GLOBAL EXISTENCE OF RENORMALIZED SOLUTIONS TO ENTROPY-DISSIPATING REACTION-DIFFUSION SYSTEMS

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Abstract. In the present work we introduce a notion of renormalized solution for reaction-diffusion systems with entropy-dissipating reactions. We establish global existence of renormalized solutions. In case of integrable reaction terms our notion of renormalized solution reduces to the usual notion of weak solution. Our existence result in particular covers all reaction-diffusion systems involving a single reversible reaction with mass-action kinetics and (possibly species-dependent) Fick-law diffusion; more generally, it covers the case of systems of reversible reactions with mass-action kinetics which satisfy the detailed balance condition. For such equations existence of any kind of solution in general was an open problem, thereby motivating the study of renormalized solutions.

1. Introduction

In this paper, we are concerned with reaction-diffusion-advection equations of the form

$$\frac{d}{dt}u_i = \nabla\cdot(A_i\nabla u_i) - \nabla\cdot(u_i\vec{b}_i) + R_i(u) \quad \forall i\in\{1,\ldots,S\}\ , \tag{1}$$

where S denotes the number of chemical species, u_i denotes the concentration of species i, A_i denotes the (species-dependent) diffusion tensor, $\vec{b}_i$ denotes the (species-dependent) drift, and R_i denotes the net production/consumption of species i due to reactions.

Consider a general reversible chemical reaction

$$\alpha_1\mathcal{A}_1 + \ldots + \alpha_S\mathcal{A}_S \rightleftharpoons \beta_1\mathcal{A}_1 + \ldots + \beta_S\mathcal{A}_S\ , \tag{2}$$

where $\mathcal{A}_i$ denote the different chemical species and where α_i, β_i are nonnegative integers. In many physically realistic situations the reaction rates then are either given by or at least approximated well by mass action kinetics, i.e. we have

$$R_i(u) = (\beta_i-\alpha_i)\left(c_1\prod_{k=1}^{S}u_k^{\alpha_k} - c_2\prod_{k=1}^{S}u_k^{\beta_k}\right)\ , \tag{3}$$

where $c_1, c_2 > 0$ are reaction constants. The simplest corresponding reaction-diffusion equation then reads

$$\frac{d}{dt}u_i = a_i\Delta u_i + (\beta_i-\alpha_i)\left(c_1\prod_{k=1}^{S}u_k^{\alpha_k} - c_2\prod_{k=1}^{S}u_k^{\beta_k}\right) \quad \forall i\in\{1,\ldots,S\} \tag{4}$$

with $a_i > 0$ denoting the species-dependent diffusion constants. To the best of our knowledge, global existence in time of any kind of solution to this equation in general has been an open problem, even for smooth initial data and e.g. periodic or homogeneous Neumann boundary conditions. The main difficulty is the lack of control of the reaction terms $R_i(u)$, which may grow very fast as $|u|\to\infty$ due to the possibly large powers of u appearing in (3); the known energy estimates are not even sufficient to guarantee boundedness of the reaction terms $R_i(u)$ in

$L^1(\Omega \times [0,T])$. Thus $R_i(u)$ is a priori not even known to define a distribution. It is therefore not obvious how to define a notion of solution for which global solutions exist.

An important consequence of the structure of the reaction terms in (3) is the existence of an S-tuple of constants $\mu_i \in \mathbb{R}$, $i \in \{1,\ldots,S\}$, such that the so-called *entropy inequality*

$$\sum_{i=1}^{S} R_i(u)(\mu_i + \log u_i) \leq 0 \quad \forall u \in \left(\mathbb{R}_0^+\right)^S \tag{5}$$

is satisfied. Note that this inequality is equivalent to requiring that the entropy estimate

$$\frac{d}{dt}\sum_{i=1}^{S}(\mu_i - 1 + \log u_i)u_i \leq 0$$

be satisfied for any solution u of the system of ODEs

$$\frac{d}{dt}u_i = R_i(u) \ \ \forall i \ .$$

For our reaction-diffusion equation (4), in case of periodic or homogeneous Neumann boundary conditions the entropy inequality (5) formally implies that

$$\frac{d}{dt}\int \sum_{i=1}^{S}(\mu_i - 1 + \log u_i)u_i \ dx \leq -c\int \sum_{i=1}^{S} |\nabla \sqrt{u_i}| \ dx \ . \tag{6}$$

In particular, no blowup of the $L \log L$ Orlicz norm of solutions may occur. However, the entropy estimate (6) is the only known energy estimate for our equation (4), apart from an $L^2(\Omega \times [0,T])$ bound discovered by Pierre and Schmitt [36]. It has therefore not been known whether there exists a notion of weak solution for which global solutions can be constructed, at least if the powers of u occuring in (3) exceed 2.

