CONVERGENCE RATES OF THE ALLEN-CAHN EQUATION TO MEAN CURVATURE FLOW: A SHORT PROOF BASED ON RELATIVE ENTROPIES

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Abstract. We give a short and self-contained proof for rates of convergence of the Allen-Cahn equation towards mean curvature flow, assuming that a classical (smooth) solution to the latter exists and starting from well-prepared initial data. Our approach is based on a relative entropy technique. In particular, it does not require a stability analysis for the linearized Allen-Cahn operator. As our analysis also does not rely on the comparison principle, we expect it to be applicable to more complex equations and systems.

1. Introduction

The Allen-Cahn equation

$$\frac{d}{dt}u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon) \tag{1}$$

– with a suitable double-well potential W like for instance $W(s) = c\,(1-s^2)^2$, $c>0$ – is the most natural diffuse-interface approximation for (two-phase) mean curvature flow: It is well-known that in the limit of vanishing interface width $\varepsilon \to 0$, the solutions u_ε to the Allen-Cahn equation (1) converge to a characteristic function $\chi : \mathbb{R}^d \times [0,T] \to \{-1,1\}$ whose interface evolves by motion by mean curvature. For a proof of this fact in the framework of Brakke solutions to mean curvature flow, we refer to [8], while for the convergence towards the viscosity solution of the level-set formulation under the assumption of non-fattening we refer to [5]. Provided that the total energy converges in the limit $\varepsilon \to 0$, one may prove that the limit is a distributional solution [10]. For a general compactness statement using the gradient-flow structure of (1) and the identification of the limit in the radially symmetric case, we refer the reader to [2]. Under the assumption of the existence of a smooth limiting evolution, rates of convergence may be derived based on a strategy of matched asymptotic expansions and the stability of the linearized Allen-Cahn operator [3, 4].

The Allen-Cahn equation corresponds to the L^2 gradient flow of the Ginzburg-Landau energy functional

$$E_\varepsilon[v] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{\varepsilon}W(v)\,\mathrm{d}x. \tag{2}$$

Solutions to the Allen-Cahn equation (1) satisfy the energy dissipation estimate

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}x = -\int_{\mathbb{R}^d} \frac{1}{\varepsilon}\left|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right|^2 \mathrm{d}x. \tag{3}$$

In the present work, we pursue a strategy of deriving a quantitative convergence result in the sharp-interface limit $\varepsilon \to 0$ based purely on the energy dissipation structure. In particular, we give a short proof for the following quantitative convergence of solutions of the Allen-Cahn equation towards a smooth solution of mean curvature flow.

Theorem 1. *Let $d \in \mathbb{N}$. Let $I(t) \subset \mathbb{R}^d$, $t \in [0,T]$, be a compact interface $I(t) = \partial\Omega(t)$ evolving smoothly by mean curvature, and let $\chi : \mathbb{R}^d \times [0,T] \to \{-1,1\}$ be the corresponding phase indicator function*

$$\chi(x,t) := \begin{cases} 1 & \text{if } x \in \Omega(t), \\ -1 & \text{if } x \notin \Omega(t). \end{cases}$$

Let W be a standard double-well potential as described below and denote by θ the corresponding one-dimensional interface profile. Let u_ε be the solution to the Allen-Cahn equation (1) *with initial data given by $u_\varepsilon(x,0) = \theta(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0)))$, where $\mathrm{dist}^\pm(x,I(0))$ is the signed distance function to $I(0)$ with the convention $\mathrm{dist}^\pm(x,I(0)) > 0$ for $x \in \Omega(0)$. Define $\psi_\varepsilon(x,t) := \int_0^{u_\varepsilon(x,t)} \sqrt{2W(s)}\,\mathrm{d}s$. Then the error estimate*

$$\sup_{t\in[0,T]} ||\psi_\varepsilon(\cdot,t) - \chi(\cdot,t)||_{L^1(\mathbb{R}^d)} \leq C\big(d,T,(I(t))_{t\in[0,T]}\big)\,\varepsilon$$

holds.

We note that this error estimate is of optimal order, as ε is the typical width of the diffuse interface in the Allen-Cahn approximation (i. e. the typical width of the region in which the function ψ_ε takes values in the range $[-1+\delta, 1-\delta]$ for any fixed $\delta > 0$).

