# APPROXIMATION OF SLIGHTLY COMPRESSIBLE FLUIDS BY THE INCOMPRESSIBLE NAVIER-STOKES EQUATION AND LINEARIZED ACOUSTICS: A POSTERIORI ESTIMATES

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Abstract. Consider a slightly compressible fluid. A formal expansion suggests that the incompressible Navier-Stokes equation combined with the equations of linearized acoustics should yield a higher-order approximation (with respect to the compressibility) of the compressible Navier-Stokes equation. In the present work, we derive rigorous a posteriori estimates for jointly the modeling error and the numerical error for this expansion of the compressible Navier-Stokes equation. In case of well-behaved solenoidal numerical solutions of the incompressible Navier-Stokes equation, we expect our estimates to provide not just fully rigorous but also practically meaningful bounds for the modeling error. Our estimates are valid for any weak solution to the compressible Navier-Stokes equation in the sense of Lions, i.e. we do not assume any additional regularity of the exact solution to the compressible Navier-Stokes equation.

## 1. Introduction

The behavior of a slightly compressible fluid is known to be approximated well in many situations by the incompressible Navier-Stokes equation

$$\frac{d}{dt}v + (v\cdot\nabla)v = -\nabla p_i + \mu_0\Delta v + f, \tag{1a}$$

$$\operatorname{div} v = 0. \tag{1b}$$

A more accurate description of a slightly compressible fluid is however provided by the compressible Navier-Stokes equation

$$\frac{d}{dt}\rho + \operatorname{div}(\rho u) = 0, \tag{2a}$$

$$\frac{d}{dt}\rho u + \operatorname{div}(\rho u\otimes u) = -\nabla p_c + \operatorname{div}\Big(\mu(\rho)(\nabla u + \nabla u^T)\Big) + \nabla(\lambda(\rho)\operatorname{div} u) + \rho f. \tag{2b}$$

The compressible Navier-Stokes equation needs to be supplemented by an equation of state relating the pressure to the density. In many physical situations, the relation

$$p_c := p_\epsilon(\rho) := \frac{\rho^\gamma - 1}{\gamma\epsilon} \tag{3}$$

may be used (for some $\gamma > 1$; note that we have fixed the reference density – i.e. the density of the fluid in the incompressible idealization – to be 1). Here, $\epsilon > 0$ denotes the compressibility of the fluid; for a slightly compressible fluid, $\epsilon$ is small.

*Key words and phrases.* Navier-Stokes equation, compressible fluid, incompressible limit, linearized acoustics, modeling error, a posteriori estimate.

It is not obvious that the incompressible Navier-Stokes equation (1) provides an approximation for the compressible Navier-Stokes equation (2) for small compressibilities $\epsilon$. Indeed, the incompressible Navier-Stokes equation arises as a singular limit from the compressible Navier-Stokes equation as $\epsilon$ tends to 0. In connection with the low known regularity of weak solutions to the compressible and incompressible Navier-Stokes equation, this makes the mathematical analysis of the limit challenging. For example, in the case of at least three spatial dimensions $d \geq 3$ only weak convergence of solutions of the compressible NSE towards a solution of the incompressible NSE as $\epsilon \to 0$ is known [28] (see [12, 13, 15, 16, 29] for related results). Results including a rate of convergence so far require at least regularity assumptions on the solution of the limit problem (i.e. the solution of the incompressible NSE) which for $d \geq 3$ go beyond the available regularity theory for the Navier-Stokes equation [22, 23, 24, 25].

Quantitative estimates for the difference between numerical solutions to the incompressible Navier-Stokes equation and the corresponding exact weak solution to the compressible Navier-Stokes equation have been derived only in the recent work [19]. The estimates of [19] are of a posteriori type, i.e. the error bounds depend on the numerical solution to the incompressible Navier-Stokes equation and the physical parameters. The *a posteriori modeling error estimates* of [19] are fully rigorous, i.e. they apply to the finite energy weak solutions to the compressible Navier-Stokes equation constructed by Lions [26] (note that only such finite energy weak solutions are known to exist globally in time). Furthermore, the estimates are also practical, i.e. the constants in the estimates are all of at most moderate magnitude and thus the estimates are meaningful also in realistic situations.

For well-behaved flows and solenoidal approximations of the velocity field in the incompressible Navier-Stokes equation, the estimates in [19] yield a bound for the error in the velocity $u - v$ of the order of $O(\sqrt{\epsilon})$. To obtain a higher-order approximation of the compressible Navier-Stokes equation, let us denote by $s := (\rho - 1 - \epsilon p_i)/\epsilon$ and $w := (u - v)/\sqrt{\epsilon}$ the rescaled deviation of the solution of the incompressible NSE (1) from the solution of the compressible NSE (2). One then observes that formally, $s$ and $w$ satisfy the equations of *linearized acoustics*

$$\frac{d}{dt}s + (v \cdot \nabla)s + \frac{\operatorname{div} w}{\sqrt{\epsilon}} = -\frac{d}{dt}p_i - (v \cdot \nabla)p_i, \tag{4a}$$

$$\frac{d}{dt}w + (v \cdot \nabla)w + \frac{\nabla s}{\sqrt{\epsilon}} = \mu_0 \Delta w + (\lambda_0 + \mu_0)\nabla \operatorname{div} w - (w \cdot \nabla)v \tag{4b}$$

up to an error of $O(\sqrt{\epsilon})$. Considering solutions to (2) and (1) as well as a solution $s$, $w$ to (4), formally $1 + \epsilon p_i + \epsilon s$ and $v + \sqrt{\epsilon} w$ should therefore provide a higher-order approximation for $\rho$ and $u$ (compared to the lowest-order approximation $\rho \approx 1$ and $u \approx v$). More precisely, we expect the bounds

$$|\rho - (1 + \epsilon p_i + \epsilon s)| = O(\epsilon^{3/2}),$$
$$|u - (v + \sqrt{\epsilon} w)| = O(\epsilon)$$

(at least in certain norms). It is the central goal of the present work to establish a corresponding rigorous a posteriori modeling error estimate.

As in [19], we consider the error functional

$$D[v, p, \rho, u] := \int_\Omega \frac{1}{2}\rho|v - u|^2 + \pi_\epsilon(\rho) - (\rho - 1)p + \Pi_\epsilon(p) \; dx$$

(where $\pi_\epsilon(\rho) = \rho \int_1^\rho s^{-2} p_\epsilon(s)\,ds$ is the internal energy density of the fluid and where $\Pi_\epsilon$ is the convex conjugate of $\pi_\epsilon(1+\cdot)$) which again enables us to take advantage of the energy dissipation of weak solutions to the compressible Navier-Stokes equation (2) in the sense of Lions: The low known regularity of weak solutions to the compressible Navier-Stokes equation prevents us from evaluating the evolution of nonlinear functionals of density and momentum density $(\rho, \rho u)$ directly using just the weak formulation of the compressible Navier-Stokes equation; yet an error functional needs to be nonlinear in the velocity $u$. The energy of the compressible NSE is of the form $\int_\Omega \frac{1}{2}\rho|u|^2 + \pi_\epsilon(\rho)\,dx$; thus, the energy dissipation inequality and our choice of the error functional enable us to derive error estimates nevertheless. For the usage of the energy dissipation inequality in a closely related context – the context of relative entropies –, see in particular [9, 14, 16, 18, 21] and the references therein; in these works, the relative entropy for the compressible Navier-Stokes equation is used to establish certain singular limits and – remarkably – weak-strong uniqueness for the compressible Navier-Stokes equation.

In contrast to the previous work [19] in which an error estimate of the rough structure

$$D[v_h, p_h, \rho, u](T) \leq O(\epsilon) + \mathcal{E}_{num}^2(T)$$

has been derived, in the present work we shall now show an estimate of the form

$$D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u](T) \leq O(\epsilon^2) + \mathcal{E}_{num}^2(T) \tag{5}$$

for any fixed $T > 0$ in case of well-behaved flows; here $v_h$, $p_h$ denote an approximate solution to the incompressible Navier-Stokes equation (1) and $s_h$, $w_h$ denote an approximate solution to the equations of linearized acoustics (4) (with $v$ and $p_i$ in (4) replaced by $v_h$ and $p_h$); $\mathcal{E}_{num}$ denotes the numerical error (of both $v_h$, $p_h$ and $s_h$, $w_h$). Again, all constants in the error estimates are of at most moderate magnitude.

Recall that the error functional $D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u]$ actually controls the total error: We have

$$D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u] \geq \int_\Omega \frac{1}{2}\rho|v_h + \sqrt{\epsilon} w_h - u|^2 + \frac{|\rho - (1 + \epsilon p_h + \epsilon s_h)|^2}{2\epsilon}\,dx$$

in case $\gamma = 2$ in the equation of state (3) and similar estimates in the general case. Therefore, our error estimate (5) captures the expected higher-order approximation property of the expansion $1 + \epsilon p_h + \epsilon s_h$ and $v_h + \sqrt{\epsilon} w_h$ for $\rho$ and $u$ correctly.

In total, we expect the a posteriori modeling error estimate in the present paper to provide a rigorous justification of the equations of linearized acoustics also in practical situations. Besides that, the a posteriori modeling error estimates in [19] and the present paper enable us to spatially locate the origin of the modeling error. Our estimates may therefore enable us in future to devise a model-adaptive numerical scheme for the simulation of slightly compressible fluids, which solves the compressible Navier-Stokes equation where necessary and the incompressible Navier-Stokes equation along with the equations of linearized acoustics where possible.

Regarding a posteriori modeling error estimates and model-adaptive numerical schemes for other problems, we would like to refer the reader to [4, 5, 7, 10, 11, 20, 31, 32, 33, 38, 39, 40, 42, 43, 44] and the references therein. Note that apart from [7, 19, 31] (the first and the last work dealing with modeling errors in combustion problems and the Stokes approximation of the incompressible Navier-Stokes equation, respectively) there have been little results on a posteriori modeling error estimates in fluid mechanics.

To estimate the numerical error, in the present work we make use of a posteriori error estimates of functional type; see e.g. [30, 35, 36, 37, 41]. For other types of a posteriori error estimates – in particular a posteriori error estimates of residual type and duality-based a posteriori error estimates – we refer the reader e.g. to [1, 2, 3, 6, 8, 34, 45, 46, 47] and the references therein.

**Notation.** Throughout the paper, we use standard notation for Sobolev spaces. By $\Omega \subset \mathbb{R}^d$ we denote a bounded Lipschitz domain.

The notation $C_c^\infty(\Omega)$ is used to refer to the set of smooth compactly supported functions in $\Omega$. As usual, $H^1(\Omega) = W^{1,2}(\Omega)$ denotes the space of functions in $L^2(\Omega)$ whose distributional derivative belongs to $L^2(\Omega;\mathbb{R}^d)$. We also use the standard notation $H^1_0(\Omega)$ for the subspace of functions in $H^1(\Omega)$ with vanishing boundary trace. The notation $W^{1,\infty}(\Omega\times I)$ is used for the space of functions in $L^\infty(\Omega\times I)$ whose distributional derivative belongs to $L^\infty(\Omega\times I;\mathbb{R}^d\times\mathbb{R})$.

The notation $L^p(\Omega;\mathbb{R}^d)$ (with $1 \leq p \leq \infty$) refers to the space of $\mathbb{R}^d$-valued measurable mappings $f:\Omega\rightarrow\mathbb{R}^d$ which satisfy $|f| \in L^p(\Omega)$. The corresponding norm is given by $||v||_{L^p(\Omega;\mathbb{R}^d)} := \left(\int_\Omega |v|^p \,dx\right)^{1/p}$.

For $a, b \in \mathbb{R}$, we denote by $a \vee b$ the maximum of $a$ and $b$. Similarly, $a \wedge b$ refers to the minimum of $a$ and $b$. We also use the abbreviations $(a)_+$ for $\max(a,0)$ and $(a)_-$ for $\max(-a,0)$.

For two matrices $A, B$ of dimension $n\times n$, we denote the sum $\sum_{i,j=1}^n A_{ij}B_{ij}$ by $A:B$. The notation $|A|_2$ is used for the spectral norm of $A$. In contrast, the notation $|A|$ denotes the Frobenius norm. For a tensor field $a \in L^2(\Omega;\mathbb{R}^{d\times d})$, by $||a||_{L^2(\Omega)}$ we denote the norm $||\,|a|\,||_{L^2(\Omega)}$.

Throughout the paper, we fix some (arbitrarily large) $T_{max} > 0$ and abbreviate $I := [0, T_{max})$. For a Banach space $X$, we use the notation $L^p([0,T];X)$ to denote the space of all (strongly Bochner) measurable mappings $f:[0,T]\rightarrow X$ which satisfy $||f||_X \in L^p([0,T])$.

For a vector field $v$, we shall use the convention $(\nabla v)_{ij} := \partial_j v_i$. Given a tensor field $a \in L^2(\Omega;\mathbb{R}^{d\times d})$, we set $\operatorname{div} a := \left(\sum_{i=1}^d \partial_i a_{1i}, \cdots, \sum_{i=1}^d \partial_i a_{di}\right)^T$. This allows us to rewrite the Laplacian acting on a vector field $v$ component-wise as $\Delta v = \operatorname{div}(\nabla v)$.

## 2. Main Results

Our assumptions on the domain and the coefficients are mostly parallel to the ones in the preceding paper [19]. More precisely, our assumptions are as follows:

(A1) Let $d = 2$ or $d = 3$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.

(A2) Suppose that $\mu:[0,\infty)\rightarrow\mathbb{R}^+$ is a bounded Lipschitz continuous mapping with $\mu_{min} := \inf_{s\geq 0}\mu(s) > 0$ and $\mu_{max} := \sup_{s\geq 0}\mu(s) < \infty$.

(A3) Assume that $\lambda:[0,\infty)\rightarrow\mathbb{R}^+$ is a bounded Lipschitz continuous mapping with $\lambda_{min} := \inf_{s\geq 0}\lambda(s) > 0$ and $\lambda_{max} := \sup_{s\geq 0}\lambda(s) < \infty$.