The general situation – having energy estimates which prevent global blowup of solutions, which however are insufficient for defining a notion of weak solution – therefore resembles the situation of the Boltzmann equation; this suggest that the concept of *renormalized solutions* as introduced by DiPerna and Lions [14, 15, 16] might apply to our case. Let us briefly explain this concept. For a single equation, given a smooth one-to-one mapping $\xi : \mathbb{R} \to \mathbb{R}$, a renormalized solution to a PDE is a function u with the property that $\xi(u)$ solves the PDE deduced by a formal application of the chain rule. For example, a renormalized solution u to the transport equation

$$\partial_t u + \nabla \cdot (u\vec{b}) = 0$$

would have to satisfy

$$\partial_t \xi(u) + \nabla \cdot (\xi(u)\vec{b}) + (\xi'(u)u - \xi(u))\nabla \cdot \vec{b} = 0$$

in a suitable weak sense. More generally, one considers a suitable family of (now no longer necessarily one-to-one) mappings ξ_k and requires that

$$\partial_t \xi_k(u) + \nabla \cdot (\xi_k(u)\vec{b}) + (\xi_k'(u)u - \xi_k(u))\nabla \cdot \vec{b} = 0$$

be satisfied for all k. Here, one e.g. may choose the ξ_k as some kind of truncation of the identity map; note that in such a case, in the latter equation all terms are well-defined distributions if u, $\vec{b}$, $\nabla \cdot \vec{b}$ just belong to L^1.

We would like to emphasize that renormalized solutions have found widespread use in the theory of PDEs, in particular also for elliptic and parabolic equations; see e.g. [1, 2, 7, 31, 39].

For certain cases of (4) with reaction rates with at most quartic growth, renormalized solutions with defect measure have been constructed by Desvillettes, Fellner, Pierre, and Vovelle [12]. However, for reactions of higher order, global existence of any kind of solution has remained an open problem. In the present paper, we introduce a suitable definition of renormalized solutions which enables us to prove global existence of solutions. In case of integrable reaction rates, we demonstrate that our notion of solution reduces to the usual notion of weak solution.

The main novelty of our notion of renormalized solutions is that our mappings ξ for which an evolution equation is imposed on $\xi(u_1, \dots, u_S)$ now depend on *all* concentrations $u_1, \dots, u_S$ at the same time, not just on a single one. This idea, albeit quite natural, apparently has not been exploited previously: in the literature only renormalized solutions for parabolic equations seem to have been considered which impose evolution equations for $\xi(u_1)$, $\xi(u_2)$, $\dots$, $\xi(u_S)$; yet such a notion of solution does not eliminate the need for growth restrictions of the reaction terms.

From now on, we shall consider a more general situation: we now deal with the full equation (1), imposing only modest restrictions on the coefficients and the domain. Our main assumption on the reaction terms R_i (apart from local Lipschitz continuity) will be the existence of constants $\mu_i \in \mathbb{R}$, $1 \leq i \leq S$, such that the *entropy inequality* (5) is satisfied. Besides the case of a single reversible reaction with mass action kinetics (as described by (2), (3)), many systems of reversible reactions with mass-action kinetics satisfy this entropy inequality and thus are covered by our result: In particular, the entropy inequality holds for all systems of N_R reactions of the form

$$\alpha_1^k \mathcal{A}_1 + \ldots + \alpha_S^k \mathcal{A}_S \rightleftharpoons \beta_1^k \mathcal{A}_1 + \ldots + \beta_S^k \mathcal{A}_S, \qquad 1 \leq k \leq N_R, \tag{7}$$

if $N_R \leq S$ holds and if the matrix $(\beta_i^k - \alpha_i^k)_{ik}$ has full rank (as shown e.g. in [23, 37]; see also [17] for further mathematical results on such systems of reactions). More generally, the entropy inequality holds for systems of reversible reactions with mass-action kinetics which satisfy the so-called condition of detailed balance.

Before sketching our technique and stating our main results, we would like to provide an overview of the existing literature for equation (1).