The assumptions required for the double-well potential W are standard: We require W to satisfy $W(1) = W(-1) = 0$ and $W(s) \geq c\min\{|s-1|^2, |s+1|^2\}$; furthermore, we require W to be twice continuously differentiable, symmetric around the origin, and subject to the normalization $\int_{-1}^1 \sqrt{2W(s)}\,\mathrm{d}s = 2$. The simplest example is the normalized standard double-well potential $W(s) := \frac{9}{8}(1-s^2)^2$. Under these assumptions, we may define the one-dimensional equilibrium profile $\theta : \mathbb{R} \to \mathbb{R}$ to be the unique odd solution of the ODE $\theta'(s) = \sqrt{2W(\theta(s))}$ with boundary conditions $\theta(\pm\infty) = \pm 1$; the profile θ then approaches its boundary values ± 1 at $\pm\infty$ with an exponential rate, see [11].

As our quantitative convergence analysis does not rely on the comparison principle, it may be applicable to more complex models, such as systems of Navier-Stokes-Allen-Cahn type [1]; note that a weak-strong uniqueness theorem for the two-fluid free boundary problem for the Navier-Stokes equation (i. e. the corresponding sharp-interface model) has already been obtained in [6]. We note that a relative entropy concept related to the one in [6] had already been employed by Jerrard and Smets [9] to deduce weak-strong uniqueness of solutions to binormal curvature flow. In the forthcoming work [7], we employ an energy-based strategy to deduce a weak-strong uniqueness theorem for multiphase mean curvature flow.

2. Definition of the Relative Energy and Gronwall Estimate

2.1. **Extending the unit normal vector field of the surface evolving by mean curvature.** Let $I = I(t)$ be a surface that evolves smoothly by motion by mean curvature. Fix $r_c > 0$ small enough depending on $(I(t))_{t\in[0,T]}$. For each

$t \in [0,T]$, we extend the inner unit normal n_I of the surface $I(t)$ to a vector field on $\mathbb{R}^d$ by defining

$$\xi(x) := \eta(\mathrm{dist}^{\pm}(x,I))\mathrm{n}_I(P_I(x)), \tag{4}$$

where the map $P_I : \mathbb{R}^d \to I = I(t)$ is the nearest point projection and where η is a cutoff satisfying

$$\eta(0) = 1, \qquad \eta(s) = 0 \ \text{ for } |s| \geq \frac{r_c}{2}, \tag{5a}$$

$$\eta(s) \leq \max\{1 - cr_c s^2, 0\}, \tag{5b}$$

$$|\eta'(s)| \leq C\min\{r_c^{-1}, r_c^{-2}|s|\}. \tag{5c}$$

For instance, one can define $\eta(s) := (1 - cr_c^2 s^2)\tilde{\eta}(s)$, where $\tilde{\eta}$ is a standard cut-off which is identically 1 in a neighborhood of 0.

The extended unit normal vector field ξ and mean curvature vector $\mathrm{H}_I(x) := \mathrm{H}_I(P_I x)\tilde{\eta}(\mathrm{dist}(x,I))$ then satisfy the PDEs

$$\frac{d}{dt}\xi = -(\mathrm{H}_I \cdot \nabla)\xi - (\nabla \mathrm{H}_I)^T\xi + O(\mathrm{dist}(x,I)), \tag{6a}$$

$$\frac{d}{dt}|\xi|^2 = -(\mathrm{H}_I \cdot \nabla)|\xi|^2 + O(\mathrm{dist}^2(x,I)), \tag{6b}$$

and

$$-\nabla \cdot \xi = \mathrm{H}_I \cdot \xi + O(\mathrm{dist}(x,I)). \tag{6c}$$

Furthermore, we have the estimate

$$|\nabla\xi| + |\mathrm{H}_I| + |\nabla \mathrm{H}_I| \leq C(I(t)). \tag{6d}$$

To see that (6a) and (6b) hold, one makes use of the formulas $\mathrm{n}_I(x) = \nabla\mathrm{dist}^{\pm}(x,I)$ and $\partial_t\mathrm{dist}^{\pm}(x,I) = -\mathrm{H}_I \cdot \mathrm{n}_I(P_I x)$ valid in a neighborhood of $I(t)$. Formula (6c) is an immediate consequence of the equality $\mathrm{H}_I = -(\nabla \cdot \mathrm{n}_I)\mathrm{n}_I$ valid on the interface $I(t)$ and the Lipschitz continuity of both sides of the equation.