(A4) Let $\gamma > \frac{3}{2}$. Suppose $\epsilon > 0$. Let $p_\epsilon(\rho)$ be given by

$$p_\epsilon(\rho) := \frac{\rho^\gamma - 1}{\gamma\epsilon}. \tag{6}$$

Note that this entails by definition of $\pi_\epsilon$

$$\pi_\epsilon(\rho) = \rho \int_1^\rho s^{-2} p_\epsilon(s)\ ds = \frac{1}{\epsilon}\left(\frac{1}{\gamma(\gamma-1)}\rho^\gamma - \frac{1}{\gamma-1}\rho + \frac{1}{\gamma}\right)$$

and by a general formula for the convex conjugate

$$\Pi'_\epsilon(s) = (\pi'_\epsilon)^{-1}(s) - 1 = ((\gamma-1)\epsilon s + 1)^{1/(\gamma-1)} - 1,$$

i.e. in particular

$$\Pi_\epsilon(s) = \frac{1}{\gamma\epsilon}((\gamma-1)\epsilon s + 1)^{\gamma/(\gamma-1)} - s - \frac{1}{\gamma\epsilon}.$$

(A5) Let the force density satisfy $f \in L^\infty(\Omega \times I; \mathbb{R}^d)$.

We shall impose the no-slip boundary condition

$$u = 0 \quad \text{on } \partial\Omega \tag{7}$$

at the lateral boundary for any $t > 0$.

Let us now state the notion of solutions to the compressible Navier-Stokes equation for which we derive our a posteriori modeling error estimates. Weak solutions to the compressible Navier-Stokes equation originally were introduced by Lions [26, 27]. The notion which we shall work with in the present paper is a slightly rephrased and weakened form of the definition in [17], which in turn is closely related to the original definition of Lions.

**Definition 1.** *Let (A1) to (A5) be satisfied. Assume that $\rho_0 \in L^\gamma(\Omega)$ is nonnegative; let $u_0$ be a measurable vector field with $\rho_0|u_0|^2 \in L^1(\Omega)$. We then call a pair $\rho \in L^\infty(I; L^\gamma(\Omega))$, $u \in L^2(I; H^1_0(\Omega; \mathbb{R}^d))$ with $\rho \geq 0$ a finite energy weak solution to the compressible Navier-Stokes equation with no-slip boundary condition on $\partial\Omega$ if for any $\psi \in C_c^\infty(\Omega \times I; \mathbb{R}^d)$ we have*

$$\begin{aligned}
&-\int_I \int_\Omega \rho u \cdot \frac{d}{dt}\psi\ dx\ dt - \int_\Omega \rho_0 u_0 \cdot \psi(.,0)\ dx \\
&= \int_I \int_\Omega \rho u \cdot (u \cdot \nabla)\psi\ dx\ dt + \int_I \int_\Omega p_\epsilon(\rho) \operatorname{div} \psi\ dx\ dt \\
&\quad - \int_I \int_\Omega \mu(\rho)(\nabla u + \nabla u^T) : \nabla \psi\ dx\ dt - \int_I \int_\Omega \lambda(\rho) \operatorname{div} u \operatorname{div} \psi\ dx\ dt \\
&\quad + \int_I \int_\Omega \rho f \cdot \psi\ dx\ dt,
\end{aligned} \tag{8}$$

*if for any $\phi \in C^\infty(\overline{\Omega} \times I)$ with $\phi(\cdot, T_{max}) \equiv 0$ we have*

$$-\int_I \int_\Omega \rho \frac{d}{dt}\phi\ dx\ dt - \int_\Omega \rho_0 \phi(.,0)\ dx - \int_I \int_\Omega \rho u \cdot \nabla \phi\ dx\ dt = 0, \tag{9}$$

*and if in addition the energy inequality*

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho(.,t_2)|u(.,t_2)|^2 + \pi_\epsilon(\rho(.,t_2))\ dx \\
&+ \int_{t_1}^{t_2} \int_\Omega \frac{\mu(\rho)}{2}|\nabla u + \nabla u^T|^2\ dx\ dt + \int_{t_1}^{t_2} \int_\Omega \lambda(\rho)|\operatorname{div} u|^2\ dx\ dt \\
&\leq \int_\Omega \frac{1}{2}\rho(.,t_1)|u(.,t_1)|^2 + \pi_\epsilon(\rho(.,t_1))\ dx + \int_{t_1}^{t_2} \int_\Omega f \cdot \rho u\ dx\ dt
\end{aligned} \tag{10}$$

*holds for a.e. $0 \leq t_1 \leq t_2$ and a.e. $t_2 \geq 0$ in case $t_1 = 0$.*

Note that by approximation, the equations (8) and (9) are also satisfied for Lipschitz test functions (with zero boundary trace in case of equation (8)).

Recall that in the case of density-independent viscosities $\mu$ and $\lambda$ and bounded domains $\Omega$ of class $C^{2,\alpha}$, weak solutions in the sense of the previous definition exist globally in time for any initial data satisfying the above restrictions [17, 26].

To state our main result, we need the following notation for constants, most of which have already been introduced in [19].

**Definition 2.** *We define*

$$C_{\pi,p} := \sup_{s\in\mathbb{R}_0^+} \frac{(|\epsilon p_\epsilon(s)|-\sqrt{2\epsilon\pi_\epsilon(s)})_+}{\epsilon\pi_\epsilon(s)}, \quad C_{\pi,\mu} := \sup_{s\in\mathbb{R}_0^+} \frac{|\mu(s)-\mu_0|^2}{\epsilon\pi_\epsilon(s)\mu_{min}},$$
$$C_{\pi,\lambda} := \sup_{s\in\mathbb{R}_0^+} \frac{|\lambda(s)-\lambda_0|^2}{2\epsilon\pi_\epsilon(s)\lambda_{min}},$$
$$C_{\pi,e} := \sup_{s\in[0,1)} \frac{(1-\sqrt{s})^2}{\epsilon\pi_\epsilon(s)}, \qquad C_{\pi,f} := \sup_{s\in\mathbb{R}_0^+} \left|\frac{(|s-1|-\sqrt{2\epsilon\pi_\epsilon(s)})_+}{(\epsilon\pi_\epsilon(s))^{5/6}}\right|^2,$$
$$C_{\pi,m,1} := \sup_{s\in\mathbb{R}_0^+} \frac{(1-\epsilon\pi_\epsilon''(s))_+}{|s-1|}, \qquad C_{\pi,m,2} := \sup_{s\in\mathbb{R}_0^+} \frac{2\min(C_{\pi,m,1}|s-1|,1)|s-1|}{\epsilon\pi_\epsilon(s)},$$

*and denote by $C_{\Omega,2,6}$ the Poincaré-Sobolev constant for the embedding $H_0^1(\Omega) \to L^6(\Omega)$, i.e. we set*

$$C_{\Omega,2,6} := \sup_{\phi\in C_c^\infty(\Omega)} \frac{||\phi||_{L^6(\Omega)}}{||\nabla\phi||_{L^2(\Omega)}}.$$

**Remark 3.** *If the equation of state is given by (6), we have*

$$C_{\pi,p} = \begin{cases} \gamma-1 & \text{for } \gamma\geq 2 \\ \gamma-1+\frac{1}{3}(2-\gamma) & \text{for } \gamma<2 \end{cases}, \quad C_{\pi,e} = \gamma,$$

$$C_{\pi,f} = \gamma^{5/3}\left(1-\sqrt{\frac{2}{\gamma}}\right)^2 \quad \text{for } \gamma\geq 2, \qquad C_{\pi,m,1} = \begin{cases} 2-\gamma & \text{for } \gamma<2 \\ 0 & \text{for } \gamma=2 \\ 1 & \text{for } 2<\gamma\leq 3 \\ \gamma-2 & \text{for } \gamma>3 \end{cases},$$

$$C_{\pi,m,2} \leq 2\gamma C_{\pi,m,1} \quad \text{for } \gamma\geq 2.$$

*Furthermore, in case $d=3$ we have*

$$C_{\Omega,2,6} = \frac{1}{\sqrt{3\pi}}\sqrt[3]{\frac{4}{\sqrt{\pi}}}.$$

Our main result – the joint a posteriori estimate for the numerical error and the modeling error for the approximation of the compressible Navier-Stokes equation by the incompressible Navier-Stokes equation and the equations of linearized acoustics – reads as follows. Note that the colored underlines are just there to relate the terms in the estimate to the corresponding terms in the proof of the estimate.

We would like to emphasize that our error estimate takes the form of an *integral inequality* for our (time-dependent) error functional $D[v_h+\sqrt{\epsilon}w_h, p_h+s_h, \rho, u]$ (see inequality (12) below). The integral inequality (12) may be solved numerically (if desired, one may even compute a guaranteed upper bound on $D[v_h+\sqrt{\epsilon}w_h, p_h+s_h, \rho, u](T)$ numerically) or one may derive an explicit upper bound for $D[v_h+\sqrt{\epsilon}w_h, p_h+s_h, \rho, u](T)$ by Remark 5 below.

**Theorem 4.** *Assume that (A1) to (A5) are satisfied. Let $v_h \in W^{1,\infty}(\Omega\times I;\mathbb{R}^d)$, $p_h \in W^{1,\infty}(\Omega\times I)$ satisfy $v_h = 0$ on $\partial\Omega\times I$. Suppose that $w_h \in W^{1,\infty}(\Omega\times I;\mathbb{R}^d)$,*

*$s_h \in W^{1,\infty}(\Omega \times I)$ satisfy $w_h = 0$ on $\partial\Omega \times I$. Let $\rho, u$ be an arbitrary finite energy weak solution to the compressible Navier-Stokes equation in the sense of Definition 1. Introduce the error functional*

$$
\begin{aligned}
(11) \quad & D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u] := \\
& \int_\Omega \frac{1}{2}\rho \left|v_h + \sqrt{\epsilon} w_h - u\right|^2 + \pi_\epsilon(\rho) - (\rho - 1)(p_h + s_h) + \Pi_\epsilon(p_h + s_h)\ dx,
\end{aligned}
$$

*with $\Pi_\epsilon$ denoting the convex conjugate of $\pi_\epsilon(1 + \cdot)$. Abbreviate*

$$
\mathcal{D}_{\pi_\epsilon(\rho)} := 2D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u] + \int_\Omega \Pi_\epsilon(2p_h + 2s_h) - 2\,\Pi_\epsilon\,(p_h + s_h)\ dx.
$$

*Let $y \in W^{1,\infty}(\Omega \times I; \mathbb{R}^{d\times d})$ and set for $\tau_7 > 0$*

$$
\begin{aligned}
\mathcal{E}^2_{num1}(T) := & \int_0^T \left|\left| \frac{d}{dt} v_h + (v_h \cdot \nabla) v_h + \nabla p_h - f - \mu_0 \operatorname{div} y \right|\right|_{L^2(\Omega)} \\
& \times \sqrt{2D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u]}\ dt \\
& + \int_0^T \frac{\mu_0^2}{4\mu_{min}\tau_7} ||\nabla v_h - y||^2_{L^2(\Omega)}\ dt.
\end{aligned}
$$

*Let $z \in W^{1,\infty}(\Omega \times I; \mathbb{R}^{d\times d})$ and set for $\tau_9, \tau_{11} > 0$*

$$
\begin{aligned}
\mathcal{E}^2_{num2}(T) := & \int_0^T \sqrt{\epsilon} \Bigg|\Bigg| \frac{d}{dt} w_h + (v_h \cdot \nabla) w_h + (w_h \cdot \nabla) v_h - \mu_0 \operatorname{div} z \\
& - (\mu_0 + \lambda_0) \nabla \operatorname{tr} z + \frac{\nabla s_h}{\sqrt{\epsilon}} \Bigg|\Bigg|_{L^2(\Omega)} \\
& \times \sqrt{2D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u]}\ dt \\
& + \int_0^T \epsilon \frac{\mu_0^2}{4\mu_{min}\tau_9} ||\nabla w_h - z||^2_{L^2(\Omega)}\ dt \\
& + \int_0^T \epsilon \frac{(\lambda_0 + \mu_0)^2}{4\lambda_{min}\tau_{11}} ||\operatorname{div} w_h - \operatorname{tr} z||^2_{L^2(\Omega)}\ dt \\
& + \int_0^T \sqrt{\epsilon} \frac{4}{3} \left|\left| \frac{d}{dt} s_h + (v_h \cdot \nabla) s_h + \frac{\operatorname{div} w_h}{\sqrt{\epsilon}} + \frac{d}{dt} p_h + (v_h \cdot \nabla) p_h \right|\right|_{L^2(\Omega)} \\
& \times \sqrt{2D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u]}\ dt.
\end{aligned}
$$

*Define for $\tau_1, \tau_2, \tau_6 > 0$*

$$
\begin{aligned}
& \mathcal{E}^2_{model}(T) := \\
& \int_0^T \epsilon \sqrt{2D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u]} \cdot ||(w_h \cdot \nabla) w_h||_{L^2(\Omega)}\ dt \\
& + \int_0^T \int_\Omega \epsilon \frac{C_{\pi,\mu}}{2\tau_1} \big(\,\Pi_\epsilon\,(2p_h + 2s_h) - 2\,\Pi_\epsilon\,(p_h + s_h)\big) |\nabla v_h + \nabla \sqrt{\epsilon} w_h|^2\ dx\ dt \\
& + \int_0^T \int_\Omega \epsilon \frac{C_{\pi,\lambda}}{2\tau_2} \big(\,\Pi_\epsilon\,(2p_h + 2s_h) - 2\,\Pi_\epsilon\,(p_h + s_h)\big) |\operatorname{div}(v_h + \sqrt{\epsilon} w_h)|^2\ dx\ dt
\end{aligned}
$$

$$+\int_0^T \epsilon\sqrt{2D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}\cdot\frac{4}{3}||w_h\cdot(\nabla p_h+\nabla s_h)||_{L^2(\Omega)}\ dt$$
$$+\int_0^T \epsilon\frac{C_{\Omega,2,6}^2}{\mu_{min}\tau_6}\left|\left|\left(\frac{d}{dt}+(v_h+\sqrt{\epsilon}w_h)\cdot\nabla\right)(v_h+\sqrt{\epsilon}w_h)-f+\nabla(p_h+s_h)\right|\right|_{L^3(\Omega)}^2\mathcal{D}_{\pi_\epsilon(\rho)}\ dt$$
$$+\int_0^T \epsilon\sqrt{2D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}\cdot\frac{4(\gamma-1)}{3}||(p_h+s_h)\operatorname{div}w_h||_{L^2(\Omega)}\ dt.$$

*Furthermore, set*

$$\mathcal{E}_{div}^2(T):=\frac{4}{3}\int_0^T\sqrt{2D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}\cdot\frac{||\operatorname{div}v_h||_{L^2(\Omega)}}{\sqrt{\epsilon}}\ dt.$$

