mathbf{b}^n||^2_{L^\infty(\Omega)}||u^n||^2_{L^2(\partial K)} \\
&\quad + Ch_K^5||\nabla\mathbf{b}^n||^2_{L^\infty(\Omega)}||\nabla\lambda_h^n||^2_{L^2(\partial K)} + Ch_K^5||\nabla\mathbf{b}^n||^2_{L^\infty(\Omega)}||\nabla u^n||^2_{L^2(\partial K)} \\
&\quad + Ch_K^5||D^2\mathbf{b}^n||^2_{L^\infty(\Omega)}||u^n||^2_{L^2(\partial K)} + Ch_K^4||D^2(\mathbf{b}^n u^n)||^2_{L^2(K)} \\
&\overset{(29)}{\leq} Ch_K^2\big(||\mathbf{b}^n||^2_{L^\infty(\Omega)} + h_K^2||\nabla\mathbf{b}^n||^2_{L^\infty(\Omega)}\big)||\mathbf{q}_h^n - \mathbf{q}^n||^2_{L^2(K)} \\
&\quad + C||\mathbf{b}^n||^2_{L^\infty(\Omega)}||u_h^n - P_h^0 u^n||^2_{L^2(K)} \\
&\quad + Ch_K^2\big(||\mathbf{b}^n||^2_{L^\infty(\Omega)} + h_K^2||\nabla\mathbf{b}^n||^2_{L^\infty(\Omega)}\big)||\mathcal{B}[\lambda_h^n] - u^n\mathbf{b}^n||^2_{L^2(K)} \\
&\quad + Ch_K^5||\nabla\mathbf{b}^n||^2_{L^\infty(\Omega)}||\nabla u^n||^2_{L^2(\partial K)} \\
&\quad + Ch_K^4||D^2\mathbf{b}^n||^2_{L^\infty(\Omega)}||u^n||^2_{L^2(K)} \\
&\quad + Ch_K^6||D^2\mathbf{b}^n||^2_{L^\infty(\Omega)}||\nabla u^n||^2_{L^2(K)} \\
&\quad + Ch_K^4||D^2(\mathbf{b}^n u^n)||^2_{L^2(K)},\end{aligned}$$

from which (26) follows using the trace estimate

$$||\nabla u^n||^2_{L^2(\partial K)} \leq Ch_K^{-1}||\nabla u^n||^2_{L^2(K)} + Ch_K||D^2u^n||^2_{L^2(K)}$$

and taking the sum over all $K \in \mathcal{T}_h$. Note that in the last inequality of the previous estimate we have used a trace estimate for u^n on K; furthermore, in the penultimate inequality we have used the fact that

$$\begin{aligned} &Q_h^1(\mathbf{b}^n u^n) - \mathbf{b}^n\, Q_h^1 u^n = Q_h^1((\mathbf{b}^n - Q_h^0\mathbf{b}^n)u^n) - (\mathbf{b}^n - Q_h^0\mathbf{b}^n)Q_h^1 u^n \\ &= Q_h^1((\mathbf{b}^n - Q_h^0\mathbf{b}^n)(u^n - Q_h^0 u^n)) - (\mathbf{b}^n - Q_h^0\mathbf{b}^n)(Q_h^1 u^n - Q_h^0 u^n) \\ &\quad - (\mathbf{b}^n - Q_h^1\mathbf{b}^n)Q_h^0 u^n \end{aligned}$$

holds, which implies for $F \in \mathcal{F}_h$ the estimate

$$\begin{aligned} ||Q_h^1(\mathbf{b}^n u^n) - \mathbf{b}^n Q_h^1 u^n||_{L^2(F)} \leq &Ch_K^2||\nabla \mathbf{b}^n||_{L^\infty(\Omega)}||\nabla u^n||_{L^2(F)} \\ &+ Ch_K^2||D^2\mathbf{b}^n||_{L^\infty(\Omega)}||u^n||_{L^2(F)}. \end{aligned} \tag{30}$$

□

The estimate of Corollary 2 can be derived by following along the lines of the proof of Lemma 4.1 in [5], using (27), (25) and Theorem 1.

5. Auxiliary Results

5.1. Approximation of products of linear polynomials on the reference face by linear polynomials.

The following result is essential for our proof of higher-order convergence of the modified BDM_1 scheme, as it provides a justification for the improved formula for the advective flux.