2.2. **The relative energy inequality.** Our argument is based on a relative energy method. As the Modica-Mortola trick will play an important role in the definition of the relative energy, we introduce the function

$$\psi_\varepsilon(x,t) := \int_0^{u_\varepsilon(x,t)} \sqrt{2W(s)}\,\mathrm{d}s. \tag{7}$$

Given a smooth solution u_ε to the Allen-Cahn equation (1) and a surface $I(t)$ which evolves smoothly by mean curvature flow, we define the relative energy $E[u_\varepsilon|I]$ as

$$E[u_\varepsilon|I] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - \xi \cdot \nabla\psi_\varepsilon\,\mathrm{d}x. \tag{8}$$

Introducing the short-hand notation

$$\mathrm{n}_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \tag{9a}$$

(with $\mathrm{n}_\varepsilon(x,t) \in \mathbb{S}^{d-1}$ arbitrary but fixed in case $|\nabla u_\varepsilon| = 0$) and writing

$$E[u_\varepsilon|I] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon|\,\mathrm{d}x + \int_{\mathbb{R}^d}(1 - \xi \cdot \mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon|\,\mathrm{d}x,$$

we see that the relative energy consists of two contributions: The first term

$$\int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon| \,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{1}{2}\Big|\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big|^2 \,\mathrm{d}x$$

controls the local lack of equipartition of energy between the terms $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2$ and $\frac{1}{\varepsilon}W(u_\varepsilon)$, while the second term

$$\int_{\mathbb{R}^d}(1-\xi\cdot \mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon| \,\mathrm{d}x \geq \frac{1}{2}\int_{\mathbb{R}^d} |\mathrm{n}_\varepsilon - \xi|^2|\nabla\psi_\varepsilon| \,\mathrm{d}x$$

controls the local deviation of the normals n_ε and n_I. Note that the latter term also controls the distance to the interface $I(t)$ (since $|\xi| \leq \max\{1-c\,\mathrm{dist}^2(x,I),0\}$).

We furthermore introduce the notation

$$\mathrm{H}_\varepsilon := -\Big(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}, \tag{9b}$$

motivated by the fact that H_ε will play a role of a curvature vector.

The key step in our analysis is the following Gronwall-type estimate for the relative energy.

Theorem 2. *Let $I(t)$, $t\in[0,T]$, be an interface evolving smoothly by mean curvature. Let u_ε be a solution to the Allen-Cahn equation* (1) *with initial data given by $u_\varepsilon(x,0) = \theta(\varepsilon^{-1}\mathrm{dist}^{\pm}(x,I(0)))$. Then for any $t\in[0,T]$ the estimate*

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] + \int_{\mathbb{R}^d} \frac{1}{4\varepsilon}\big|\mathrm{H}_\varepsilon - \mathrm{H}_I\,\varepsilon|\nabla u_\varepsilon|\big|^2 + \frac{1}{4\varepsilon}\big|\mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon - (-\nabla\cdot\xi)\sqrt{2W(u_\varepsilon)}\big|^2 \,\mathrm{d}x \\ \leq C(d,I(t))E[u_\varepsilon|I]\end{aligned}$$

holds.

2.3. **Coercivity properties of the relative energy functional.** For the proof of the Gronwall-type inequality of Theorem 2, we shall need the following coercivity properties of the relative energy.

Lemma 3. *We have the estimates*

$$\int_{\mathbb{R}^d}\Big(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big)^2 \,\mathrm{d}x \leq 2E[u_\varepsilon|I], \tag{10a}$$

$$\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2|\nabla\psi_\varepsilon| \,\mathrm{d}x \leq 2E[u_\varepsilon|I], \tag{10b}$$

$$\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2\varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x \leq 12E[u_\varepsilon|I], \tag{10c}$$

$$\int_{\mathbb{R}^d}\min\{\mathrm{dist}^2(x,I),1\}\Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big) \,\mathrm{d}x \leq CE[u_\varepsilon|I]. \tag{10d}$$

Proof. We complete the square to get

$$E[u_\varepsilon|I] = \int_{\mathbb{R}^d}\frac{1}{2}\Big(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big)^2 + (1-\xi\cdot\mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon| \,\mathrm{d}x.$$

In particular, we directly obtain (10a) and (10b) by $|\xi|\leq 1$. By the property (5b) of the cutoff η (and hence $1-\xi\cdot\mathrm{n}_\varepsilon \geq cr_c\,\mathrm{dist}^2(x,I)$), we deduce (10d) with $|\nabla\psi_\varepsilon|$ instead of the energy density, which we may replace upon using (10a).