*Then for any $T>0$ and any $\tau_1,\cdots,\tau_{11}>0$ the following a posteriori estimate holds, provided that we have $c_{div}:=\lambda_{min}(1-\tau_2-\tau_4-\tau_{11})+\mu_{min}(1-\tau_3)>0$ and $1-\tau_1-\tau_5-\sum_{i=6}^{10}\tau_i>0$:*

$$\begin{aligned}
&D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]\Big|_0^T\\
&\quad+\mu_{min}\left(1-\tau_1-\tau_5-\sum_{i=6}^{10}\tau_i\right)\int_0^T\int_\Omega|\nabla u-\nabla(v_h+\sqrt{\epsilon}w_h)|^2\ dx\ dt\\
&\quad+c_{div}\int_0^T\int_\Omega|\operatorname{div}(v_h+\sqrt{\epsilon}w_h)-\operatorname{div}u|^2\ dx\ dt\\
&\leq\mathcal{E}_{num1}^2(T)+\mathcal{E}_{num2}^2(T)+\mathcal{E}_{model}^2(T)+\mathcal{E}_{div}^2(T)+\mathcal{E}_{h.o.t.}^2(T)\\
&\quad+\int_0^T 2\sup_{x\in\Omega}|\nabla v_h+\sqrt{\epsilon}\nabla w_h|_2\ D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]\ dt,
\end{aligned}\tag{12}$$

*where*

$$\begin{aligned}
&\mathcal{E}_{h.o.t.}^2(T):=\\
&\frac{\epsilon^3C_{\pi,e}C_{\Omega,2,6}^2}{4\mu_{min}\tau_5}\int_0^T||(w_h\cdot\nabla)w_h||_{L^3(\Omega)}^2\mathcal{D}_{\pi_\epsilon(\rho)}\ dt\\
&+\frac{\epsilon C_{\pi,\mu}}{\tau_1}\int_0^T\sup_{x\in\Omega}|\nabla v_h+\nabla\sqrt{\epsilon}w_h|^2\ D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]\ dt\\
&+\frac{\epsilon C_{\pi,\lambda}}{\tau_2}\int_0^T\sup_{x\in\Omega}|\operatorname{div}(v_h+\sqrt{\epsilon}w_h)|^2\ D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]\ dt\\
&+\int_0^T\int_\Omega\left(\frac{\mu_0^2}{4\mu_{min}\tau_3}+\frac{\lambda_0^2}{4\lambda_{min}\tau_4}\right)|\operatorname{div}v_h|^2\ dx\ dt\\
&+\epsilon^{5/3}\frac{C_{\pi,f}C_{\Omega,2,6}^2}{2\mu_{min}\tau_6}\int_0^T\mathcal{D}_{\pi_\epsilon(\rho)}^{5/3}\sup_{x\in\Omega}\left|\left(\frac{d}{dt}+(v_h+\sqrt{\epsilon}w_h)\cdot\nabla\right)(v_h+\sqrt{\epsilon}w_h)\right.\\
&\qquad\qquad\left.-f+\nabla(p_h+s_h)\right|^2\ dt\\
&+\epsilon^{3/2}\frac{4C_{\pi,m,2}}{3}\int_0^T\mathcal{D}_{\pi_\epsilon(\rho)}\sup_{x\in\Omega}|w_h\cdot(\nabla p_h+\nabla s_h)|\ dt
\end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{\epsilon}\frac{8C_{\pi,m,1}}{3}\int_0^T\int_\Omega |\,\Pi'_\epsilon\,(p_h+s_h)|^2|w_h\cdot(\nabla p_h+\nabla s_h)|\;dx\;dt\\
&+ \sqrt{\epsilon}(\gamma-1)\int_0^T \sup_{x\in\Omega}|(\operatorname{div} w_h)_-|\cdot D[v_h+\sqrt{\epsilon}w_h, p_h+s_h,\rho,u]\;dt\\
&+ \frac{\epsilon^{3/2}(4\gamma-4)C_{\pi,m,2}}{3}\int_0^T \sup_{x\in\Omega}|(p_h+s_h)\operatorname{div} w_h|\cdot\mathcal{D}_{\pi_\epsilon(\rho)}\;dt\\
&+ \frac{\sqrt{\epsilon}(8\gamma-8)C_{\pi,m,1}}{3}\int_0^T\int_\Omega |(p_h+s_h)\operatorname{div} w_h||\,\Pi'_\epsilon\,(p_h+s_h)|^2\;dx\;dt\\
&+ \frac{4}{3}C_{\pi,m,2}\int_0^T \mathcal{D}_{\pi_\epsilon(\rho)}\sup_{x\in\Omega}|\operatorname{div} v_h|\;dt\\
&+ \frac{8C_{\pi,m,1}}{3}\int_0^T\int_\Omega \epsilon^{-1}|\,\Pi'_\epsilon\,(p_h+s_h)|^2|\operatorname{div} v_h|\;dx\;dt\\
&+ (\gamma-1)\int_0^T \mathcal{D}_{\pi_\epsilon(\rho)}\sup_{x\in\Omega}|(\operatorname{div} v_h)_-|\;dt\\
&+ \int_0^T\int_\Omega \big(\,\Pi_\epsilon\,(p_h+s_h)-\epsilon^{-1}\,\Pi'_\epsilon\,(p_h+s_h)+p_h+s_h\big)\operatorname{div} v_h\;dx\;dt\\
&+ \int_0^T\int_\Omega \operatorname{div}\sqrt{\epsilon}w_h\Big((p_h+s_h)-\epsilon^{-1}\,\Pi'_\epsilon\,(p_h+s_h)\\
&\qquad\qquad -(\gamma-1)(p_h+s_h)\,\Pi'_\epsilon\,(p_h+s_h)+\gamma\,\Pi_\epsilon\,(p_h+s_h)\Big)\;dx\;dt\\
&+ \frac{8C_{\pi,m,1}}{3}\int_0^T\int_\Omega \left|\frac{d}{dt}s_h+(v_h\cdot\nabla)s_h+\frac{\operatorname{div} w_h}{\sqrt{\epsilon}}+\frac{d}{dt}p_h+(v_h\cdot\nabla)p_h\right|\\
&\qquad\qquad \times|\,\Pi'_\epsilon\,(p_h+s_h)|^2\;dx\;dt\\
&+ \epsilon\frac{4C_{\pi,m,2}}{3}\int_0^T \mathcal{D}_{\pi_\epsilon(\rho)}\sup_x\left|\frac{d}{dt}s_h+(v_h\cdot\nabla)s_h+\frac{\operatorname{div} w_h}{\sqrt{\epsilon}}+\frac{d}{dt}p_h+(v_h\cdot\nabla)p_h\right|\;dt\\
&+ \epsilon\frac{C^2_{\Omega,2,6}C_{\pi,e}}{4\mu_{min}\tau_8}\int_0^T \mathcal{D}_{\pi_\epsilon(\rho)}\left\|\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div} y\right\|^2_{L^3(\Omega)}\;dt\\
&+ \epsilon^2\frac{C^2_{\Omega,2,6}C_{\pi,e}}{4\mu_{min}\tau_{10}}\int_0^T \mathcal{D}_{\pi_\epsilon(\rho)}\Bigg\|\frac{d}{dt}w_h+(v_h\cdot\nabla)w_h+(w_h\cdot\nabla)v_h\\
&\qquad\qquad -\mu_0\operatorname{div} z-(\mu_0+\lambda_0)\nabla\operatorname{tr} z+\frac{\nabla s_h}{\sqrt{\epsilon}}\Bigg\|^2_{L^3(\Omega)}\;dt.
\end{aligned}
$$

We observe that we have indeed derived an integral inequality for the time-dependent error functional $D[v_h+\sqrt{\epsilon}w_h, p_h+s_h,\rho,u]$, the coefficients of the integral inequality being entirely computable in terms of the physical parameters $\epsilon$, $\gamma$, $\mu$, $\lambda$ and the approximate solutions $v_h$, $p_h$ and $w_h$, $s_h$.

Let us review the different contributions to the error. The term $\mathcal{E}^2_{h.o.t.}$ consists of higher-order terms; in practical applications, we expect it to be negligible. The term $\mathcal{E}^2_{num1}$ bounds the (squared) numerical error of the approximate solution to the incompressible Navier-Stokes equation $v_h$, $p_h$, while the term $\mathcal{E}^2_{num2}$ is an estimate for the (squared) numerical error in our approximate solution to the equations of linearized acoustics $w_h$, $s_h$.

The term $\mathcal{E}^2_{div}$ is a result of a harmful interaction of model simplification and discretization. In practice, this term requires our approximate solution to the incompressible Navier-Stokes equation $v_h$, $p_h$ to be (almost) solenoidal. Numerically, this may be accomplished by using divergence-free elements or – as in [19] – by postprocessing of the solution.

The last term on the right-hand side of (12) captures a possible instability of our flow: Depending on the flow, small perturbations in the flow may grow exponentially in time. This term only amplifies errors which are already present; it does not introduce new errors by itself. However, as seen in the numerical examples in [19] it is often the main contribution to the overall error. Obtaining an improved estimate for the possible instability of the solution to the incompressible Navier-Stokes equation will be the subject of future work.

Note that for $d = 3$, even for well-behaved discretizations in theory it is not guaranteed *a priori* that the integral $\int_0^T 2\sup_{x\in\Omega} |\nabla v_h + \sqrt{\epsilon}\nabla w_h|_2 \, dt$ (by the exponential of which our error estimates are amplified by the last term in (12)) remains bounded uniformly in the discretization parameter $h$ (and therefore, it is not guaranteed *a priori* that our estimates are meaningful for small $h$): For well-behaved discretizations the approximate solution $v_h$ inherits the regularity from the corresponding exact solution to the incompressible Navier-Stokes equation (though in general not more than Lipschitz regularity should be expected for $v_h$). Existence of strong solutions to the incompressible Navier-Stokes equation in three spatial dimensions is, of course, the famous open problem; however, it is generally considered to be likely that for most data strong solutions exist globally. Morally speaking, in our estimates we exploit the weak-strong uniqueness principle for the incompressible Navier-Stokes equation to obtain rigorous quantitative error estimates by imposing the strong regularity assumption on the *approximate solution* $v_h$, $p_h$. This has the advantage that the check whether the approximate solution $v_h$, $p_h$ has the required regularity can be performed *a posteriori*.

The leading-order part of the modeling error is accounted for by the term $\mathcal{E}^2_{model}$. The compressible Navier-Stokes equation is a nonlinear equation in $\rho$ and $u$; therefore, the linearized approximation around the incompressible limit (i.e. around the solution of the incompressible Navier-Stokes equation) – by which the equations of linearized acoustics are derived – necessarily introduces errors. For example, the green terms in $\mathcal{E}^2_{model}$ account for the possible dependence of the viscosities $\mu$ and $\lambda$ on the density $\rho$.

Note that the tensor fields $y$ and $z$ in our estimates for the numerical error need to be chosen appropriately to obtain an efficient a posteriori error estimate. One should think of $y$ and $z$ as Lipschitz continuous approximations of the (possibly discontinuous) tensor fields $\nabla v_h$ and $\nabla w_h$. In practice, one may choose $y$ and $z$ by minimizing the convex functionals defined by the error terms in the class of functions of a suitable finite element space, i.e. by solving a finite-dimensional convex optimization problem. In most cases, it is even sufficient to minimize a quadratic functional similar to the error terms in some finite element space, which amounts to solving a linear system of equations. In order to save computational time, one might also use an appropriate (higher-order) flux reconstruction to construct $y$ and $z$ from $\nabla v_h$ and $\nabla w_h$.

To cast the integral inequality (12) into a more explicit form, observe that in case of solenoidal velocity fields $v_h$, up to higher-order terms the inequality (12) takes the form

$$\begin{aligned}
&D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]\Big|_0^T\\
&\leq \int_0^T (\epsilon A(t)+e_{num,a})\sqrt{D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}\ dt\\
&\quad+\int_0^T B(t)D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]\ dt\\
&\quad+\int_0^T \epsilon^2 E(t)+e_{num,b}^2\ dt
\end{aligned}$$

for certain ($\epsilon$-independent) functions $A,B,E\in L^1([0,T])$. Here, $e_{num,a}$ and $e_{num,b}$ denote different contributions to the numerical error estimate. This reformulation gives rise to the following remark.

**Remark 5.** *For well-behaved flows and solenoidal velocity fields $v_h$, the modeling error estimates stated in Theorem 4 are indeed of order $\epsilon$ on any fixed time interval $[0,T]$: An upper bound for the solution to the integral equation*

$$D(t)\Big|_0^T=\int_0^T \sqrt{\hat{A}(t)\,D(t)}+\hat{B}(t)\,D(t)\ dt+\hat{E}(T)$$

*is given by*

$$D^{up}(t):=\left(\sqrt{D(0)+\hat{E}(t)}+\frac{1}{2}\int_0^t\sqrt{\hat{A}(\tau)}\ d\tau\right)^2\exp\left(\int_0^t\hat{B}(\tau)\ d\tau\right),$$

*which implies the estimate $D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u](t)\lesssim\epsilon^2+\int_0^t e_{num,a}^2(\tau)+e_{num,b}^2(\tau)\ d\tau$ on any fixed time interval for well-behaved $v_h$, $p_h$, $w_h$, $s_h$.*

## 3. A posteriori estimates for the numerical error

For the proof of our main result, we use the following a posteriori error estimates of functional type for the numerical error in the incompressible Navier-Stokes equation and the equations of linearized acoustics. For the reader's convenience, we have used colored underlines to mark the terms; terms with the same color are generally rearranged together in the subsequent computations (in particular in the proof of Theorem 4). Exceptions to this rule will be mentioned in the accompanying text.

