Contribution to Oberwolfach Report

Optimal estimates on interface propagation in thin-film flow

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Lower bounds on the propagation of the free boundary are well-established in the theory of second-order degenerate parabolic equations like the porous medium equation

$$u_t = \Delta u^m \ .$$

For example, it is known that for large times the support of a nonnegative solution to the porous medium equation on $\mathbb{R}^d$ almost coincides with the support of the corresponding self-similar solution. Such results are typically derived using the comparison principle or Harnack inequalities.

However, for higher-order degenerate parabolic equations like the thin-film equation

$$u_t = -\nabla \cdot (u^n \nabla \Delta u)$$

no estimates from below on front propagation have been available at all: Given some point on the initial free boundary, it has not even been known whether the free boundary ever moves near this point.

In the recent papers [3, 4], we devise a technique for the derivation of lower bounds on front propagation for higher-order parabolic equations, allowing for the first time for the derivation of lower bounds on contact line propagation for the thin-film equation. The key ingredient of our approach are new monotonicity formulas for the thin-film equation of the form

$$\frac{d}{dt} \int u^{1+\alpha} |x - x_0|^\gamma \ dx \geq c \int u^{1+\alpha+n} |x - x_0|^{\gamma - 4} \ dx$$

(for certain $\alpha \in (-1, 0)$ and $\gamma < 0$); these formulas are valid as as long as the support of u does not touch the singularity of the weight at x_0. Combining these formulas with a differential inequality argument due to Chipot and Sideris [2], we obtain estimates from below on support propagation. More precisely, we derive upper bounds on waiting times, sufficient criteria for immediate forward motion of the free boundary, as well as lower bounds on asymptotic support propagation rates.

In the case of one spatial dimension and $n \in (2, 3)$, our upper bounds on waiting times read as follows: given initial data u_0 with $\operatorname{supp} u_0 \subset [0, \infty)$ and $u_0(x) \geq S x_+^{\frac{4}{n}}$ in some neighbourhood of 0, the left free boundary will start moving forward at time $c(n) S^{-n}$ the latest. If the initial data even satisfy $\lim_{x \downarrow 0} x^{-\frac{4}{n}} u_0(x) = \infty$, the left interface starts moving forward instantaneously. These upper bounds on waiting times coincide up to a constant factor with the known lower bounds on waiting times [7] and are therefore optimal. The sufficient condition for immediate forward motion of the interface is also sharp.

In the borderline case $n = 2$, we obtain upper bounds on waiting times and sufficient conditions for immediate forward motion of the interface which are optimal up to a logarithmic correction term.

In the multidimensional case (at least for $d \leq 3$), similar assertions can be derived for $n \in [2,3)$ if the initial free boundary locally is regular enough. The proof proceeds by using a singular weight which is adapted to the shape of the initial support and a cutoff argument, resulting in an almost monotonicity formula.

The expected waiting-time behaviour for $n < 2$ is more complex; see [1] for conjectures obtained by formal asymptotic analysis. Our approach yields some limited rigorous results for $n \in (1,2)$ [3, 5].

Regarding lower bounds on asymptotic support propagation rates, we prove the following assertion: Given $n \in (1.5,3)$ and $x_s \in \operatorname{supp} u_0$, a solution to the thin-film equation on $\mathbb{R}^d$ satisfies for all $t \geq 0$

$$B_{R(t)}(x_s) \subset \operatorname{supp} u(.,t) \ ,$$

where

$$R(t) := c(d,n)||u_0||_{L^1}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}} - \operatorname{diam}(\operatorname{supp} u_0) \ .$$

This in particular implies that for large t, the support of a solution to the thin-film equation contains a ball whose diameter is of the same order as the diameter of the corresponding self-similar solution.

Finally, we would like to point out that our method is not limited to the thin-film equation, but is flexible enough to be applied to other higher-order nonnegativity-preserving parabolic equations: For example, in the case of the Derrida-Lebowitz-Speer-Spohn equation

$$u_t = -\nabla \cdot \left(u \nabla \frac{\Delta \sqrt{u}}{\sqrt{u}} \right)$$

an adaption of our ansatz can be used to prove infinite speed of propagation [6].

References

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