Employing Young's inequality in the form of

$$\begin{aligned}\varepsilon|\nabla u_\varepsilon|^2 &= |\nabla\psi_\varepsilon| + \sqrt{\varepsilon}|\nabla u_\varepsilon|\left(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)\\ &\le |\nabla\psi_\varepsilon| + \frac{1}{2}\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2}\left(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2, \qquad (11)\end{aligned}$$

absorption and $|\mathrm{n}_\varepsilon - \xi| \le 2$ yield

$$\begin{aligned}&\int_{\mathbb{R}^d} |\mathrm{n}_\varepsilon - \xi|^2 \varepsilon|\nabla u_\varepsilon|^2\,dx\\ &\le 2\int_{\mathbb{R}^d} |\mathrm{n}_\varepsilon - \xi|^2|\nabla\psi_\varepsilon|\,\mathrm{d}x + 4\int_{\mathbb{R}^d}\left(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2\mathrm{d}x.\end{aligned}$$

By (10a) and (10b), this shows (10c). ☐

2.4. **Time evolution of the relative energy functional.** The main step in the proof of Theorem 2 is the derivation of the following formula; by estimating the right-hand side using the abovementioned coercivity properties and equations (6a)–(6c), we will derive the Gronwall-type inequality of Theorem 2.

Lemma 4. *Let u_ε be a solution to the Allen-Cahn equation* (1) *and let $I = I(t)$ be a smooth solution to mean curvature flow. Let ξ be as defined in* (4)*. The time evolution of the relative energy is then given by*

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] = &-\int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\big|\mathrm{H}_\varepsilon - \mathrm{H}_I\,\varepsilon|\nabla u_\varepsilon|\big|^2 + \frac{1}{2\varepsilon}\big|\mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon - (-\nabla\cdot\xi)\sqrt{2W(u_\varepsilon)}\big|^2\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d} |\mathrm{H}_I|^2\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + |\nabla\cdot\xi|^2\frac{1}{\varepsilon}W(u_\varepsilon) + \mathrm{H_I}\cdot\mathrm{n}_\varepsilon(\nabla\cdot\xi)|\nabla\psi_\varepsilon|\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon|\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^d}\nabla\mathrm{H}_I : \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2 - |\nabla\psi_\varepsilon|)\,\mathrm{d}x\\ &-\int_{\mathbb{R}^d}\nabla\mathrm{H}_I : (\mathrm{n}_\varepsilon - \xi)\otimes(\mathrm{n}_\varepsilon - \xi)|\nabla\psi_\varepsilon|\,\mathrm{d}x \qquad (12)\\ &+\int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I(1 - \xi\cdot\mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon|\,\mathrm{d}x\\ &-\int_{\mathbb{R}^d}|\nabla\psi_\varepsilon|(\mathrm{n}_\varepsilon - \xi)\cdot\left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi + (\nabla\mathrm{H}_I)^T\xi\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^d}|\nabla\psi_\varepsilon|\xi\cdot\left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi\right)\mathrm{d}x.\end{aligned}$$

Proof. By direct computation, we obtain

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] &= \frac{d}{dt}\int_{\mathbb{R}^d}\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - \xi\cdot\nabla\psi_\varepsilon\,\mathrm{d}x\\ &\overset{(3),(1)}{=} -\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\left|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right|^2\mathrm{d}x\\ &\quad -\int_{\mathbb{R}^d}\nabla\psi_\varepsilon\cdot\frac{d}{dt}\xi\,\mathrm{d}x + \int_{\mathbb{R}^d}\sqrt{2W(u_\varepsilon)}\Big(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\Big)\nabla\cdot\xi\,\mathrm{d}x.\end{aligned}$$

With the definitions (9a) and (9b), we obtain

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] = &\int_{\mathbb{R}^d} -\frac{1}{\varepsilon}|\mathrm{H}_\varepsilon|^2 + \mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon\,(-\nabla\cdot\xi)\,\frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}\nabla\mathrm{H}_I:\xi\otimes\mathrm{n}_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}(\mathrm{H}_I\cdot\nabla)\xi\,\cdot\nabla\psi_\varepsilon\,\mathrm{d}x\\ &-\int_{\mathbb{R}^d}\nabla\psi_\varepsilon\cdot\left(\frac{d}{dt}\xi+(\mathrm{H}_I\cdot\nabla)\xi+(\nabla\mathrm{H}_I)^T\xi\right)\mathrm{d}x.\end{aligned}$$