**Lemma 6.** *Let $v_h,w_h\in W^{1,\infty}(\Omega\times I;\mathbb{R}^d)$, $p_h\in W^{1,\infty}(\Omega\times I)$ and let $u\in L^2(I;H^1(\Omega;\mathbb{R}^d))$. Suppose that $v_h+\sqrt{\epsilon}w_h-u\in L^2(I;H_0^1(\Omega;\mathbb{R}^d))$. It then holds for any $y\in W^{1,\infty}(\Omega\times I;\mathbb{R}^{d\times d})$ and any $T>0$ that*

$$\begin{aligned}
&\int_0^T\int_\Omega (v_h+\sqrt{\epsilon}w_h-u)\cdot\frac{d}{dt}v_h\ dx\ dt\\
&\leq -\underline{\int_0^T\int_\Omega (v_h+\sqrt{\epsilon}w_h-u)\cdot(v_h\cdot\nabla)v_h\ dx\ dt}\\
&\quad-\underline{\int_0^T\int_\Omega \mu_0\nabla v_h:\nabla(v_h+\sqrt{\epsilon}w_h-u)\ dx\ dt}\\
&\quad-\underline{\int_0^T\int_\Omega \nabla p_h\cdot(v_h+\sqrt{\epsilon}w_h-u)\ dx\ dt}
\end{aligned}\tag{13}$$

$$+\int_0^T\int_\Omega f\cdot(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt$$
$$+A_{residual,1},$$

*where*

$$\begin{aligned}A_{residual,1}:=\int_0^T&\bigg|\bigg|(v_h+\sqrt{\epsilon}w_h-u)\\&\cdot\left(\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div}y\right)\bigg|\bigg|_{L^1(\Omega)}\;dt\\&+\int_0^T\mu_0||\nabla(v_h+\sqrt{\epsilon}w_h-u)||_{L^2(\Omega)}||\nabla v_h-y||_{L^2(\Omega)}\;dt.\end{aligned}$$

*Proof.* We add the integrals on the right-hand side of (13) to the integral $\int_0^T\int_\Omega(v_h+\sqrt{\epsilon}w_h-u)\cdot\frac{d}{dt}v_h\;dx\;dt$ and subtract them again; then we subtract the term $\mu_0\int_0^T\int_\Omega\nabla(v_h+\sqrt{\epsilon}w_h-u):y+(v_h+\sqrt{\epsilon}w_h-u)\cdot\operatorname{div}y\;dx\;dt$ (note that this term is zero) and employ standard estimates. □

**Lemma 7.** *Let $v_h,w_h\in W^{1,\infty}(\Omega\times I;\mathbb{R}^d)$, $u\in L^2(I;H^1(\Omega;\mathbb{R}^d))$, $p_h,s_h\in W^{1,\infty}(\Omega\times I)$. Suppose that $v_h+\sqrt{\epsilon}w_h-u\in L^2(I;H^1_0(\Omega;\mathbb{R}^d))$. We then have for any $z\in W^{1,\infty}(\Omega\times I;\mathbb{R}^{d\times d})$ and any $T>0$ that*

$$\begin{aligned}&\int_0^T\int_\Omega(v_h+\sqrt{\epsilon}w_h-u)\cdot\frac{d}{dt}w_h\;dx\;dt\\&\leq-\int_0^T\int_\Omega(v_h+\sqrt{\epsilon}w_h-u)\cdot(v_h\cdot\nabla)w_h\;dx\;dt\\&\quad-\int_0^T\int_\Omega(v_h+\sqrt{\epsilon}w_h-u)\cdot(w_h\cdot\nabla)v_h\;dx\;dt\\&\quad-\int_0^T\int_\Omega\mu_0\nabla w_h:\nabla(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt\\&\quad-\int_0^T\int_\Omega(\lambda_0+\mu_0)\operatorname{div}w_h\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt\\&\quad-\int_0^T\int_\Omega\frac{1}{\sqrt{\epsilon}}\nabla s_h\cdot(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt\\&\quad+A_{residual,2},\end{aligned}\tag{14}$$

*where*

$$\begin{aligned}A_{residual,2}:=\int_0^T&\bigg|\bigg|\bigg(\frac{d}{dt}w_h+(v_h\cdot\nabla)w_h+(w_h\cdot\nabla)v_h-\mu_0\operatorname{div}z\\&-(\mu_0+\lambda_0)\nabla\operatorname{tr}z+\frac{\nabla s_h}{\sqrt{\epsilon}}\bigg)\cdot(v_h+\sqrt{\epsilon}w_h-u)\bigg|\bigg|_{L^1(\Omega)}\;dt\\&+\int_0^T\mu_0||\nabla(v_h+\sqrt{\epsilon}w_h-u)||_{L^2(\Omega)}||\nabla w_h-z||_{L^2(\Omega)}\\&\quad+(\lambda_0+\mu_0)||\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)||_{L^2(\Omega)}||\operatorname{div}w_h-\operatorname{tr}z||_{L^2(\Omega)}\;dt.\end{aligned}$$

*Proof.* We start from the estimate $0 \leq 0$ and add and subtract all space-time integrals occurring in (14). Adding the equation

$$\begin{aligned}0 = &-\int_\Omega (\mu_0 \operatorname{div} z + (\mu_0+\lambda_0)\nabla \operatorname{tr} z)\cdot (v_h + \sqrt{\epsilon} w_h - u)\ dx \\ &- \int_\Omega \mu_0 z : \nabla(v_h + \sqrt{\epsilon} w_h - u)\ dx \\ &- \int_\Omega (\mu_0+\lambda_0) \operatorname{tr} z \operatorname{div}(v_h + \sqrt{\epsilon} w_h - u)\ dx\end{aligned}$$

and using Hölder's inequality, the desired result is established. □

**Lemma 8.** *Let $v_h, w_h \in W^{1,\infty}(\Omega\times I;\mathbb{R}^d)$, $p_h, s_h \in W^{1,\infty}(\Omega\times I)$, and $\rho \in L^\infty(I; L^\gamma(\Omega))$. It then holds for any $T>0$ that*

$$\begin{aligned}&\int_0^T \int_\Omega \big(-(\rho-1) + \Pi_\epsilon'(p_h+s_h)\big)\cdot \frac{d}{dt} s_h\ dx\ dt \\ &\leq \int_0^T \int_\Omega \big((\rho-1) - \Pi_\epsilon'(p_h+s_h)\big)\cdot (v_h\cdot\nabla) s_h\ dx\ dt \\ &\quad + \int_0^T \int_\Omega \big((\rho-1) - \Pi_\epsilon'(p_h+s_h)\big) \frac{\operatorname{div} w_h}{\sqrt{\epsilon}}\ dx\ dt \\ &\quad + \int_0^T \int_\Omega \big((\rho-1) - \Pi_\epsilon'(p_h+s_h)\big) \left(\frac{d}{dt} p_h + (v_h\cdot\nabla)p_h\right)\ dx\ dt \\ &\quad + A_{residual,3},\end{aligned} \tag{15}$$

*where*

$$\begin{aligned}A_{residual,3} := \int_0^T \int_\Omega &\left| \frac{d}{dt} s_h + (v_h\cdot\nabla)s_h + \frac{\operatorname{div} w_h}{\sqrt{\epsilon}} + \frac{d}{dt}p_h + (v_h\cdot\nabla)p_h \right| \\ &\times \left| \rho - 1 - \Pi_\epsilon'(p_h+s_h) \right|\ dx\ dt.\end{aligned}$$

*Proof.* Trivial. □

## 4. A posteriori modeling error estimates

In order to derive our a posteriori modeling error estimates, let us first prove two lemmas for the evolution of the difference between $D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u]$ and $D[v_h, p_h, \rho, u]$.

**Lemma 9.** *Suppose that the assumptions of Theorem 4 are satisfied. We then have for almost every $T \geq 0$*

$$\begin{aligned}&\int_\Omega \rho\ (v_h - u)\cdot \sqrt{\epsilon} w_h + \frac{\rho}{2} |\sqrt{\epsilon} w_h|^2\ dx \Big|_0^T \\ &- \int_0^T \int_\Omega \rho\ \sqrt{\epsilon} w_h \cdot \frac{d}{dt} v_h\ dx\ dt \\ &- \int_0^T \int_\Omega (\rho-1)\ (v_h + \sqrt{\epsilon} w_h - u)\cdot \frac{d}{dt}\sqrt{\epsilon} w_h\ dx\ dt \\ &\leq -\int_0^T \int_\Omega (v_h + \sqrt{\epsilon} w_h - u)\cdot (v_h\cdot\nabla)\sqrt{\epsilon} w_h\ dx\ dt\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_\Omega (v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)v_h\,dx\,dt\\
&+\int_0^T\int_\Omega \rho\,\sqrt{\epsilon}w_h\cdot(u\cdot\nabla)v_h\,dx\,dt\\
&+\int_0^T\int_\Omega \rho\,(v_h+\sqrt{\epsilon}w_h-u)\cdot(u\cdot\nabla)\sqrt{\epsilon}w_h\,dx\,dt\\
&-\int_0^T\int_\Omega \mu_0\nabla\sqrt{\epsilon}w_h:\nabla(v_h+\sqrt{\epsilon}w_h-u)\,dx\,dt\\
&-\int_0^T\int_\Omega (\lambda_0+\mu_0)\operatorname{div}\sqrt{\epsilon}w_h\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\,dx\,dt\\
&+\int_0^T\int_\Omega \mu(\rho)(\nabla u+\nabla u^T):\nabla\sqrt{\epsilon}w_h\,dx\,dt+\int_0^T\int_\Omega \lambda(\rho)\operatorname{div}u\operatorname{div}\sqrt{\epsilon}w_h\,dx\,dt\\
&-\int_0^T\int_\Omega \nabla s_h\cdot(v_h+\sqrt{\epsilon}w_h-u)\,dx\,dt-\int_0^T\int_\Omega p_\epsilon(\rho)\operatorname{div}\sqrt{\epsilon}w_h\,dx\,dt\\
&-\int_0^T\int_\Omega \rho f\cdot\sqrt{\epsilon}w_h\,dx\,dt\\
&+\sqrt{\epsilon}A_{residual,2}.
\end{aligned}
$$

*Proof.* Testing the continuity equation (9) with $v_h\cdot\sqrt{\epsilon}w_h+\frac{1}{2}|\sqrt{\epsilon}w_h|^2$, we infer

$$
\begin{aligned}
&\int_\Omega \rho\,\left(v_h\cdot\sqrt{\epsilon}w_h+\frac{1}{2}|\sqrt{\epsilon}w_h|^2\right)\,dx\Big|_0^T\\
&-\int_0^T\int_\Omega \rho\,\sqrt{\epsilon}w_h\cdot\frac{d}{dt}v_h\,dx\,dt-\int_0^T\int_\Omega \rho\,(v_h+\sqrt{\epsilon}w_h)\cdot\frac{d}{dt}\sqrt{\epsilon}w_h\,dx\,dt\\
&=\int_0^T\int_\Omega \sqrt{\epsilon}w_h\cdot(\rho u\cdot\nabla)v_h\,dx\,dt+\int_0^T\int_\Omega (v_h+\sqrt{\epsilon}w_h)\cdot(\rho u\cdot\nabla)\sqrt{\epsilon}w_h\,dx\,dt.
\end{aligned}
$$

Testing the momentum equation (8) with $-\sqrt{\epsilon}w_h$ gives

$$
\begin{aligned}
&-\int_\Omega \rho u\cdot\sqrt{\epsilon}w_h\,dx\Big|_0^T\\
&+\int_0^T\int_\Omega \rho u\cdot\frac{d}{dt}\sqrt{\epsilon}w_h\,dx\,dt\\
&=-\int_0^T\int_\Omega \rho u\cdot(u\cdot\nabla)\sqrt{\epsilon}w_h\,dx\,dt-\int_0^T\int_\Omega p_\epsilon(\rho)\operatorname{div}\sqrt{\epsilon}w_h\,dx\,dt\\
&\quad+\int_0^T\int_\Omega \mu(\rho)(\nabla u+\nabla u^T):\nabla\sqrt{\epsilon}w_h\,dx\,dt+\int_0^T\int_\Omega \lambda(\rho)\operatorname{div}u\operatorname{div}\sqrt{\epsilon}w_h\,dx\,dt\\
&\quad-\int_0^T\int_\Omega \rho f\cdot\sqrt{\epsilon}w_h\,dx\,dt.
\end{aligned}
$$

Taking the sum of the two formulas as well as (14) multiplied by $\sqrt{\epsilon}$ yields the desired result. $\square$

**Lemma 10.** *Suppose that the assumptions of Theorem 4 are satisfied. We then have for almost every $T \geq 0$*

$$
\begin{aligned}
&\int_\Omega -(\rho-1)s_h + \Pi_\epsilon(p_h+s_h) - \Pi_\epsilon(p_h)\ dx\Big|_0^T \\
&\leq \int_0^T \int_\Omega \left(\Pi_\epsilon'(p_h+s_h) - \Pi_\epsilon'(p_h)\right) \frac{d}{dt}p_h\ dx\ dt \\
&\quad - \int_0^T \int_\Omega \rho u \cdot \nabla s_h\ dx\ dt \\
&\quad + \int_0^T \int_\Omega \big((\rho-1) - \Pi_\epsilon'(p_h+s_h)\big) \cdot (v_h\cdot\nabla)s_h\ dx\ dt \\
&\quad + \int_0^T \int_\Omega \big((\rho-1) - \Pi_\epsilon'(p_h+s_h)\big) \frac{\operatorname{div} w_h}{\sqrt{\epsilon}}\ dx\ dt \\
&\quad + \int_0^T \int_\Omega \big((\rho-1) - \Pi_\epsilon'(p_h+s_h)\big) \left(\frac{d}{dt}p_h + (v_h\cdot\nabla)p_h\right)\ dx\ dt \\
&\quad + A_{residual,3}.
\end{aligned}
$$

*Proof.* Testing the continuity equation (9) with $-s_h$, we infer

$$
\begin{aligned}
&\int_\Omega -(\rho-1)s_h + \Pi_\epsilon(p_h+s_h) - \Pi_\epsilon(p_h)\ dx\Big|_0^T \\
&- \int_0^T \int_\Omega -(\rho-1)\cdot\frac{d}{dt}s_h\ dx\ dt - \int_0^T \int_\Omega \Pi_\epsilon'(p_h+s_h)\frac{d}{dt}s_h\ dx\ dt \\
&- \int_0^T \int_\Omega \left(\Pi_\epsilon'(p_h+s_h) - \Pi_\epsilon'(p_h)\right)\frac{d}{dt}p_h\ dx\ dt \\
&= -\int_0^T \int_\Omega \rho u\cdot\nabla s_h\ dx\ dt.
\end{aligned}
$$

Adding the formula in Lemma 8, the lemma is established. □

Recall the following two results from [19].