For diffusion coefficients A_i and advection velocities b_i which are *independent* of the species i, existence of globally bounded solutions in case of reactions of the form (7) and mass-action kinetics has been shown by Mincheva and Siegel [30]; for a more general result in this direction see the paper by Kräutle [26]. If all A_i coincide, the quantity $\sum_{i=1}^S (\mu_i - 1 + \log u_i) u_i$ is a subsolution of some parabolic partial differential equation; in [30] this fact has been utilized to apply a maximum principle, while in [26] Moser iteration has been used to derive L^∞ bounds for the solution. However, as soon as the diffusion coefficients A_i depend on the species (which is the case in most physically relevant situations), this method is not applicable anymore as the abovementioned quantity ceases to be a subsolution. For diffusion coefficients which are close to each other, Canizo, Desvillettes and Fellner [5] nevertheless have proven existence of weak solutions (i.e. with integrable reaction terms) by duality estimates.

It is therefore the interplay of reactions and species-dependent diffusion which is responsible for the difficulty of our problem. The results of Pierre and Schmitt

[35, 36] suggest that in general, given only the entropy inequality (5), for *species-dependent* diffusion coefficients we should perhaps not expect existence of bounded solutions: In [35] it is shown that the L^∞ norm of solutions to reaction-diffusion equations of the form (1) with $\vec{b_i} = 0$ and species-dependent diffusion may blow up in finite time, even if the so-called dissipation of mass condition

$$\sum_{i=1}^{S} R_i(u) \leq 0 \quad \forall u \in \left(\mathbb{R}_0^+\right)^S \tag{8}$$

is satisfied (for existence results given the dissipation of mass condition, see the overview by Pierre [34]). Although the dissipation of mass condition is not equivalent to our entropy condition, the amount of control on solutions provided by these conditions does not differ much: the dissipation of mass condition allows controlling the norm $\sup_t ||u(.,t)||_{L^1}$, while the entropy condition gives control of the quantity $\sup_t ||u(.,t) \log u(.,t)||_{L^1}$. Therefore it seems likely that the entropy condition does not prevent finite-time blowup of the L^∞ norm of solutions either.

Whether globally bounded solutions exist under more restrictive conditions on the structure (but not the order) of the reaction terms – e.g. for a single reversible reaction with mass-action kinetics (4) – is an open problem.

If we have some uniform L^1 a-priori bound for the reaction terms for approximate problems, weak solutions exist in many cases as shown by Pierre [33] (even without the entropy condition).

In case of reaction rates satisfying the mass dissipation condition, an L^2 estimate has been derived by Pierre and Schmitt [35, 36] by a duality approach and subsequently been exploited to prove existence of weak solutions for different cases in which the reaction rates have at most quadratic growth; see also the papers [5, 9, 12]. Note that for reaction rates with low-order growth and/or low spatial dimensions, weak solutions or even bounded (and thus smooth) solutions can often be obtained; see the previous references and e.g. [4, 6, 19, 20, 24].

For general reaction rates satisfying the mass dissipation condition, global existence of any kind of solution is also an open problem (which we do not address in the present paper); see the overview article by Pierre [34].

For reaction-diffusion equations with mass-action kinetics in the framework of semiconductor models, we refer the reader to [18, 22] and the references therein.

Note that the entropy estimate also allows for analyzing large-time behavior of solutions to our reaction-diffusion system; see e.g. [8, 9, 10, 11, 29].

In a recent work, Mielke [28] has shown that formally, reaction-diffusion equations with reactions of the form (7) and mass action kinetics can even be regarded as a gradient flow of the entropy with respect to a dissipation functional related to the Wasserstein distance (see also [27]). However, a rigorous analysis in this direction in general is still lacking. It would be an interesting question to decide whether the renormalized solutions constructed in the present paper may also be obtained by Otto calculus (as introduced in [25, 32]).

Before stating our main results, let us provide a sketch of our method:

- Our notion of renormalized solution requires that for a renormalized solution u, for *any* $\xi \in C^\infty([0,\infty)^S)$ with compactly supported derivatives the function $\xi(u)$ must satisfy the equation deduced by a formal application of the chain rule.