We exploit the symmetry of the Hessian $\nabla^2\psi_\varepsilon$

$$\begin{aligned}&\int_{\mathbb{R}^d}(\mathrm{H}_I\cdot\nabla)\xi\,\cdot\nabla\psi_\varepsilon\,\mathrm{d}x\\ &=-\int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I\;\xi\cdot\nabla\psi_\varepsilon\,\mathrm{d}x-\int_{\mathbb{R}^d}\mathrm{H}_I\otimes\xi:\nabla^2\psi_\varepsilon\,\mathrm{d}x\\ &=\int_{\mathbb{R}^d}(\nabla\cdot\xi\;\mathrm{H}_I-\nabla\cdot\mathrm{H}_I\;\xi)\cdot\nabla\psi_\varepsilon\,\mathrm{d}x+\int_{\mathbb{R}^d}(\xi\cdot\nabla)\mathrm{H}_I\cdot\nabla\psi_\varepsilon\,\mathrm{d}x,\end{aligned}$$

which yields

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] = &\int_{\mathbb{R}^d} -\frac{1}{\varepsilon}|\mathrm{H}_\varepsilon|^2 + \mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon\,(-\nabla\cdot\xi)\,\frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}\nabla\mathrm{H}_I:\xi\otimes\mathrm{n}_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}(\nabla\cdot\xi\;\mathrm{H}_I-\nabla\cdot\mathrm{H}_I\;\xi)\cdot\nabla\psi_\varepsilon\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}(\xi\cdot\nabla)\mathrm{H}_I\cdot\mathrm{n}_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x\\ &-\int_{\mathbb{R}^d}\nabla\psi_\varepsilon\cdot\left(\frac{d}{dt}\xi+(\mathrm{H}_I\cdot\nabla)\xi+(\nabla\mathrm{H}_I)^T\xi\right)\mathrm{d}x.\end{aligned}$$

The computation (13) below then implies by adding zero

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] = &\int_{\mathbb{R}^d} -\frac{1}{\varepsilon}|\mathrm{H}_\varepsilon|^2 + \mathrm{H}_\varepsilon\cdot\mathrm{H}_I\,|\nabla u_\varepsilon| + \mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon\,(-\nabla\cdot\xi)\,\frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I|\nabla\psi_\varepsilon|\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)-|\nabla\psi_\varepsilon|\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^d}\nabla\mathrm{H}_I:\mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2-|\nabla\psi_\varepsilon|)\,\mathrm{d}x\\ &-\int_{\mathbb{R}^d}\nabla\mathrm{H}_I:(\mathrm{n}_\varepsilon-\xi)\otimes(\mathrm{n}_\varepsilon-\xi)|\nabla\psi_\varepsilon|\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}(\nabla\cdot\xi\;\mathrm{H}_I-\nabla\cdot\mathrm{H}_I\;\xi)\cdot\nabla\psi_\varepsilon\,\mathrm{d}x\\ &+\int_{\mathbb{R}^d}(\xi\cdot\nabla)\mathrm{H}_I\cdot\xi|\nabla\psi_\varepsilon|\,\mathrm{d}x\end{aligned}$$

$$-\int_{\mathbb{R}^d} \nabla\psi_\varepsilon \cdot \left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi + (\nabla \mathrm{H}_I)^T\xi\right)\mathrm{d}x.$$

Completing the squares and adding zero, we obtain (12). □

2.5. **Auxiliary computation.** In the above computation, we have made use of the formula

$$\begin{aligned}(13)\qquad &\int_{\mathbb{R}^d} \nabla \mathrm{H}_I : \mathrm{n}_\varepsilon \otimes \mathrm{n}_\varepsilon |\nabla\psi_\varepsilon|\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \mathrm{H}_\varepsilon\cdot \mathrm{H}_I|\nabla u_\varepsilon|\,\mathrm{d}x + \int_{\mathbb{R}^d}\nabla\cdot \mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^d}\nabla \mathrm{H}_I : \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2 - |\nabla\psi_\varepsilon|)\,\mathrm{d}x.\end{aligned}$$

Indeed, due to definition (9b) we have

$$-\int_{\mathbb{R}^d} \mathrm{H}_\varepsilon\cdot\mathrm{H}_I|\nabla u_\varepsilon|\,\mathrm{d}x = \int_{\mathbb{R}^d}\left(\varepsilon\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\right)\mathrm{H}_I\cdot\nabla u_\varepsilon\,\mathrm{d}x.$$

Using the identity $\sum_{i=1}^d \partial_i\partial_i u_\varepsilon \partial_j u_\varepsilon = \sum_{i=1}^d(\partial_i(\partial_i u_\varepsilon\partial_j u_\varepsilon)) - \frac{1}{2}\partial_j|\nabla u_\varepsilon|^2$ we calculate

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(\varepsilon\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\right)\mathrm{H}_I\cdot\nabla u_\varepsilon\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\sum_{i,j=1}^d(\varepsilon\partial_i\partial_i u_\varepsilon\partial_j u_\varepsilon \mathrm{H}_{I,j}) - \frac{1}{\varepsilon}\mathrm{H}_I\cdot\nabla\left(W(u_\varepsilon)\right)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\sum_{i,j=1}^d(-\varepsilon\partial_i\mathrm{H}_{I,j}\partial_i u_\varepsilon\partial_j u_\varepsilon) + \nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right)\,\mathrm{d}x.\end{aligned}$$