**Lemma 11** (cf. [19, Lemma 8])**.** *Let $\pi_\epsilon : \mathbb{R} \to [0,\infty]$ be a strictly convex function with $\pi_\epsilon(1) = 0$. Denote by $\Pi_\epsilon$ the convex conjugate of $\pi_\epsilon(1+\cdot)$. Then the estimates*

$$
\pi_\epsilon(\rho) \leq 2\Big(\pi_\epsilon(\rho) - (\rho-1)p + \Pi_\epsilon(p)\Big) + \Pi_\epsilon(2p) - 2\,\Pi_\epsilon\,(p)
$$

*and*

$$
\begin{aligned}
\frac{|\rho-1-\Pi_\epsilon'(p)|^2}{2\epsilon} \leq{}& \pi_\epsilon(\rho) - (\rho-1)p + \Pi_\epsilon(p) \\
&+ \frac{|\rho-1-\Pi_\epsilon'(p)|^2}{2\epsilon}\max\Bigg(\sup_{s\in[1\wedge\rho,1\vee\rho]}(1-\epsilon\pi_\epsilon''(s))_+, \\
&\qquad \sup_{s\in[1\wedge 1+\Pi_\epsilon'(p),1\vee 1+\Pi_\epsilon'(p)]}(1-\epsilon\pi_\epsilon''(s))_+\Bigg)
\end{aligned}
$$

*are satisfied for any $\rho\in[0,\infty)$ and any $p\in\mathbb{R}$.*

The following result corresponds to the estimate in [19, Equation (21)] with one exception: In the present estimate, the first term on the right-hand side (i.e. the yellow term) is stated in its original form (see the estimates preceding [19, Equation (21)]).

**Lemma 12.** *Suppose that the assumptions of Theorem 4 are satisfied. We then have for any $T \geq 0$*

$$
\begin{aligned}
D[v_h,p_h,\rho,u]\Big|_0^T &= \int_\Omega \frac{1}{2}\rho|v_h-u|^2+\pi_\epsilon(\rho)-(\rho-1)p_h+\Pi_\epsilon(p_h)\;dx\Big|_0^T \\
&\leq -\int_0^T\int_\Omega \rho\,((v_h-u)\otimes(v_h-u)):\nabla v_h\;dx\;dt \\
&\quad+\int_0^T\int_\Omega (\rho-1)(v_h-u)\cdot\left(\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f\right)\;dx\;dt \\
&\quad-\int_0^T\int_\Omega \frac{\mu(\rho)}{2}\Big|(\nabla v_h+\nabla v_h^T)-(\nabla u+\nabla u^T)\Big|^2\;dx\;dt \\
&\quad-\int_0^T\int_\Omega \lambda(\rho)|\operatorname{div} v_h-\operatorname{div} u|^2\;dx\;dt \qquad (16)\\
&\quad+\int_0^T\int_\Omega (\mu(\rho)-\mu_0)\nabla v_h:\Big((\nabla v_h+\nabla v_h^T)-(\nabla u+\nabla u^T)\Big)\;dx\;dt \\
&\quad+\int_0^T\int_\Omega (\lambda(\rho)-\lambda_0)(\operatorname{div} v_h-\operatorname{div} u)\operatorname{div} v_h\;dx\;dt \\
&\quad+\int_0^T\int_\Omega (\mu_0+\lambda_0)\operatorname{div}(v_h-u)\operatorname{div} v_h\;dx\;dt \\
&\quad-\int_0^T\int_\Omega \big(\rho-1-\Pi_\epsilon'(p_h)\big)\left(\frac{d}{dt}p_h+(v_h\cdot\nabla)p_h\right)\;dx\;dt \\
&\quad+\int_0^T\int_\Omega \big(p_h+\Pi_\epsilon(p_h)\big)\operatorname{div} v_h\;dx\;dt \\
&\quad-\int_0^T\int_\Omega p_\epsilon(\rho)\operatorname{div} v_h\;dx\;dt+A_{residual},
\end{aligned}
$$

*where $A_{residual}$ is given by*

$$
\begin{aligned}
A_{residual} := &\int_0^T \left|\left|(v_h-u)\cdot\left(\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div} y\right)\right|\right|_{L^1(\Omega)}\;dt \\
&+\int_0^T \mu_0||\nabla(v_h-u)||_{L^2(\Omega)}||\nabla v_h-y||_{L^2(\Omega)}\;dt.
\end{aligned}
$$

We are now in position to prove our main result.

*Proof of Theorem 4.* By replacing the a posteriori estimate for the numerical error in [19, Lemma 7] in the derivation of (16) by our estimate in Lemma 6, we obtain an estimate for $D[v_h,p_h,\rho,u]$ which almost coincides with the estimate (16); however now $A_{residual}$ is replaced by $A_{residual,1}$ and additionally the terms

$$
\begin{aligned}
&-\int_0^T\int_\Omega \sqrt{\epsilon}w_h\cdot\frac{d}{dt}v_h\,dx\,dt-\int_0^T\int_\Omega \sqrt{\epsilon}w_h\cdot(v_h\cdot\nabla)v_h\,dx\,dt\\
&-\int_0^T\int_\Omega \mu_0\nabla v_h:\nabla\sqrt{\epsilon}w_h\,dx\,dt-\int_0^T\int_\Omega \nabla p_h\cdot\sqrt{\epsilon}w_h\,dx\,dt\\
&+\int_0^T\int_\Omega f\cdot\sqrt{\epsilon}w_h\,dx\,dt
\end{aligned}
$$

appear on the right-hand side. Adding the formula from Lemma 9 to this new formula, we deduce

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\rho|v_h+\sqrt{\epsilon}w_h-u|^2+\pi_\epsilon(\rho)-(\rho-1)p_h+\Pi_\epsilon(p_h)\,dx\Big|_0^T\\
&\le Yellow+Green+A_{residual,1}+\sqrt{\epsilon}A_{residual,2}\\
&\quad+\int_0^T\int_\Omega(\rho-1)\,(v_h+\sqrt{\epsilon}w_h-u)\cdot\frac{d}{dt}\sqrt{\epsilon}w_h\,dx\,dt\\
&\quad+\int_0^T\int_\Omega(\rho-1)\sqrt{\epsilon}w_h\cdot\frac{d}{dt}v_h\,dx\,dt\\
&\quad+\int_0^T\int_\Omega(\rho-1)(v_h-u)\cdot\left(\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f\right)dx\,dt\\
&\quad-\int_0^T\int_\Omega p_\epsilon(\rho)\operatorname{div}(v_h+\sqrt{\epsilon}w_h)\,dx\,dt\\
&\quad-\int_0^T\int_\Omega \nabla s_h\cdot(v_h+\sqrt{\epsilon}w_h-u)\,dx\,dt\\
&\quad-\int_0^T\int_\Omega\big(\rho-1-\Pi_\epsilon'(p_h)\big)\left(\frac{d}{dt}p_h+(v_h\cdot\nabla)p_h\right)dx\,dt\\
&\quad+\int_0^T\int_\Omega\big(p_h+\Pi_\epsilon(p_h)\big)\operatorname{div}v_h+p_h\operatorname{div}\sqrt{\epsilon}w_h\,dx\,dt\\
&\quad-\int_0^T\int_\Omega(\rho-1)f\cdot\sqrt{\epsilon}w_h\,dx\,dt,
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
Yellow:=&-\int_0^T\int_\Omega\rho\,((v_h-u)\otimes(v_h-u)):\nabla v_h\,dx\,dt\\
&+\int_0^T\int_\Omega(\rho-1)\sqrt{\epsilon}w_h\cdot(v_h\cdot\nabla)v_h\,dx\,dt\\
&+\int_0^T\int_\Omega(\rho-1)(v_h+\sqrt{\epsilon}w_h-u)\cdot(v_h\cdot\nabla)\sqrt{\epsilon}w_h\,dx\,dt\\
&-\int_0^T\int_\Omega\rho\,(v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)v_h\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega (\rho-1)(v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)v_h\;dx\;dt\\
&+\int_0^T\int_\Omega \rho\;\sqrt{\epsilon}w_h\cdot((u-v_h)\cdot\nabla)v_h\;dx\;dt\\
&+\int_0^T\int_\Omega \rho\;(v_h+\sqrt{\epsilon}w_h-u)\cdot((u-v_h)\cdot\nabla)\sqrt{\epsilon}w_h\;dx\;dt
\end{aligned}
$$

and

$$
\begin{aligned}
Green :=&-\int_0^T\int_\Omega \frac{\mu(\rho)}{2}\Big|(\nabla v_h+\nabla v_h^T)-(\nabla u+\nabla u^T)\Big|^2\;dx\;dt\\
&-\int_0^T\int_\Omega \lambda(\rho)|\operatorname{div} v_h-\operatorname{div} u|^2\;dx\;dt\\
&+\int_0^T\int_\Omega (\mu(\rho)-\mu_0)\nabla v_h:\Big((\nabla v_h+\nabla v_h^T)-(\nabla u+\nabla u^T)\Big)\;dx\;dt\\
&+\int_0^T\int_\Omega (\lambda(\rho)-\lambda_0)(\operatorname{div} v_h-\operatorname{div} u)\operatorname{div} v_h\;dx\;dt\\
&+\int_0^T\int_\Omega (\mu_0+\lambda_0)\operatorname{div}(v_h-u)\operatorname{div} v_h\;dx\;dt\\
&-\int_0^T\int_\Omega \mu_0\nabla\sqrt{\epsilon}w_h:\nabla(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt\\
&-\int_0^T\int_\Omega (\lambda_0+\mu_0)\operatorname{div}\sqrt{\epsilon}w_h\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt\\
&+\int_0^T\int_\Omega (\mu(\rho)(\nabla u+\nabla u^T)-\mu_0\nabla v_h):\nabla\sqrt{\epsilon}w_h\;dx\;dt\\
&+\int_0^T\int_\Omega \lambda(\rho)\operatorname{div} u\operatorname{div}\sqrt{\epsilon}w_h\;dx\;dt.
\end{aligned}
$$

Rearranging and using the identities

$$
\begin{aligned}
&\int_0^T\int_\Omega \mu_0\operatorname{div}\sqrt{\epsilon}w_h\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt\\
&-\int_0^T\int_\Omega \mu_0(\nabla\sqrt{\epsilon}w_h)^T:\nabla(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt=0
\end{aligned}
$$

and

$$
\int_0^T\int_\Omega \mu_0\operatorname{div} v_h\operatorname{div}\sqrt{\epsilon}w_h\;dx\;dt-\int_0^T\int_\Omega \mu_0(\nabla v_h)^T:\nabla\sqrt{\epsilon}w_h\;dx\;dt=0,
$$

we obtain

$$
\begin{aligned}
Green =&-\int_0^T\int_\Omega \frac{\mu(\rho)}{2}\Big|(\nabla v_h+\nabla v_h^T)-(\nabla u+\nabla u^T)\Big|^2\;dx\;dt\\
&-\int_0^T\int_\Omega \lambda(\rho)|\operatorname{div} v_h-\operatorname{div} u|^2\;dx\;dt
\end{aligned}
$$

$$+\int_0^T\int_\Omega(\mu(\rho)-\mu_0)\nabla v_h:\Big((\nabla v_h+\nabla v_h^T)-(\nabla u+\nabla u^T)\Big)\;dx\;dt$$
$$+\int_0^T\int_\Omega(\lambda(\rho)-\lambda_0)(\operatorname{div}v_h-\operatorname{div}u)\operatorname{div}v_h\;dx\;dt$$
$$+\int_0^T\int_\Omega(\mu_0+\lambda_0)\operatorname{div}(v_h-u)\operatorname{div}v_h\;dx\;dt$$
$$-\int_0^T\int_\Omega\mu(\rho)(\nabla\sqrt{\epsilon}w_h+\nabla\sqrt{\epsilon}w_h^T):\nabla(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt$$
$$+\int_0^T\int_\Omega(\mu(\rho)-\mu_0)(\nabla\sqrt{\epsilon}w_h+\nabla\sqrt{\epsilon}w_h^T):\nabla(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt$$
$$-\int_0^T\int_\Omega\lambda(\rho)\operatorname{div}\sqrt{\epsilon}w_h\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt$$
$$+\int_0^T\int_\Omega(\lambda(\rho)-\lambda_0)\operatorname{div}\sqrt{\epsilon}w_h\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\;dx\;dt$$
$$+\int_0^T\int_\Omega\mu(\rho)((\nabla u+\nabla u^T)-(\nabla v_h+\nabla v_h^T)):\nabla\sqrt{\epsilon}w_h\;dx\;dt$$
$$+\int_0^T\int_\Omega(\mu(\rho)-\mu_0)(\nabla v_h+\nabla v_h^T):\nabla\sqrt{\epsilon}w_h\;dx\;dt$$
$$+\int_0^T\int_\Omega\mu_0\operatorname{div}v_h\operatorname{div}\sqrt{\epsilon}w_h\;dx\;dt$$
$$+\int_0^T\int_\Omega\lambda(\rho)\operatorname{div}u\operatorname{div}\sqrt{\epsilon}w_h\;dx\;dt$$
$$-\int_0^T\int_\Omega\lambda(\rho)\operatorname{div}v_h\operatorname{div}\sqrt{\epsilon}w_h\;dx\;dt$$
$$+\int_0^T\int_\Omega(\lambda(\rho)-\lambda_0)\operatorname{div}v_h\operatorname{div}\sqrt{\epsilon}w_h\;dx\;dt$$
$$+\int_0^T\int_\Omega\lambda_0\operatorname{div}v_h\operatorname{div}\sqrt{\epsilon}w_h\;dx\;dt.$$

Simplifying, we get

$$\begin{aligned}&Green\\&=-\int_0^T\int_\Omega\frac{\mu(\rho)}{2}\Big|(\nabla v_h+\nabla\sqrt{\epsilon}w_h+\nabla v_h^T+\nabla\sqrt{\epsilon}w_h^T)-(\nabla u+\nabla u^T)\Big|^2\;dx\;dt\\&\quad-\int_0^T\int_\Omega\lambda(\rho)|\operatorname{div}v_h+\operatorname{div}\sqrt{\epsilon}w_h-\operatorname{div}u|^2\;dx\;dt\\&\quad+\int_0^T\int_\Omega(\mu(\rho)-\mu_0)(\nabla v_h+\nabla\sqrt{\epsilon}w_h)\\&\qquad\qquad:\Big((\nabla v_h+\nabla\sqrt{\epsilon}w_h+\nabla v_h^T+\nabla\sqrt{\epsilon}w_h^T)-(\nabla u+\nabla u^T)\Big)\;dx\;dt\end{aligned}$$

$$+\int_0^T\int_\Omega (\lambda(\rho)-\lambda_0)\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\operatorname{div}(v_h+\sqrt{\epsilon}w_h)\;dx\;dt$$
$$+\int_0^T\int_\Omega (\mu_0+\lambda_0)\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\operatorname{div}v_h\;dx\;dt.$$