- To prove existence of renormalized solutions, we first show existence of solutions u^ϵ for a regularized approximating problem using a standard Galerkin approach. The regularization is chosen in such a way that the entropy condition is preserved. The entropy estimate therefore yields a uniform (with respect to ϵ) bound on u^ϵ; more precisely, we have

$$\sup_{t\in(0,T)} \sum_{i=1}^{S} \int_\Omega u_i^\epsilon(\log u_i^\epsilon + \mu_i - 1)\ dx + \sum_{i=1}^{S} \int_0^T \int_\Omega |\nabla \sqrt{u_i^\epsilon}|^2\ dx\ dt \le C(T)\ .$$

- In the second step, compactness of the solutions u^ϵ to the approximate problem in $L^1([0,T];L^1(\Omega))$ is shown. We proceed by applying the Aubin-Lions lemma to the truncations $\varphi_i^E(u^\epsilon)$ of the approximating solutions, where $\varphi_i^E : (\mathbb{R}_0^+)^S \to \mathbb{R}$ is a truncation of the mapping $v \mapsto v_i$. The function $\varphi_i^E(v)$ equals v_i in case $\sum_{i=1}^S v_i < E$; for $\sum_{i=1}^S v_i \ge 2E$ the function $\varphi_i^E(v)$ is constant.

  The required bounds on the spatial derivatives of $\varphi_i^E(u^\epsilon)$ are inferred from the entropy estimate (and the chain rule), while the bounds on the time derivative are deduced by deriving an evolution equation for $\varphi_i^E(u^\epsilon)$.

  Having proven compactness of the families $\varphi_i^E(u^\epsilon)$ in $L^1([0,T];L^1(\Omega))$ for all E, we deduce compactness of u^ϵ; here we additionally need the $L\log L$ bound inferred from the entropy estimate.
- We then would like to pass to the limit $\epsilon \to 0$ in the evolution equation for $\xi(u^\epsilon)$ for any choice of $\xi \in C^\infty([0,\infty)^S)$ with compactly supported derivatives. The compactness of the u^ϵ in $L^1([0,T];L^1(\Omega))$ and the uniform bounds on the u^ϵ deduced from the entropy estimate enable us to pass to the limit in most terms of the equation.

  However, due to the nonlinearity of ξ, terms involving products of derivatives of u^ϵ appear in the equation for $\xi(u^\epsilon)$ (note that we only know weak convergence of $\nabla\sqrt{u_i^\epsilon}$ in $L^2([0,T];L^2(\Omega))$). We do not know how to show convergence of these terms directly and therefore have to resort to a different strategy.
- Instead of passing to the limit in the equation for $\xi(u^\epsilon)$, we pass to the limit in the equation for the truncated densities $\varphi_i^E(u^\epsilon)$. Of course, φ_i^E is just a special choice of ξ; therefore the issue with the terms involving products of derivatives of u^ϵ remains. However, we can now show that these terms – which by our uniform bounds on u^ϵ must (weak-$*$) converge to some signed measure in the limit $\epsilon \to 0$ – become negligible in the limit of infinite truncation height $E \to \infty$ (see below). We therefore obtain an equation for the truncated densities $\varphi_i^E(u)$ which is only satisfied up to some error, with the error converging to zero as $E \to \infty$.
- To bound the error in the equation for the truncated densities $\varphi_i^E(u)$, we first notice that it is bounded by $C\sum_{j=1}^S \mathcal{E}_j^E$, where

$$\mathcal{E}_i^E := \lim_{\epsilon\to 0} \int_0^T \int_{\{E\le\sum_j u_j^\epsilon<2E\}} |\nabla\sqrt{u_i^\epsilon}|^2\ dx\ dt\ .$$

The latter quantity tends to zero in the limit of infinite truncation height $E \to \infty$, as we have

$$\sum_{k=0}^{\infty} \lim_{\epsilon\to 0} \int_0^T \int_{\{k \leq \sum_j u_j^\epsilon < k+1\}} |\nabla \sqrt{u_i^\epsilon}|^2 \ dx \ dt$$
$$\leq \liminf_{\epsilon\to 0} \sum_{k=0}^{\infty} \int_0^T \int_{\{k \leq \sum_j u_j^\epsilon < k+1\}} |\nabla \sqrt{u_i^\epsilon}|^2 \ dx \ dt$$
$$= \liminf_{\epsilon\to 0} \int_0^T \int_\Omega |\nabla \sqrt{u_i^\epsilon}|^2 \ dx \ dt \leq C(T).$$

- In the last step, starting with the equations for the truncated densities $\varphi_i^E(u)$ (which are only satisfied up to some error), we derive an equation for $\xi(\varphi^E(u))$, which again is only satisfied up to some error. Here $\varphi^E(u)$ denotes the vector with entries $\varphi_1^E(u)$, ..., $\varphi_S^E(u)$. Then passing to the limit $E \to \infty$, we obtain the desired exact equation for $\xi(u)$.