Recalling the abbreviation $\mathrm{n}_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ we get

$$\begin{aligned}&-\int_{\mathbb{R}^d}\mathrm{H}_\varepsilon\cdot\mathrm{H}_I|\nabla u_\varepsilon|\,\mathrm{d}x \\ (14)\qquad &= \int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right) - \nabla\mathrm{H_I} : (\mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon)\,\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x.\end{aligned}$$

With the goal of replacing the expressions $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}$ and $\varepsilon|\nabla u_\varepsilon|^2$ by $|\nabla\psi_\varepsilon|$ we rewrite the identity (14) as (13).

2.6. **Derivation of the Gronwall inequality.**

Proof of Theorem 2. Using the estimates of Lemma 3 we can control the terms on the right-hand side of the identity (12). Using (6a), (6b), and the bound $||\nabla H_I||_{L^\infty} \leq C(I(t))$, the last four lines of (12) may be estimated by

$$C(I(t))\int_{\mathbb{R}^d}\min\{\mathrm{dist}^2(x,I),1\}|\nabla\psi_\varepsilon| + |\mathrm{n}_\varepsilon - \xi|^2|\nabla\psi_\varepsilon| + (1-\mathrm{n}_\varepsilon\cdot\xi)|\nabla\psi_\varepsilon|\,\mathrm{d}x,$$

which by (10b) and (10d) is bounded by $C(I(t))E[u_\varepsilon|I]$.

The third line on the right-hand side of (12) can be estimated as

$$(15)\qquad \int_{\mathbb{R}^d}\left|\nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon|\right)\right|\,\mathrm{d}x \leq \|\nabla\cdot\mathrm{H}_I\|_\infty E[u_\varepsilon|I].$$

Thus, it only remains to estimate the second and the fourth term on the right-hand side of (12).

Concerning the second term, we use the fact that $(\xi\cdot\nabla)\mathrm{H}_I\equiv 0$ holds in a neighborhood of $I(t)$, Young's inequality, and (7) to deduce

$$\begin{aligned}
&\int_{\mathbb{R}^d}\big|\nabla\mathrm{H}_I:\mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2-|\nabla\psi_\varepsilon|)\big|\,\mathrm{d}x\\
&\le\int_{\mathbb{R}^d}\big|\nabla\mathrm{H}_I:\mathrm{n}_\varepsilon\otimes(\mathrm{n}_\varepsilon-\xi)\big(\varepsilon|\nabla u_\varepsilon|^2-|\nabla\psi_\varepsilon|\big)\big|\,\mathrm{d}x\\
&\quad+C\int_{\mathbb{R}^d}\min\{\mathrm{dist}^2(x,I),1\}\big(\varepsilon|\nabla u_\varepsilon|^2+|\nabla\psi_\varepsilon|\big)\,\mathrm{d}x\\
&\le\|\nabla\mathrm{H}_I\|_\infty\left(\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2\varepsilon|\nabla u_\varepsilon|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^d}\left(\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2\mathrm{d}x\right)^{\frac12}\\
&\quad+C\int_{\mathbb{R}^d}\min\{\mathrm{dist}^2(x,I),1\}\big(\varepsilon|\nabla u_\varepsilon|^2+|\nabla\psi_\varepsilon|\big)\,\mathrm{d}x.
\end{aligned}$$

Consequently, Lemma 3 implies that the fourth line on the right-hand side of (12) is bounded by $CE[u_\varepsilon|I]$.

It only remains to bound the term in the second line of the right-hand side of (12). To this aim, we complete the square and estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d}|\mathrm{H}_I|^2\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+|\nabla\cdot\xi|^2\frac{1}{\varepsilon}W(u_\varepsilon)+\mathrm{H}_I\cdot\mathrm{n}_\varepsilon\nabla\cdot\xi|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\frac12\left|\sqrt{\varepsilon}|\nabla u_\varepsilon|\mathrm{H}_I+\frac{1}{\sqrt{\varepsilon}}\nabla\cdot\xi\sqrt{2W(u_\varepsilon)}\mathrm{n}_\varepsilon\right|^2\mathrm{d}x\\
&\le\frac32\int_{\mathbb{R}^d}\left|(\nabla\cdot\xi)\mathrm{n}_\varepsilon\Big(\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big)\right|^2\mathrm{d}x\\
&\quad+\frac32\int_{\mathbb{R}^d}\left|(\nabla\cdot\xi)(\xi-\mathrm{n}_\varepsilon)\sqrt{\varepsilon}|\nabla u_\varepsilon|\right|^2\mathrm{d}x\\
&\quad+\frac32\int_{\mathbb{R}^d}\left|\big(\mathrm{H}_I+(\nabla\cdot\xi)\xi\big)\sqrt{\varepsilon}|\nabla u_\varepsilon|\right|^2\mathrm{d}x.
\end{aligned}$$