Rearranging the yellow terms yields

$$\begin{aligned}
Yellow = &-\int_0^T\int_\Omega \rho\,\left((v_h+\sqrt{\epsilon}w_h-u)\otimes(v_h+\sqrt{\epsilon}w_h-u)\right):\nabla(v_h+\sqrt{\epsilon}w_h)\;dx\;dt\\
&+\int_0^T\int_\Omega (v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)\sqrt{\epsilon}w_h\;dx\;dt\\
&+\int_0^T\int_\Omega (\rho-1)(v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)\sqrt{\epsilon}w_h\;dx\;dt\\
&+\int_0^T\int_\Omega (\rho-1)\sqrt{\epsilon}w_h\cdot(v_h\cdot\nabla)v_h\;dx\;dt\\
&+\int_0^T\int_\Omega (\rho-1)(v_h+\sqrt{\epsilon}w_h-u)\cdot(v_h\cdot\nabla)\sqrt{\epsilon}w_h\;dx\;dt\\
&+\int_0^T\int_\Omega (\rho-1)(v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)v_h\;dx\;dt.
\end{aligned}$$

Going back to formula (17), we add the gray term to the colorless terms and the last four terms in the previous formula for $Yellow$ to the colorless terms. Furthermore, we insert the rearranged expression for $Green$ in the formula and simplify the blue terms using integration by parts. This gives

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho|v_h+\sqrt{\epsilon}w_h-u|^2+\pi_\epsilon(\rho)-(\rho-1)p_h+\Pi_\epsilon(p_h)\;dx\Big|_0^T\\
&\leq -\int_0^T\int_\Omega \rho\,\left((v_h+\sqrt{\epsilon}w_h-u)\otimes(v_h+\sqrt{\epsilon}w_h-u)\right):\nabla(v_h+\sqrt{\epsilon}w_h)\;dx\;dt\\
&\quad+\int_0^T\int_\Omega (v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)\sqrt{\epsilon}w_h\;dx\;dt\\
&\quad+\int_0^T\int_\Omega (\rho-1)(v_h+\sqrt{\epsilon}w_h-u)\\
&\qquad\qquad\cdot\left(\frac{d}{dt}v_h+\frac{d}{dt}\sqrt{\epsilon}w_h+((v_h+\sqrt{\epsilon}w_h)\cdot\nabla)(v_h+\sqrt{\epsilon}w_h)-f\right)\;dx\;dt\\
&\quad+\int_0^T\int_\Omega (\rho-1)(v_h-u)\cdot\nabla p_h\;dx\;dt\\
&\quad-\int_0^T\int_\Omega \frac{\mu(\rho)}{2}\left|(\nabla v_h+\nabla\sqrt{\epsilon}w_h+\nabla v_h^T+\nabla\sqrt{\epsilon}w_h^T)-(\nabla u+\nabla u^T)\right|^2\;dx\;dt\\
&\quad-\int_0^T\int_\Omega \lambda(\rho)|\operatorname{div}v_h+\operatorname{div}\sqrt{\epsilon}w_h-\operatorname{div}u|^2\;dx\;dt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^T \int_\Omega (\mu(\rho) - \mu_0)(\nabla v_h + \nabla \sqrt{\epsilon} w_h) \\
&\qquad : \Big( (\nabla v_h + \nabla \sqrt{\epsilon} w_h + \nabla v_h^T + \nabla \sqrt{\epsilon} w_h^T) - (\nabla u + \nabla u^T) \Big) \, dx \, dt \\
&+ \int_0^T \int_\Omega (\lambda(\rho) - \lambda_0) \operatorname{div}(v_h + \sqrt{\epsilon} w_h - u) \operatorname{div}(v_h + \sqrt{\epsilon} w_h) \, dx \, dt \\
&+ \int_0^T \int_\Omega (\mu_0 + \lambda_0) \operatorname{div}(v_h + \sqrt{\epsilon} w_h - u) \operatorname{div} v_h \, dx \, dt \\
&- \int_0^T \int_\Omega p_\epsilon(\rho) \operatorname{div}(v_h + \sqrt{\epsilon} w_h) \, dx \, dt - \int_0^T \int_\Omega \nabla s_h \cdot (v_h + \sqrt{\epsilon} w_h - u) \, dx \, dt \\
&- \int_0^T \int_\Omega (\rho - 1) \left( \frac{d}{dt} p_h + (v_h \cdot \nabla) p_h \right) \, dx \, dt + \int_0^T \int_\Omega \Pi_\epsilon'(p_h) \frac{d}{dt} p_h \, dx \, dt \\
&+ \int_0^T \int_\Omega p_h (\operatorname{div} v_h + \operatorname{div} \sqrt{\epsilon} w_h) \, dx \, dt + A_{residual,1} + \sqrt{\epsilon} A_{residual,2}.
\end{aligned}
$$

Adding the formula from Lemma 10 and observing that several cancellations occur, we infer

$$
\begin{aligned}
&\int_\Omega \frac{1}{2} \rho |v_h + \sqrt{\epsilon} w_h - u|^2 + \pi_\epsilon(\rho) - (\rho - 1)(p_h + s_h) + \Pi_\epsilon(p_h + s_h) \, dx \Big|_0^T \\
&\leq - \int_0^T \int_\Omega \rho \left( (v_h + \sqrt{\epsilon} w_h - u) \otimes (v_h + \sqrt{\epsilon} w_h - u) \right) : \nabla (v_h + \sqrt{\epsilon} w_h) \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (v_h + \sqrt{\epsilon} w_h - u) \cdot (\sqrt{\epsilon} w_h \cdot \nabla) \sqrt{\epsilon} w_h \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (\rho - 1)(v_h + \sqrt{\epsilon} w_h - u) \\
&\qquad \cdot \left( \frac{d}{dt} v_h + \frac{d}{dt} \sqrt{\epsilon} w_h + ((v_h + \sqrt{\epsilon} w_h) \cdot \nabla)(v_h + \sqrt{\epsilon} w_h) - f \right) \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (\rho - 1)(v_h - u) \cdot \nabla p_h \, dx \, dt \\
&\quad - \int_0^T \int_\Omega \frac{\mu(\rho)}{2} \left| (\nabla v_h + \nabla \sqrt{\epsilon} w_h + \nabla v_h^T + \nabla \sqrt{\epsilon} w_h^T) - (\nabla u + \nabla u^T) \right|^2 \, dx \, dt \\
&\quad - \int_0^T \int_\Omega \lambda(\rho) |\operatorname{div} v_h + \operatorname{div} \sqrt{\epsilon} w_h - \operatorname{div} u|^2 \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (\mu(\rho) - \mu_0)(\nabla v_h + \nabla \sqrt{\epsilon} w_h) \\
&\qquad : \Big( (\nabla v_h + \nabla \sqrt{\epsilon} w_h + \nabla v_h^T + \nabla \sqrt{\epsilon} w_h^T) - (\nabla u + \nabla u^T) \Big) \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (\lambda(\rho) - \lambda_0) \operatorname{div}(v_h + \sqrt{\epsilon} w_h - u) \operatorname{div}(v_h + \sqrt{\epsilon} w_h) \, dx \, dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega (\mu_0+\lambda_0)\operatorname{div}(v_h+\sqrt{\epsilon}w_h-u)\operatorname{div} v_h \,dx\,dt\\
&+\int_0^T\int_\Omega \frac{\rho-1-\epsilon p_\epsilon(\rho)}{\epsilon}\operatorname{div}\sqrt{\epsilon}w_h \,dx\,dt-\int_0^T\int_\Omega p_\epsilon(\rho)\operatorname{div} v_h \,dx\,dt\\
&-\int_0^T\int_\Omega \Pi_\epsilon'(p_h+s_h)\frac{\operatorname{div} w_h}{\sqrt{\epsilon}} \,dx\,dt\\
&-\int_0^T\int_\Omega \nabla s_h\cdot(v_h+\sqrt{\epsilon}w_h) \,dx\,dt+\int_0^T\int_\Omega (\rho-1)(v_h-u)\cdot\nabla s_h \,dx\,dt\\
&-\int_0^T\int_\Omega \Pi_\epsilon'(p_h+s_h)(v_h\cdot\nabla)s_h \,dx\,dt\\
&-\int_0^T\int_\Omega \Pi_\epsilon'(p_h+s_h)(v_h\cdot\nabla)p_h \,dx\,dt\\
&+\int_0^T\int_\Omega p_h(\operatorname{div} v_h+\operatorname{div}\sqrt{\epsilon}w_h) \,dx\,dt\\
&+A_{residual,1}+\sqrt{\epsilon}A_{residual,2}+A_{residual,3}.
\end{aligned}
$$

Now add the second orange term to the second colorless term. We then add the first blue term to the last orange term and integrate by parts in the resulting term (writing down the final result without a color). Furthermore, we integrate by parts in the first orange term and split the result into two colorless contributions. Finally adding the third red term and the second blue term to the colorless terms, we deduce

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\rho|v_h+\sqrt{\epsilon}w_h-u|^2+\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h)\,dx\Big|_0^T\\
&\leq -\int_0^T\int_\Omega \rho\,\left((v_h+\sqrt{\epsilon}w_h-u)\otimes(v_h+\sqrt{\epsilon}w_h-u)\right):\nabla(v_h+\sqrt{\epsilon}w_h)\,dx\,dt\\
&\quad+\int_0^T\int_\Omega (v_h+\sqrt{\epsilon}w_h-u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)\sqrt{\epsilon}w_h\,dx\,dt\\
&\quad+\int_0^T\int_\Omega (\rho-1)(v_h+\sqrt{\epsilon}w_h-u)\\
&\qquad\qquad\cdot\left(\frac{d}{dt}v_h+\frac{d}{dt}\sqrt{\epsilon}w_h+((v_h+\sqrt{\epsilon}w_h)\cdot\nabla)(v_h+\sqrt{\epsilon}w_h)-f\right)\,dx\,dt\\
&\quad+\int_0^T\int_\Omega (\rho-1)(v_h-u)\cdot(\nabla p_h+\nabla s_h)\,dx\,dt\\
&\quad-\int_0^T\int_\Omega \frac{\mu(\rho)}{2}\left|(\nabla v_h+\nabla\sqrt{\epsilon}w_h+\nabla v_h^T+\nabla\sqrt{\epsilon}w_h^T)-(\nabla u+\nabla u^T)\right|^2\,dx\,dt\\
&\quad-\int_0^T\int_\Omega \lambda(\rho)|\operatorname{div} v_h+\operatorname{div}\sqrt{\epsilon}w_h-\operatorname{div} u|^2\,dx\,dt
\end{aligned}
$$

$$+\int_0^T \int_\Omega (\mu(\rho)-\mu_0)(\nabla v_h + \nabla\sqrt{\epsilon}w_h)$$
$$: \Big((\nabla v_h + \nabla\sqrt{\epsilon}w_h + \nabla v_h^T + \nabla\sqrt{\epsilon}w_h^T) - (\nabla u + \nabla u^T)\Big)\ dx\ dt$$
$$+\int_0^T \int_\Omega (\lambda(\rho)-\lambda_0)\operatorname{div}(v_h + \sqrt{\epsilon}w_h - u)\operatorname{div}(v_h + \sqrt{\epsilon}w_h)\ dx\ dt$$
$$+\int_0^T \int_\Omega (\mu_0+\lambda_0)\operatorname{div}(v_h + \sqrt{\epsilon}w_h - u)\operatorname{div} v_h\ dx\ dt$$
$$+\int_0^T \int_\Omega \frac{\rho - 1 - \epsilon p_\epsilon(\rho)}{\epsilon}\operatorname{div}\sqrt{\epsilon}w_h\ dx\ dt$$
$$-\int_0^T \int_\Omega \frac{\rho-1}{\epsilon}\operatorname{div} v_h\ dx\ dt - \int_0^T \int_\Omega \frac{\epsilon p_\epsilon(\rho) - (\rho-1)}{\epsilon}\operatorname{div} v_h\ dx\ dt$$
$$+\int_0^T \int_\Omega \Pi_\epsilon(p_h + s_h)\operatorname{div} v_h\ dx\ dt$$
$$+\int_0^T \int_\Omega \left(p_h + s_h - \frac{\Pi'_\epsilon(p_h+s_h)}{\epsilon}\right)\operatorname{div}\sqrt{\epsilon}w_h\ dx\ dt$$
$$+\int_0^T \int_\Omega (p_h + s_h)\operatorname{div} v_h\ dx\ dt$$
$$+ A_{residual,1} + \sqrt{\epsilon}A_{residual,2} + A_{residual,3}.$$

In the next step, we shift terms from the red terms to the colorless terms and the other way around. Observing that

$$\int_0^T \int_\Omega \Pi'_\epsilon(p_h+s_h)\sqrt{\epsilon}w_h \cdot \nabla(p_h+s_h)\ dx\ dt + \int_0^T \int_\Omega \Pi_\epsilon(p_h+s_h)\operatorname{div}\sqrt{\epsilon}w_h\ dx\ dt = 0$$

and using the fact that $\epsilon p_\epsilon(\rho) - (\rho-1) = (\gamma-1)\epsilon\pi_\epsilon(\rho)$ (recall (A4)), we infer