Throughout the paper, we shall use standard notation for Sobolev spaces. Furthermore we abbreviate $I := [0, \infty)$. By $L^p_{loc}(I; X)$ we denote the set of all mappings $u : I \to X$ which belong to $L^p([0, T]; X)$ for all $T > 0$. By $\vec{n}$ we denote the outer unit normal vector to a domain $\Omega \subset \mathbb{R}^d$. The k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. By $RM(A)$ we denote the space of Radon measures on A with the total variation norm $||\cdot||_{RM(A)}$. As usual, the Sobolev space $H^1(\Omega)$ is defined to consist of all functions in $L^2(\Omega)$ whose weak first derivatives also belong to $L^2(\Omega)$. The space of smooth compactly supported functions on a set A is denoted by $C^\infty_{cpt}(A)$. By $C^{0,1}(A)$ we refer to the class of Lipschitz continuous functions on A. The notations $\overline{A}$ and A° are used for the closure and the interior of a set A, respectively; ∂A refers to the boundary of the set A. Given a matrix $A \in \mathbb{R}^{n\times m}$, by $|A|$ we denote its spectral norm.

2. Main Results

Besides the entropy condition (5) we shall impose the following (modest) restrictions on our domain, the coefficients, and the reaction rates:

- (A1) Let $d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.
- (A2) Assume that $A_i \in [L^\infty(I; L^\infty(\Omega))]^{d\times d}$. We denote $\max_i \sup_{x,t} |A_i|$ by Λ.
- (A3) Suppose that there exists $\lambda > 0$ such that for all i, all $x \in \Omega$, all $t \in I$ and all $v \in \mathbb{R}^d$ we have $\langle A_i(x,t)v, v\rangle \geq \lambda |v|^2$.
- (A4) Assume that $\vec{b}_i \in [L^\infty(I; L^\infty(\Omega))]^d$ and that the trace of $\vec{b}_i$ on the spatial boundary $\partial\Omega \times I$ exists.
- (A5) Let $R_i : \left(\mathbb{R}_0^+\right)^S \to \mathbb{R}$ be locally Lipschitz for all $1 \leq i \leq S$.
- (A6) Assume that $R_i(v) \geq 0$ for any $v \in (\mathbb{R}_0^+)^S$ with $v_i = 0$.

The last condition (A6) serves to guarantee nonnegativity of all concentrations u_i (as negative concentrations are absurd from a physical point of view). It has a simple physical interpretation: if species i is not present, it cannot be consumed by reactions.

We impose the following boundary conditions:

- (B1) Assume that there exist disjoint open subsets Γ_{In}, Γ_{Out} of $\partial\Omega$ such that the decomposition $\overline{\Gamma}_{In} \cup \overline{\Gamma}_{Out} = \partial\Omega$ holds.

(B2) Let $g_i \in L^\infty(\Gamma_{In} \times I)$ be nonnegative bounded functions. Then we impose the boundary condition $\vec{n} \cdot (A_i \nabla u_i - u_i \vec{b}_i) = g_i$ on $\Gamma_{In} \times I$.

(B3) Assume that $\vec{n} \cdot \vec{b}_i \geq 0$ holds on Γ_{Out}. We then impose the condition $\vec{n} \cdot A_i \nabla u_i = 0$ on $\Gamma_{Out} \times I$.

Condition (B2) (which prescribes the (in-)flow at the boundary) is a typical inflow boundary condition on Γ_{In}, while condition (B3) (which implies that the diffusive flux at the boundary vanishes) is a typical outflow boundary condition on Γ_{Out}; see e.g. [26]. The condition that the advection velocity be pointed outward on Γ_{Out} is imposed in order to control the evolution of the total entropy. The condition that the (in)flux g_i be positive on Γ_{In} is necessary to ensure nonnegativity of the solution.