Inserting the estimates (6c) and (6d) and using the fact that $H_I=(H_I\cdot\xi)\xi+O(\mathrm{dist}^2(x,I))$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}|\mathrm{H}_I|^2\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+|\nabla\cdot\xi|^2\frac{1}{\varepsilon}W(u_\varepsilon)+\mathrm{H}_I\cdot\mathrm{n}_\varepsilon\nabla\cdot\xi|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^d}\left|\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right|^2\mathrm{d}x\\
&\quad+C\int_{\mathbb{R}^d}\min\{\mathrm{dist}^2(x,I),1\}\varepsilon|\nabla u_\varepsilon|^2+|\mathrm{n}_\varepsilon-\xi|^2\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x.
\end{aligned}$$

By Lemma 3, we see that these terms are estimated by $CE[u_\varepsilon|I]$. □

3. Estimate for the Interface Error

We now derive the interface error estimate of Theorem 1.

Proof of Theorem 1. Step 1: Estimate for the relative energy. In view of Theorem 2, in order to prove

(16)
$$\sup_{t\in[0,T]} E[u_\varepsilon|I] + \int_0^T \int_{\mathbb{R}^d} \frac{1}{\varepsilon}\big|\mathrm{H}_\varepsilon - \mathrm{H}_I\,\varepsilon|\nabla u_\varepsilon|\big|^2 + \frac{1}{\varepsilon}\big|\mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon - (-\nabla\cdot\xi)\sqrt{2W(u_\varepsilon)}\big|^2\,\mathrm{d}x\,\mathrm{d}t \\ \leq C(d,T,(I(t))_{t\in[0,T]})\varepsilon^2$$

it only remains to show that the initial relative energy satisfies $E[u_\varepsilon|I](0)\leq C(d,I(0))\varepsilon^2$. To this aim, we compute using $u_\varepsilon(x,0)=\theta(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0)))$ and the fact that $\nabla\mathrm{dist}^\pm(x,I(0))\cdot\xi = |\nabla\mathrm{dist}^\pm(x,I(0))||\xi|\geq|\xi|^2$

$$\begin{aligned} E[u_\varepsilon|I](0) \leq \int_{\mathbb{R}^d} &\frac{|\xi|^2}{2\varepsilon}|\theta'(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0)))|^2 + \frac{|\xi|^2}{\varepsilon}W(\theta(\varepsilon^{-1}\,\mathrm{dist}^\pm(x,I(0))) \\ &- \frac{1}{\varepsilon}\sqrt{2W(\theta(\varepsilon^{-1}\,\mathrm{dist}^\pm(x,I(0)))}\,\theta'(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0)))|\xi|^2\,\mathrm{d}x \\ + \int_{\mathbb{R}^d} &\frac{1}{2\varepsilon}(1-|\xi|^2)\Big(|\theta'(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0)))|^2 + W(\theta(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0))))\Big)\,\mathrm{d}x. \end{aligned}$$

Using the defining equation $\theta'(s)=\sqrt{2W(\theta(s))}$ as well as the fact that $|\theta'(s)|$ decays exponentially in s and that $|\xi|^2\geq 1-c\,\mathrm{dist}^2(x,I)$, we deduce $E[u_\varepsilon|\chi](0)\leq C(d,I(0))\varepsilon^2$.

Step 2: Interface error estimate. We now perform an additional computation to obtain a more explicit control on the interface error. We may write

$$\partial_t\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\partial_t u_\varepsilon \overset{(1),(9b)}{=} -\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon.$$

Choosing $\tau:\mathbb{R}\to[-1,1]$ to be a smooth monotone truncation of the identity map (with $\tau(s)\geq\min\{s,\frac12\}$ for $s>0$ and $\tau(s)\leq\max\{s,-\frac12\}$ for $s<0$) and fixing $s_0>0$ to be determined later, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d}\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x \\ &\quad+\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\partial_t\mathrm{dist}^\pm(x,I)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d}\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x \\ &\quad-\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\mathrm{H}_I\cdot\nabla\Big(\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\Big)\,\mathrm{d}x \\ &\quad+\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\big(\partial_t\mathrm{dist}^\pm(x,I)+\mathrm{H}_I\cdot\nabla\mathrm{dist}^\pm(x,I)\big)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d}(\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon - \nabla\psi_\varepsilon\cdot H_I)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x \end{aligned}$$

$$+\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\nabla\cdot\mathrm{H}_I\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\big(\partial_t\mathrm{dist}^\pm(x,I)+H_I\cdot\nabla\mathrm{dist}^\pm(x,I)\big)\,\mathrm{d}x$$

where in the last step we have used integration by parts and $\tau(\mathrm{dist}^\pm(x,I(t))=0$ on $\operatorname{supp}\nabla\chi(\,\cdot\,,t)$.