(18)

$$\int_\Omega \frac{1}{2}\rho|v_h + \sqrt{\epsilon}w_h - u|^2 + \pi_\epsilon(\rho) - (\rho-1)(p_h+s_h) + \Pi_\epsilon(p_h+s_h)\ dx\Big|_0^T$$
$$\le -\int_0^T \int_\Omega \rho\ ((v_h + \sqrt{\epsilon}w_h - u)\otimes(v_h + \sqrt{\epsilon}w_h - u)) : \nabla(v_h + \sqrt{\epsilon}w_h)\ dx\ dt$$
$$+\int_0^T \int_\Omega (v_h + \sqrt{\epsilon}w_h - u)\cdot(\sqrt{\epsilon}w_h\cdot\nabla)\sqrt{\epsilon}w_h\ dx\ dt$$
$$+\int_0^T \int_\Omega (\rho-1)(v_h + \sqrt{\epsilon}w_h - u)\cdot\bigg(\frac{d}{dt}v_h + \frac{d}{dt}\sqrt{\epsilon}w_h$$
$$+ ((v_h + \sqrt{\epsilon}w_h)\cdot\nabla)(v_h + \sqrt{\epsilon}w_h) - f + \nabla p_h + \nabla s_h\bigg)\ dx\ dt$$
$$-\int_0^T \int_\Omega (\rho - 1 - \Pi'_\epsilon(p_h+s_h))\sqrt{\epsilon}w_h\cdot(\nabla p_h + \nabla s_h)\ dx\ dt$$
$$-\int_0^T \int_\Omega \frac{\mu(\rho)}{2}\left|(\nabla v_h + \nabla\sqrt{\epsilon}w_h + \nabla v_h^T + \nabla\sqrt{\epsilon}w_h^T) - (\nabla u + \nabla u^T)\right|^2\ dx\ dt$$

$$
\begin{aligned}
&- \int_0^T \int_\Omega \lambda(\rho) |\operatorname{div} v_h + \operatorname{div} \sqrt{\epsilon} w_h - \operatorname{div} u|^2 \, dx \, dt \\
&+ \int_0^T \int_\Omega (\mu(\rho) - \mu_0)(\nabla v_h + \nabla \sqrt{\epsilon} w_h) \\
&\qquad\qquad : \Big( (\nabla v_h + \nabla \sqrt{\epsilon} w_h + \nabla v_h^T + \nabla \sqrt{\epsilon} w_h^T) - (\nabla u + \nabla u^T) \Big) \, dx \, dt \\
&+ \int_0^T \int_\Omega (\lambda(\rho) - \lambda_0) \operatorname{div}(v_h + \sqrt{\epsilon} w_h - u) \operatorname{div}(v_h + \sqrt{\epsilon} w_h) \, dx \, dt \\
&+ \int_0^T \int_\Omega (\mu_0 + \lambda_0) \operatorname{div}(v_h + \sqrt{\epsilon} w_h - u) \operatorname{div} v_h \, dx \, dt \\
&+ \int_0^T \int_\Omega (1-\gamma) \Big( \pi_\epsilon(\rho) - (\rho - 1)(p_h + s_h) + \Pi_\epsilon(p_h + s_h) \Big) \operatorname{div} \sqrt{\epsilon} w_h \, dx \, dt \\
&+ \int_0^T \int_\Omega (1-\gamma) \big( \rho - 1 - \Pi_\epsilon'(p_h + s_h) \big) (p_h + s_h) \operatorname{div} \sqrt{\epsilon} w_h \, dx \, dt \\
&- \int_0^T \int_\Omega \frac{\rho - 1 - \Pi_\epsilon'(p_h + s_h)}{\epsilon} \operatorname{div} v_h \, dx \, dt \\
&- \int_0^T \int_\Omega \frac{\epsilon p_\epsilon(\rho) - (\rho - 1) - \epsilon \, \Pi_\epsilon \, (p_h + s_h) + \Pi_\epsilon'(p_h + s_h) - \epsilon (p_h + s_h)}{\epsilon} \operatorname{div} v_h \, dx \, dt \\
&+ \int_0^T \int_\Omega \frac{(\epsilon p_h + \epsilon s_h) - \Pi_\epsilon'(p_h + s_h) - (\gamma - 1)\epsilon (p_h + s_h) \, \Pi_\epsilon' \, (p_h + s_h) + \gamma \epsilon \, \Pi_\epsilon \, (p_h + s_h)}{\epsilon} \operatorname{div} \sqrt{\epsilon} w_h \, dx \, dt \\
&+ A_{residual,1} + \sqrt{\epsilon} A_{residual,2} + A_{residual,3}.
\end{aligned}
$$

Let us provide an estimate for the yellow terms. Note that we have by Lemma 11

$$
\int_\Omega \pi_\epsilon(\rho) \, dx \leq \mathcal{D}_{\pi_\epsilon(\rho)},
$$

where

$$
\mathcal{D}_{\pi_\epsilon(\rho)} := 2D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u] + \int_\Omega \Pi_\epsilon(2p_h + 2s_h) - 2 \, \Pi_\epsilon \, (p_h + s_h) \, dx.
$$

Recalling Definition 2, we see that the yellow terms are bounded from above by

$$
\begin{aligned}
&\int_0^T \sup_x |\nabla (v_h + \sqrt{\epsilon} w_h)|_2 \int_\Omega \rho \left| v_h + \sqrt{\epsilon} w_h - u \right|^2 \, dx \, dt \\
&+ \int_0^T \int_\Omega \sqrt{\rho} |v_h + \sqrt{\epsilon} w_h - u| \cdot |(\sqrt{\epsilon} w_h \cdot \nabla) \sqrt{\epsilon} w_h| \, dx \, dt, \\
&+ \int_0^T \int_\Omega \chi_{\rho<1} |1 - \sqrt{\rho}| |v_h + \sqrt{\epsilon} w_h - u| \cdot |(\sqrt{\epsilon} w_h \cdot \nabla) \sqrt{\epsilon} w_h| \, dx \, dt \\
&\leq \int_0^T 2 \sup_x |\nabla (v_h + \sqrt{\epsilon} w_h)|_2 \, D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u] \, dt \\
&\quad + \int_0^T \sqrt{2D[v_h + \sqrt{\epsilon} w_h, p_h + s_h, \rho, u]} \cdot \big| \big| (\sqrt{\epsilon} w_h \cdot \nabla) \sqrt{\epsilon} w_h \big| \big|_{L^2(\Omega)} \, dt
\end{aligned}
$$

$$+\frac{\epsilon^3 C_{\pi,e} C_{\Omega,2,6}^2}{4\mu_{min}\tau_5}\int_0^T ||(w_h\cdot\nabla)w_h||_{L^3(\Omega)}^2 \mathcal{D}_{\pi_\epsilon(\rho)}\ dt$$
$$+\tau_5\mu_{min}\int_0^T\int_\Omega |\nabla(v_h+\sqrt{\epsilon}w_h-u)|^2\ dx\ dt,$$

where in the last step we have applied Hölder's inequality and Young's inequality to the third term and used the Sobolev embedding $H_0^1(\Omega)\to L^6(\Omega)$ (recall also the definition of $C_{\pi,e}$ in Definition 2). Note that the new four terms are present on the right-hand side of (12), with the exception of the last term which is present on the left-hand side.

The green terms in formula (18) may be estimated from above by

$$-\int_0^T\int_\Omega \frac{\mu_{min}}{2}(1-\tau_1)\Big|(\nabla v_h+\nabla\sqrt{\epsilon}w_h+\nabla v_h^T+\nabla\sqrt{\epsilon}w_h^T)-(\nabla u+\nabla u^T)\Big|^2\ dx\ dt$$
$$-\int_0^T\int_\Omega(\lambda_{min}(1-\tau_2-\tau_4)-\mu_{min}\tau_3)\,|\operatorname{div} v_h+\operatorname{div}\sqrt{\epsilon}w_h-\operatorname{div} u|^2\ dx\ dt$$
$$+\int_0^T\int_\Omega\frac{|\mu(\rho)-\mu_0|^2}{2\mu_{min}\tau_1}|\nabla v_h+\nabla\sqrt{\epsilon}w_h|^2\ dx\ dt$$
$$+\int_0^T\int_\Omega\frac{|\lambda(\rho)-\lambda_0|^2}{4\lambda_{min}\tau_2}|\operatorname{div}(v_h+\sqrt{\epsilon}w_h)|^2\ dx\ dt$$
$$+\int_0^T\int_\Omega\left(\frac{\mu_0^2}{4\mu_{min}\tau_3}+\frac{\lambda_0^2}{4\lambda_{min}\tau_4}\right)|\operatorname{div} v_h|^2\ dx\ dt.$$

Note that (since we have $\int_\Omega \nabla q:(\nabla q)^T\ dx=\int_\Omega|\operatorname{div} q|^2\ dx$ for any $q\in H_0^1(\Omega;\mathbb{R}^d)$) the first and the second term in this estimate are equal to the terms

$$-\int_0^T\int_\Omega\mu_{min}(1-\tau_1)|\nabla v_h+\nabla\sqrt{\epsilon}w_h-\nabla u|^2\ dx\ dt$$
$$-\int_0^T\int_\Omega(\lambda_{min}(1-\tau_2-\tau_4)+\mu_{min}(1-\tau_3))\,|\operatorname{div} v_h+\operatorname{div}\sqrt{\epsilon}w_h-\operatorname{div} u|^2\ dx\ dt$$

which are present on the left-hand side of (12), while the other terms in the previous formula are bounded (due to Lemma 11) by

$$\int_0^T\int_\Omega\frac{C_{\pi,\mu}\epsilon}{\tau_1}\Big[(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))$$
$$+\frac{1}{2}(\Pi_\epsilon(2p_h+2s_h)-2\,\Pi_\epsilon\,(p_h+s_h))\Big]|\nabla v_h+\nabla\sqrt{\epsilon}w_h|^2\ dx\ dt$$
$$+\int_0^T\int_\Omega\frac{C_{\pi,\lambda}\epsilon}{\tau_2}\Big[(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))$$
$$+\frac{1}{2}(\Pi_\epsilon(2p_h+2s_h)-2\,\Pi_\epsilon\,(p_h+s_h))\Big]|\operatorname{div}(v_h+\sqrt{\epsilon}w_h)|^2\ dx\ dt$$
$$+\int_0^T\int_\Omega\left(\frac{\mu_0^2}{4\mu_{min}\tau_3}+\frac{\lambda_0^2}{4\lambda_{min}\tau_4}\right)|\operatorname{div} v_h|^2\ dx\ dt.$$

The last term is present in $\mathcal{E}^2_{h.o.t.}$, while the first two terms are bounded by corresponding terms on the right-hand side of (12) (note that each of the two terms has been split into two terms, one in $\mathcal{E}^2_{model}$ and one in $\mathcal{E}^2_{h.o.t.}$).

The second formula of Lemma 11 implies

$$\begin{aligned}&|\rho-1-\Pi'_\epsilon(p_h+s_h)|^2\\ &\leq 2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))\\ &\quad+C_{\pi,m,1}\max(|\rho-1|,|\,\Pi'_\epsilon(p_h+s_h)|)|\rho-1-\Pi'_\epsilon(p_h+s_h)|^2\end{aligned}$$

which yields

$$\begin{aligned}&|\rho-1-\Pi'_\epsilon(p_h+s_h)|\\ &\leq\sqrt{2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))}\\ &\quad+\sqrt{\min(1,C_{\pi,m,1}\max(|\rho-1|,|\,\Pi'_\epsilon(p_h+s_h)|))}|\rho-1-\Pi'_\epsilon(p_h+s_h)|\end{aligned}$$

and therefore by Young's inequality and absorption

$$\begin{aligned}&\frac{3}{4}|\rho-1-\Pi'_\epsilon(p_h+s_h)|\\ &\leq\sqrt{2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))}\\ &\quad+\min(1,C_{\pi,m,1}\max(|\rho-1|,\Pi'_\epsilon(p_h+s_h)))|\rho-1-\Pi'_\epsilon(p_h+s_h)|.\end{aligned}$$

This gives

$$\begin{aligned}&|\rho-1-\Pi'_\epsilon(p_h+s_h)|\\ &\leq\frac{4}{3}\sqrt{2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))}\\ &\quad+\frac{8}{3}\max\left\{\min\{C_{\pi,m,1}|\rho-1|,1\}|\rho-1|,C_{\pi,m,1}|\,\Pi'_\epsilon(p_h+s_h)|^2\right\}\\ &\leq\frac{4}{3}\sqrt{2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))}\\ &\quad+\frac{4}{3}\max\left\{C_{\pi,m,2}\epsilon\pi_\epsilon(\rho),2C_{\pi,m,1}|\,\Pi'_\epsilon(p_h+s_h)|^2\right\}.\end{aligned}\tag{19}$$

Estimating the first two colorless terms in formula (18) using Hölder's inequality, the Sobolev embedding $H^1_0(\Omega)\to L^6(\Omega)$, and Young's inequality for the first term and by applying the previous estimate to the second term, we get

$$\begin{aligned}&\int_0^T\frac{C^2_{\Omega,2,6}}{4\mu_{min}\tau_6}\left|\left|(\rho-1)\left(\left(\frac{d}{dt}+(v_h+\sqrt{\epsilon}w_h)\cdot\nabla\right)(v_h+\sqrt{\epsilon}w_h)-f+\nabla(p_h+s_h)\right)\right|\right|^2_{L^{6/5}(\Omega)}dt\\ &+\mu_{min}\tau_6\int_0^T\int_\Omega|\nabla(v_h+\sqrt{\epsilon}w_h-u)|^2\,dx\,dt\\ &+\int_0^T\int_\Omega\frac{4}{3}\sqrt{2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))}|\sqrt{\epsilon}w_h\cdot(\nabla p_h+\nabla s_h)|\,dx\,dt\\ &+\int_0^T\int_\Omega\frac{4}{3}\max\left\{C_{\pi,m,2}\epsilon\pi_\epsilon(\rho),2C_{\pi,m,1}|\,\Pi'_\epsilon(p_h+s_h)|^2\right\}|\sqrt{\epsilon}w_h\cdot(\nabla p_h+\nabla s_h)|\,dx\,dt.\end{aligned}$$

Note that the second term is present on the left-hand side of (12), while the other terms are bounded from above by (here we use the fact that $|\rho-1|\leq\sqrt{2\epsilon\pi_\epsilon(\rho)}+$

$\left(|\rho-1|-\sqrt{2\epsilon\pi_\epsilon(\rho)}\right)_+ \leq \sqrt{2\epsilon\pi_\epsilon(\rho)}+\sqrt{C_{\pi,f}}(\epsilon\pi_\epsilon(\rho))^{5/6}$ and the first estimate in Lemma 11)

$$\begin{aligned}
&\int_0^T \epsilon\frac{C^2_{\Omega,2,6}}{\mu_{min}\tau_6}\left|\left|\left(\frac{d}{dt}+(v_h+\sqrt{\epsilon}w_h)\cdot\nabla\right)(v_h+\sqrt{\epsilon}w_h)-f+\nabla(p_h+s_h)\right|\right|^2_{L^3(\Omega)}\mathcal{D}_{\pi_\epsilon(\rho)}\;dt\\
&+\int_0^T \frac{C^2_{\Omega,2,6}}{2\mu_{min}\tau_6}C_{\pi,f}\epsilon^{5/3}\mathcal{D}^{5/3}_{\pi_\epsilon(\rho)}\\
&\qquad\times\sup_x\left|\left(\frac{d}{dt}+(v_h+\sqrt{\epsilon}w_h)\cdot\nabla\right)(v_h+\sqrt{\epsilon}w_h)-f+\nabla(p_h+s_h)\right|^2\;dt\\
&+\int_0^T \frac{4}{3}\sqrt{2\epsilon D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}||\sqrt{\epsilon}w_h\cdot(\nabla p_h+\nabla s_h)||_{L^2(\Omega)}\;dt\\
&+\int_0^T \frac{4C_{\pi,m,2}}{3}\epsilon\mathcal{D}_{\pi_\epsilon(\rho)}\sup_{x\in\Omega}|\sqrt{\epsilon}w_h\cdot(\nabla p_h+\nabla s_h)|\;dt\\
&+\int_0^T\int_\Omega \frac{8C_{\pi,m,1}}{3}|\,\Pi'_\epsilon\,(p_h+s_h)|^2|\sqrt{\epsilon}w_h\cdot(\nabla p_h+\nabla s_h)|\;dx\;dt.
\end{aligned}$$

The last two terms and the second term are present in $\mathcal{E}^2_{h.o.t.}$, while the first and the third term are included in $\mathcal{E}^2_{model}$.