Proposition 4. *Let $m, Q \in \mathcal{P}_1(\hat{F})$ be two first-order polynomials defined on the reference face*

$$\hat{F} = \begin{cases} [0,1] & \text{if } d = 2, \\ \text{conv}\{(0,0),(1,0),(0,1)\} & \text{if } d = 3, \end{cases}$$

where conv *denotes the convex hull.*

Let further A denote the first-order polynomial on $\hat{F}$ obtained by setting

$$A(x) := m(x)Q(x) \text{ for } \begin{cases} x \in \{\frac{1}{3}, \frac{2}{3}\} & \text{if } d = 2, \\ x \in \{(0.5,0),(0.5,0.5),(0,0.5)\} & \text{if } d = 3. \end{cases}$$

Then it holds that

$$||mQ - A||_{L^2(\hat{F})} \leq C||\nabla m||_{L^\infty(\hat{F})}||\nabla Q||_{L^2(\hat{F})}, \tag{31}$$

where the constant $C > 0$ is independent of m, Q and A.

Recall that the modification of the scheme introduced in [7] corresponds to replacing the direct (first-order) approximation u_h^n of our transported quantity u by a reconstruction of the transported quantity obtained by using the Lagrange multiplier λ_h^n. In the proof of Theorem 1 above, we have applied Proposition 4 to show that the reconstruction $\mathcal{B}[\lambda_h^n]$ approximates the advective flux with second-order accuracy, provided that the Lagrange multipliers λ_h^n are a second-order accurate approximation of the projected concentration $Q_h^1 u^n$ on the edges.

Proof. We define an interpolation operator $\mathcal{I}^1 : H^2(\hat{F}) \to \mathcal{P}_1(\hat{F})$ by requiring

$$\mathcal{I}^1 w(x) = w(x)$$

for $x \in \{\frac{1}{3}, \frac{2}{3}\}$ if $d = 2$ and $x \in \{(0.5, 0), (0.5, 0.5), (0, 0.5)\}$ if $d = 3$. By standard techniques, we obtain

$$||\mathcal{I}^1 w - w||_{L^2(\hat{F})} \le C||D^2 w||_{L^2(\hat{F})}. \tag{32}$$

Since $A = \mathcal{I}^1(mQ)$, estimate (31) follows from (32) and the fact that m and Q are linear polynomials on $\hat{F}$ (the latter fact implying that $D^2(mQ) = \nabla m \otimes \nabla Q + \nabla Q \otimes \nabla m$).

□

6. Computational results

In this section we illustrate our theoretical results and compare the classical and the modified BDM_1 scheme with the help of computational experiments. Our numerical results show that the error bounds derived above are sharp. Further numerical tests, in which the modified scheme was applied to nonlinear reactive transport problems, are contained in [2, 7].

6.1. **Numerical results for a 2D test problem.** In the first numerical test, problem (1) is solved on the unit square $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ on the time interval $[0, 1]$. For the diffusion coefficient and the velocity field, we choose $a = \mathbb{1}_2$ and $\mathbf{b} = (0, -1)^\top$, respectively. Moreover, homogeneous Dirichlet conditions are imposed on the boundary $\partial\Omega$. The source term f is prescribed so that the analytical solution of the problem reads

$$u(x, y, t) = x(1 - x)y(1 - y)e^{-t}.$$

In each refinement step, the mesh (which for the first test consists only of two triangles) is uniformly refined and the errors

$$\begin{aligned} E^{(1)}_{h,\tau} &= \tau^{1/2} \left(\sum_{n=1}^{N} \|\boldsymbol{q}(\cdot, t_n) - \boldsymbol{q}^n_h\|^2_{L^2(\Omega)} \right)^{1/2}, \\ E^{(2)}_{h,\tau} &= \max_{1\le n\le N} \|u(\cdot, t_n) - u^n_h\|_{L^2(\Omega)}, \\ E^{(3)}_{h,\tau} &= \max_{1\le n\le N} \|P^0_h u(\cdot, t_n) - u^n_h\|_{L^2(\Omega)} \end{aligned}$$

are computed. All computations are run for a constant time step size $\tau = 0.001$, which is chosen sufficiently small to ensure that the time discretization error is negligible compared to the space discretization error. The experimental orders of convergence are determined by

$$\mathrm{EOC}^{(i)}_{h,\tau} = \log_2 \left(\frac{E^{(i)}_{2h,\tau}}{E^{(i)}_{h,\tau}} \right), \quad i \in \{1, 2, 3\}.$$

The results of the computations are listed in Tables 1 and 2. As expected, we obtain optimal second order convergence for the flux variable if the advective fluxes are approximated using the Lagrange multipliers, whereas only suboptimal first order convergence is observed when the classical scheme is used. Moreover, the scalar unknowns are approximated with first order accuracy for both schemes, the magnitude of the errors being of almost equal size. For the approximation of the projection $P_h^0 u$ of the scalar variable into the space of piecewise constants by our numerical solution u_h^n, we observe the usual (second-order) superconvergence result both in the case of the classical scheme and in the case of the modified scheme.