This may be rewritten using the definition of ψ_ε and n_ε as

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^d}\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}(\mathrm{H}_\varepsilon-\mathrm{H}_I\varepsilon|\nabla u_\varepsilon|)\cdot\mathrm{n}_\varepsilon\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\nabla\cdot\mathrm{H}_I\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\big(\partial_t\mathrm{dist}^\pm(x,I)+H_I\cdot\nabla\mathrm{dist}^\pm(x,I)\big)\,\mathrm{d}x.
\end{aligned}$$

Since $\partial_t\mathrm{dist}^\pm(x,I)=-\mathrm{H}_I\cdot\nabla\mathrm{dist}^\pm(x,I)$ holds in a neighborhood of the interface, the last integral vanishes identically if we chose $s_0>0$ sufficiently small. Using Cauchy-Schwarz we deduce

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(\psi_\varepsilon-\chi)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&\leq\int_{\mathbb{R}^d}\varepsilon^{-1}\big|\mathrm{H}_\varepsilon-\mathrm{H}_I\varepsilon|\nabla u_\varepsilon|\big|^2\,\mathrm{d}x+\int_{\mathbb{R}^d}\varepsilon^{-1}2W(u_\varepsilon)\Big|\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\Big|^2\,\mathrm{d}x\\
&\quad+||(\nabla\cdot\mathrm{H}_I)_-||_{L^\infty}\int_{\mathbb{R}^d}|\psi_\varepsilon-\chi|\Big|\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\Big|\,\mathrm{d}x.
\end{aligned}$$

By the Gronwall inequality and (16) (note that the relative entropy $E[u_\varepsilon|I](t)$ controls $c\int_{\mathbb{R}^d}\varepsilon^{-1}W(u_\varepsilon)\,\mathrm{dist}^2(x,I)\,\mathrm{d}x)$, this shows that

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|\psi_\varepsilon-\chi|\min\{\mathrm{dist}(x,I),1\}\,\mathrm{d}x\leq C(d,T,(I(t))_{t\in[0,T]})\varepsilon^2. \tag{17}$$

Note that Fubini's theorem for the square $[0,\delta]^2$ yields

$$\begin{aligned}
&\Big(\int_0^\delta|\psi_\varepsilon(w+y\mathrm{n}_I(w),t)-\chi(w+y\mathrm{n}_I(w),t)|\,\mathrm{d}y\Big)^2\\
&\leq2\int_0^\delta|\psi_\varepsilon(w+y\mathrm{n}_I(w),t)-\chi(w+y\mathrm{n}_I(w),t)|\int_0^y2\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

This allows to estimate for a small δ-neighborhood of $I(t)$

$$
\begin{aligned}
&\left(\int_{I(t)+B_\delta} |\psi_\varepsilon(x,t)-\chi(x,t)|\,\mathrm{d}x\right)^2 \\
&\leq C(d,I(t))\bigg(\int_{I(t)}\int_0^\delta |\psi_\varepsilon(w+y\mathrm{n}_I(w),t)-\chi(w+y\mathrm{n}_I(w),t)|\,\mathrm{d}y \\
&\qquad\qquad + \int_0^\delta |\psi_\varepsilon(w-y\mathrm{n}_I(w),t)-\chi(w-y\mathrm{n}_I(w),t)|\,\mathrm{d}y\,\mathrm{d}S(w)\bigg)^2 \\
&\leq C(d,I(t))\int_{I(t)}\int_{-\delta}^\delta |\psi_\varepsilon(w+y\mathrm{n}_I(w),t)-\chi(w+y\mathrm{n}_I(w),t)| \\
&\qquad\qquad\times \operatorname{dist}(w+y\mathrm{n}_I(w),I(t))\,\mathrm{d}y\,\mathrm{d}S(w) \\
&\leq C(d,I(t))\int_{I(t)+B_\delta} |\psi_\varepsilon(x,t)-\chi(x,t)|\operatorname{dist}(x,I)\,\mathrm{d}x,
\end{aligned}
$$

which in view of (17) yields Theorem 1. □

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