A bound for the red terms in (18) is given by

$$\begin{aligned}
&\int_0^T(\gamma-1)\sqrt{\epsilon}\sup_x(\operatorname{div}w_h)_-D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]\;dt\\
&+\int_0^T\int_\Omega(\gamma-1)\cdot\frac{4}{3}\sqrt{2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))}|(p_h+s_h)\operatorname{div}\sqrt{\epsilon}w_h|\;dx\;dt\\
&+\int_0^T\int_\Omega(\gamma-1)\cdot\frac{4}{3}\max\left\{C_{\pi,m,2}\epsilon\pi_\epsilon(\rho),2C_{\pi,m,1}|\,\Pi'_\epsilon\,(p_h+s_h)|^2\right\}\\
&\qquad\qquad\times|(p_h+s_h)\operatorname{div}\sqrt{\epsilon}w_h|\;dx\;dt.
\end{aligned}$$

Note that the first term is contained in $\mathcal{E}^2_{h.o.t.}$, while the last two terms are bounded by

$$\begin{aligned}
&\int_0^T(\gamma-1)\cdot\frac{4}{3}\epsilon||(p_h+s_h)\operatorname{div}w_h||_{L^2(\Omega)}\sqrt{2D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}\;dt\\
&+\int_0^T(\gamma-1)\cdot\frac{4}{3}\sup_x|(p_h+s_h)\operatorname{div}\sqrt{\epsilon}w_h|\cdot C_{\pi,m,2}\epsilon\mathcal{D}_{\pi_\epsilon(\rho)}\;dt\\
&+\int_0^T\int_\Omega(\gamma-1)\frac{8C_{\pi,m,1}}{3}|\,\Pi'_\epsilon\,(p_h+s_h)|^2|(p_h+s_h)\operatorname{div}\sqrt{\epsilon}w_h|\;dx\;dt,
\end{aligned}$$

the latter two terms being included in $\mathcal{E}^2_{h.o.t.}$ and the first one being part of $\mathcal{E}^2_{model}$.

Finally, among the last three colorless terms in (18) the last one is present in $\mathcal{E}^2_{h.o.t.}$ and the other two are bounded by (recall that $\epsilon p_\epsilon(\rho)-(\rho-1)=(\gamma-1)\epsilon\pi_\epsilon(\rho)$)

$$\begin{aligned}
&\int_0^T\int_\Omega\frac{4}{3}\epsilon^{-1}\sqrt{2\epsilon(\pi_\epsilon(\rho)-(\rho-1)(p_h+s_h)+\Pi_\epsilon(p_h+s_h))}|\operatorname{div}v_h|\;dx\;dt\\
&+\int_0^T\int_\Omega\frac{4}{3}\max\left\{C_{\pi,m,2}\pi_\epsilon(\rho),2C_{\pi,m,1}\epsilon^{-1}|\,\Pi'_\epsilon\,(p_h+s_h)|^2\right\}|\operatorname{div}v_h|\;dx\;dt
\end{aligned}$$

$$-\int_0^T \int_\Omega (\gamma-1)\pi_\epsilon(\rho) \operatorname{div} v_h \; dx \; dt$$
$$+\int_0^T \int_\Omega \frac{\epsilon\,\Pi_\epsilon\,(p_h+s_h) - \Pi_\epsilon'(p_h+s_h) + \epsilon(p_h+s_h)}{\epsilon} \operatorname{div} v_h \; dx \; dt.$$

The last term is present in $\mathcal{E}^2_{h.o.t.}$, while the other terms are estimated from above by

$$\int_0^T \frac{4}{3}\sqrt{2D[v_h+\sqrt{\epsilon}w_h, p_h+s_h,\rho,u]}\cdot \frac{||\operatorname{div} v_h||_{L^2(\Omega)}}{\sqrt{\epsilon}} \; dt$$
$$+\int_0^T \frac{4}{3} C_{\pi,m,2}\mathcal{D}_{\pi_\epsilon(\rho)} \sup_x |\operatorname{div} v_h| \; dt$$
$$+\int_0^T \int_\Omega \frac{8C_{\pi,m,1}}{3}\epsilon^{-1} |\,\Pi_\epsilon'\,(p_h+s_h)|^2 |\operatorname{div} v_h| \; dx \; dt$$
$$+\int_0^T (\gamma-1)\mathcal{D}_{\pi_\epsilon(\rho)} \sup_x (\operatorname{div} v_h)_- \; dt.$$

The first term is included in $\mathcal{E}^2_{div}$, while the other terms are present in $\mathcal{E}^2_{h.o.t.}$.

It remains to deal with the numerical errors $A_{residual,1}$, $A_{residual,2}$ and $A_{residual,3}$. Making use of estimate (19), we see that an upper bound for the term $A_{residual,3}$ is given by

$$\int_0^T \int_\Omega \left| \frac{d}{dt}s_h + (v_h\cdot\nabla)s_h + \frac{\operatorname{div} w_h}{\sqrt{\epsilon}} + \frac{d}{dt}p_h + (v_h\cdot\nabla)p_h \right|$$
$$\times \frac{4}{3}\sqrt{2\epsilon(\pi_\epsilon(\rho) - (\rho-1)(p_h+s_h) + \Pi_\epsilon(p_h+s_h))} \; dx \; dt$$
$$+\int_0^T \int_\Omega \left| \frac{d}{dt}s_h + (v_h\cdot\nabla)s_h + \frac{\operatorname{div} w_h}{\sqrt{\epsilon}} + \frac{d}{dt}p_h + (v_h\cdot\nabla)p_h \right|$$
$$\times \frac{4}{3}\max\left\{C_{\pi,m,2}\epsilon\pi_\epsilon(\rho), 2C_{\pi,m,1}|\,\Pi_\epsilon'\,(p_h+s_h)|^2\right\} \; dx \; dt$$

which in turn is bounded by

$$\sqrt{\epsilon}\int_0^T \frac{4}{3}\left|\left| \frac{d}{dt}s_h + (v_h\cdot\nabla)s_h + \frac{\operatorname{div} w_h}{\sqrt{\epsilon}} + \frac{d}{dt}p_h + (v_h\cdot\nabla)p_h \right|\right|_{L^2(\Omega)}$$
$$\times \sqrt{2D[v_h+\sqrt{\epsilon}w_h, p_h+s_h,\rho,u]} \; dt$$
$$+\frac{8C_{\pi,m,1}}{3}\int_0^T \int_\Omega \left| \frac{d}{dt}s_h + (v_h\cdot\nabla)s_h + \frac{\operatorname{div} w_h}{\sqrt{\epsilon}} + \frac{d}{dt}p_h + (v_h\cdot\nabla)p_h \right|$$
$$\times |\,\Pi_\epsilon'\,(p_h+s_h)|^2 \; dx \; dt$$
$$+\int_0^T \frac{4}{3}C_{\pi,m,2}\epsilon\mathcal{D}_{\pi_\epsilon(\rho)} \sup_x \left| \frac{d}{dt}s_h + (v_h\cdot\nabla)s_h + \frac{\operatorname{div} w_h}{\sqrt{\epsilon}} + \frac{d}{dt}p_h + (v_h\cdot\nabla)p_h \right| dt.$$

Note that the latter two terms are present in $\mathcal{E}^2_{h.o.t.}$, while the first term is included in $\mathcal{E}^2_{num2}$.

An upper bound for the term $A_{residual,1}$ is given by

$$\int_0^T \left|\left|(1-\sqrt{\rho})_+(v_h+\sqrt{\epsilon}w_h-u)\cdot\left(\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div}y\right)\right|\right|_{L^1(\Omega)}\,dt$$
$$+\int_0^T \left|\left|\sqrt{\rho}(v_h+\sqrt{\epsilon}w_h-u)\cdot\left(\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div}y\right)\right|\right|_{L^1(\Omega)}\,dt$$
$$+\tau_7\mu_{min}\int_0^T ||\nabla(v_h+\sqrt{\epsilon}w_h-u)||^2_{L^2(\Omega)}\,dt$$
$$+\frac{\mu_0^2}{4\mu_{min}\tau_7}\int_0^T ||\nabla v_h-y||^2_{L^2(\Omega)}\,dt.$$

The penultimate term is present on the left-hand side of (12); the last term is included in $\mathcal{E}^2_{num1}$. The first two terms are bounded by

$$\int_0^T ||(1-\sqrt{\rho})_+||_{L^2(\Omega)}||v_h+\sqrt{\epsilon}w_h-u||_{L^6(\Omega)}$$
$$\times\left|\left|\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div}y\right|\right|_{L^3(\Omega)}\,dt$$
$$+\int_0^T \sqrt{2D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}$$
$$\times\left|\left|\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div}y\right|\right|_{L^2(\Omega)}\,dt.$$

The latter term is included in $\mathcal{E}^2_{num1}$, while the former term is bounded by

$$\int_0^T \left(\epsilon C_{\pi,e}\int_\Omega \pi_\epsilon(\rho)\,dx\right)^{1/2}\cdot C_{\Omega,2,6}||\nabla(v_h+\sqrt{\epsilon}w_h-u)||_{L^2(\Omega)}$$
$$\times\left|\left|\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div}y\right|\right|_{L^3(\Omega)}\,dt$$

which in turn is bounded by

$$\tau_8\mu_{min}\int_0^T ||\nabla(v_h+\sqrt{\epsilon}w_h-u)||^2_{L^2(\Omega)}\,dt$$
$$+\int_0^T \epsilon\frac{C^2_{\Omega,2,6}C_{\pi,e}}{4\mu_{min}\tau_8}\mathcal{D}_{\pi_\epsilon(\rho)}\left|\left|\frac{d}{dt}v_h+(v_h\cdot\nabla)v_h+\nabla p_h-f-\mu_0\operatorname{div}y\right|\right|^2_{L^3(\Omega)}\,dt.$$

The first term is present on the left-hand side of (12), while the second term is included in $\mathcal{E}^2_{h.o.t.}$.

Estimating the first term in $A_{residual,2}$ completely analogously to the first term in $A_{residual,1}$, we see that $\sqrt{\epsilon}A_{residual,2}$ is bounded by

$$\sqrt{\epsilon}\int_0^T \sqrt{2D[v_h+\sqrt{\epsilon}w_h,p_h+s_h,\rho,u]}$$
$$\times\left|\left|\frac{d}{dt}w_h+(v_h\cdot\nabla)w_h+(w_h\cdot\nabla)v_h\right.\right.$$
$$\left.\left.-\mu_0\operatorname{div}z-(\mu_0+\lambda_0)\nabla\operatorname{tr}z+\frac{\nabla s_h}{\sqrt{\epsilon}}\right|\right|_{L^2(\Omega)}\,dt$$

$$
\begin{aligned}
&+ \tau_{10}\mu_{min} \int_0^T ||\nabla(v_h + \sqrt{\epsilon} w_h - u)||_{L^2(\Omega)}^2 \, dt \\
&+ \int_0^T \epsilon^2 \frac{C_{\Omega,2,6}^2 C_{\pi,e}}{4\mu_{min}\tau_{10}} \mathcal{D}_{\pi_\epsilon(\rho)} \bigg|\bigg| \frac{d}{dt} w_h + (v_h \cdot \nabla) w_h + (w_h \cdot \nabla) v_h \\
&\qquad\qquad - \mu_0 \operatorname{div} z - (\mu_0 + \lambda_0) \nabla \operatorname{tr} z + \frac{\nabla s_h}{\sqrt{\epsilon}} \bigg|\bigg|_{L^3(\Omega)}^2 \, dt \\
&+ \tau_9 \mu_{min} \int_0^T ||\nabla(v_h + \sqrt{\epsilon} w_h - u)||_{L^2(\Omega)}^2 \, dt \\
&+ \frac{\epsilon \mu_0^2}{4\mu_{min}\tau_9} \int_0^T ||\nabla w_h - z||_{L^2(\Omega)}^2 \, dt \\
&+ \lambda_{min}\tau_{11} \int_0^T ||\operatorname{div}(v_h + \sqrt{\epsilon} w_h - u)||_{L^2(\Omega)}^2 \, dt \\
&+ \frac{\epsilon(\lambda_0 + \mu_0)^2}{4\lambda_{min}\tau_{11}} \int_0^T ||\operatorname{div} w_h - \operatorname{tr} z||_{L^2(\Omega)}^2 \, dt.
\end{aligned}
$$

Note that the second, the fourth, and the penultimate term are included on the left-hand side of (12). The first, the fifth, and the last term are included in $\mathcal{E}_{num2}^2$. Finally, the third term is included in $\mathcal{E}_{h.o.t.}^2$. □

## 5. Concluding Remarks

In total, we have obtained rigorous a posteriori estimates for jointly the numerical error and the modeling error for the higher-order approximation of the compressible Navier-Stokes equation by the incompressible Navier-Stokes equation and the equations of linearized acoustics. For well-behaved flows and solenoidal approximations of the solution to the incompressible Navier-Stokes equation, our estimates are of optimal order.

The implementation of our error estimates, possibly including model adaptivity, will be the subject of future work. Another topic of interest could be to consider the problem on the full space $\Omega := \mathbb{R}^d$: For the approximation of the compressible Navier-Stokes equation by the incompressible Navier-Stokes equation, we have shown in the present paper that the leading-order part of the modeling error satisfies the equations of linearized acoustics, which are a wave equation with wave speed $1/\sqrt{\epsilon}$. We therefore expect this leading-order part of the error to be carried away quickly from the region of interest in case $\Omega = \mathbb{R}^d$. This might lead to an improved explicit error estimate in this case (compared to the estimates in [19]) even without solving the equations of linearized acoustics explicitly.

## References

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