6.2. **Numerical results for a 3D test problem.** Finally, we present a numerical example to demonstrate that the modified scheme works also in three space dimensions. More precisely, we solve problem (1) on the unit cube $\Omega = (0,1)^3 \subset \mathbb{R}^3$ on the time interval $[0,1]$ for constant diffusion $a = \mathbb{1}_3$, a constant velocity field $\mathbf{b} = (0,-1,0)^\top$ and homogeneous Dirichlet boundary conditions. Similarly to the previous section, the right hand side f is prescribed so that

$$u(x,y,z,t) = x(1-x)y(1-y)z(1-z)e^{-t}$$

represents the analytical solution. The computations are carried out for a constant time step size $\tau = 0.05$ and the mesh that initially consists of five simplices is uniformly refined in each refinement step. The results of the computations are listed in Tables 3 and 4 and clearly confirm our theoretical findings. As in the two-dimensional case, the modified scheme provides optimal second order convergence for the flux variable, whereas the standard scheme is only of first order. The magnitude of the errors for the approximation of the scalar unknown and its projection into the space of piecewise constant polynomials are almost equal for both schemes.

triangles	$E_{h,\tau}^{(1)}$	EOC	$E_{h,\tau}^{(2)}$	EOC	$E_{h,\tau}^{(3)}$	EOC
$2 \cdot 4^0$	4.36e-02		2.91e-02		1.80e-02	
$2 \cdot 4^1$	2.01e-02	1.12	1.60e-02	0.86	6.37e-03	1.50
$2 \cdot 4^2$	5.94e-03	1.76	8.57e-03	0.90	1.88e-03	1.76
$2 \cdot 4^3$	1.56e-03	1.93	4.36e-03	0.97	4.94e-04	1.93
$2 \cdot 4^4$	3.96e-04	1.98	2.19e-03	0.99	1.24e-04	1.99
$2 \cdot 4^5$	9.88e-05	2.00	1.10e-03	1.00	3.07e-05	2.02
$2 \cdot 4^6$	2.39e-05	2.04	5.48e-04	1.00	7.18e-06	2.10

TABLE 1. L^2-errors and experimental orders of convergence (EOC) for the modified scheme (2D example)

triangles	$E_{h,\tau}^{(1)}$	EOC	$E_{h,\tau}^{(2)}$	EOC	$E_{h,\tau}^{(3)}$	EOC
$2 \cdot 4^0$	4.13e-02		2.91e-02		1.79e-02	
$2 \cdot 4^1$	2.05e-02	1.01	1.60e-02	0.86	6.34e-03	1.50
$2 \cdot 4^2$	6.91e-03	1.57	8.57e-03	0.90	1.87e-03	1.76
$2 \cdot 4^3$	2.51e-03	1.46	4.36e-03	0.97	4.91e-04	1.93
$2 \cdot 4^4$	1.09e-03	1.21	2.19e-03	0.99	1.23e-04	1.99
$2 \cdot 4^5$	5.19e-04	1.07	1.10e-03	1.00	3.05e-05	2.02
$2 \cdot 4^6$	2.56e-04	1.02	5.48e-04	1.00	7.12e-06	2.10

TABLE 2. L^2-errors and experimental orders of convergence (EOC) for the classical scheme (2D example)

tetrahedra	$E_{h,\tau}^{(1)}$	EOC	$E_{h,\tau}^{(2)}$	EOC	$E_{h,\tau}^{(3)}$	EOC
$5 \cdot 8^0$	1.26e-02		5.04e-03		4.21e-03	
$5 \cdot 8^1$	4.18e-03	1.60	2.48e-03	1.02	1.62e-03	1.38
$5 \cdot 8^2$	1.64e-03	1.35	1.60e-03	0.63	5.03e-04	1.69
$5 \cdot 8^3$	4.46e-04	1.88	8.33e-04	0.94	1.34e-04	1.91
$5 \cdot 8^4$	1.14e-04	1.96	4.23e-04	0.98	3.39e-05	1.98
$5 \cdot 8^5$	2.83e-05	2.01	2.13e-04	0.99	8.22e-06	2.05
$5 \cdot 8^6$	6.54e-06	2.11	1.07e-04	0.99	1.74e-06	2.24

TABLE 3. L^2-errors and experimental orders of convergence (EOC) for the modified scheme (3D example)

tetrahedra	$E_{h,\tau}^{(1)}$	EOC	$E_{h,\tau}^{(2)}$	EOC	$E_{h,\tau}^{(3)}$	EOC
$5 \cdot 8^0$	1.25e-02		5.04e-03		4.21e-03	
$5 \cdot 8^1$	4.14e-03	1.60	2.48e-03	1.02	1.63e-03	1.37
$5 \cdot 8^2$	1.80e-03	1.20	1.60e-03	0.63	5.04e-04	1.69
$5 \cdot 8^3$	6.19e-04	1.54	8.33e-04	0.94	1.34e-04	1.91
$5 \cdot 8^4$	2.51e-04	1.30	4.23e-04	0.98	3.40e-05	1.99
$5 \cdot 8^5$	1.17e-04	1.10	2.13e-04	0.99	8.22e-06	2.05
$5 \cdot 8^6$	5.74e-05	1.02	1.07e-04	0.99	1.74e-06	2.24

TABLE 4. L^2-errors and experimental orders of convergence (EOC) for the classical scheme (3D example)

7. Conclusion

In this work, we analyzed a modified BDM_1 mixed hybrid finite element scheme for advection-diffusion problems in two and three spatial dimensions. By using the Lagrange multipliers – which are introduced during hybridization – for the discretization of the advective flux, second order convergence of the total flux variable can be restored. The classical scheme BDM_1 scheme, which makes use of the cell-wise constant approximations of the scalar unknown in the discretization of the advective term, fails to display this optimal-order convergence behavior if an advective term in divergence form is present: In the case of the classical scheme, the accuracy of the approximation for the flux is limited by the (first-order) accuracy of the approximation for the scalar unknown, while the Lagrange multipliers represent a second-order approximation of the scalar unknown on the faces.

References

[1] D. N. Arnold and F. Brezzi. Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. *RAIRO Modél. Math. Anal. Numér.*, 19(1):7–32, 1985.

[2] M. Bause, F. Brunner, P. Knabner, and F. A. Radu. An improved optimal order mixed finite element method for semilinear transport problems. In A. Cangiani et al., editor, *Numerical Mathematics and Advances Applications. Proceedings of ENUMATH 2011*, pages 247–255, 2013.

[3] D. Boffi, F. Brezzi, and M. Fortin. *Mixed Finite Element Methods and Applications*. Springer, 2010.

[4] Daniele Boffi, Franco Brezzi, Leszek F. Demkowicz, Ricardo G. Durán, Richard S. Falk, and Michel Fortin. *Mixed finite elements, compatibility conditions, and applications*, volume 1939

of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin; Fondazione C.I.M.E., Florence, 2008. Lectures given at the C.I.M.E. Summer School held in Cetraro, June 26–July 1, 2006, Edited by Boffi and Lucia Gastaldi.

[5] F. Brezzi, J. Douglas, and L.D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numerische Mathematik*, 47:217–235, 1985.

[6] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer, New York, 1991.

[7] F. Brunner, F. A. Radu, M. Bause, and P. Knabner. Optimal order convergence of a modified BDM_1 mixed finite element scheme for reactive transport in porous media. *Advances in Water Resources*, 35:163–171, 2012.

[8] F. Brunner, F. A. Radu, and P. Knabner. Analysis of an upwind-mixed finite element method for transport problems. *SIAM J. Numer. Anal.*, 52:83–102, 2014.

[9] A. Demlow. Suboptimal and optimal convergence in mixed finite element methods. *SIAM J. Numer. Anal.*, 39:1938–1953, 2002.

[10] Alan Demlow. Localized pointwise error estimates for mixed finite element methods. *Mathematics of Computation*, 73(248):1623–1653 (electronic), 2004.

[11] J. Douglas, Jr. and J. E. Roberts. Global Estimates for Mixed Methods for Second Order Elliptic Equations. *Mathematics of Computation*, 44(169):39–52, 1985.

[12] J. Douglas, Jr. and J. E. Roberts. Mixed finite element methods for second order elliptic problems. *Mat. Aplic. Comp.*, 1(1):91–103, 1982.

[13] J.E. Roberts and J.-M. Thomas. Mixed and hybrid methods. In *Finite Element Methods (Part 1)*, volume 2 of *Handbook of Numerical Analysis*, pages 523 – 639. Elsevier, 1991.

[14] M. Vohralik. A posteriori error estimates for lowest-order mixed finite element discretizations of convection-diffusion-reaction equations. *SIAM J. Numer. Anal.*, 45:1570–1599, 2007.

[15] A. Younes, P. Ackerer, and F. Delay. Mixed finite elements for solving 2-d diffusion-type equations. *Rev. Geophys.*, 48, 2010.

Department of Mathematics, University of Erlangen-Nürnberg, 91058 Erlangen, Germany, E-Mail: brunner@math.fau.de

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany, E-Mail: julian.fischer@mis.mpg.de

Department of Mathematics, University of Erlangen-Nürnberg, 91058 Erlangen, Germany, E-Mail: knabner@math.fau.de