We introduce the following notion of renormalized solution for equation (1):

Definition 1. *Suppose that (A1) to (A6) hold. Let $(u_0)_i \in L^1(\Omega)$, $1 \leq i \leq S$, be nonnegative. We say that nonnegative functions $u_i \in L^\infty_{loc}(I; L^1(\Omega))$ with $\sqrt{u_i} \in L^2_{loc}(I; H^1(\Omega))$, $1 \leq i \leq S$, are a renormalized solution to the reaction-diffusion-advection equation (1) with initial data u_0 and boundary conditions (B1) to (B3) if for every smooth function $\xi : (\mathbb{R}_0^+)^S \to \mathbb{R}$ with compactly supported derivative $D\xi$ and for every $\psi \in C^\infty(\overline{\Omega} \times I)$ the equation*

$$
\begin{aligned}
&\int_\Omega \xi(u(.,T))\psi(.,T)\, dx - \int_\Omega \xi(u_0)\psi(.,0)\, dx - \int_0^T \int_\Omega \xi(u) \frac{d}{dt}\psi\, dx\, dt \\
&= - \sum_{i,j=1}^S \int_0^T \int_\Omega \psi \partial_i \partial_j \xi(u) (A_i \nabla u_i) \cdot \nabla u_j\, dx\, dt \\
&\quad - \sum_{i=1}^S \int_0^T \int_\Omega \partial_i \xi(u) (A_i \nabla u_i) \cdot \nabla \psi\, dx\, dt \\
&\quad + \sum_{i,j=1}^S \int_0^T \int_\Omega \psi \partial_i \partial_j \xi(u) u_i \vec{b}_i \cdot \nabla u_j\, dx\, dt + \sum_{i=1}^S \int_0^T \int_\Omega \partial_i \xi(u) u_i \vec{b}_i \cdot \nabla \psi\, dx\, dt \\
&\quad + \sum_{i=1}^S \int_0^T \int_\Omega \partial_i \xi(u) R_i(u) \psi\, dx\, dt \\
&\quad + \sum_{i=1}^S \int_0^T \int_{\Gamma_{In}} g_i \psi \partial_i \xi(u)\, d\mathcal{H}^{d-1}\, dt - \sum_{i=1}^S \int_0^T \int_{\Gamma_{Out}} \vec{n} \cdot \vec{b}_i\; u_i\; \psi \partial_i \xi(u)\, d\mathcal{H}^{d-1}\, dt
\end{aligned} \tag{9}
$$

is satisfied for a.e. $T > 0$.

Note that due to the fact that $D\xi$ is compactly supported, all terms in formula (9) are well-defined.

Our main result reads as follows:

Theorem 2. *Assume that conditions (A1) to (A6) and (B1) are satisfied; suppose that g_i and $\vec{b}_i$ meet the conditions in (B2) and (B3). Assume that the reaction rates satisfy the entropy inequality (5). Let $(u_0)_i \in L^1(\Omega)$ be nonnegative functions with $\sum_{i=1}^S \int_\Omega (u_0)_i \log(u_0)_i\, dx < \infty$.*

Then there exists a global (in time) renormalized solution u to equation (1) with initial data u_0 and boundary conditions (B1) to (B3); the solution has the additional regularity $u_i \log u_i \in L^\infty_{loc}(I; L^1(\Omega))$.

We now would like to discuss the relation of our notion of renormalized solutions to the "classical" notion of weak solutions.

To this aim, we introduce a family φ_i^E of truncations of the mapping $v \mapsto v_i$, which truncate v_i when $\sum_{k=1}^S v_k$ becomes too large. More precisely, we let $\varphi_i^E : \left(\mathbb{R}_0^+\right)^S \rightarrow \mathbb{R}_0^+$, $E \in \mathbb{N}$, be a family of functions subject to the following conditions:

(E1) Let $\varphi_i^E \in C^2((\mathbb{R}_0^+)^S)$.
(E2) Assume that there exists $K_1 > 0$ so that $\sqrt{v_j}\sqrt{v_k}|\partial_j\partial_k\varphi_i^E(v)| \leq K_1$ holds for all j, k, E and all $v \in (\mathbb{R}_0^+)^S$.
(E3) Suppose that for every E the set $\operatorname{supp} D\varphi_i^E$ is bounded.
(E4) Assume that $\lim_{E\rightarrow\infty} \partial_j\varphi_i^E(v) = \delta_{ij}$ holds for all $v \in (\mathbb{R}_0^+)^S$ and all j (where δ_{ij} denotes the Kronecker delta).
(E5) Suppose that there exists $K_2 > 0$ such that $|\partial_j\varphi_i^E(v)| \leq K_2$ holds for every $v \in (\mathbb{R}_0^+)^S$, every E, and every j.
(E6) Assume that $\varphi_i^E(v) = v_i$ holds for any $v \in (\mathbb{R}_0^+)^S$ with $\sum_{j=1}^S v_j < E$.
(E7) Suppose that we have $\lim_{E\rightarrow\infty} \sup_{|v|\leq K} |\partial_j\partial_k\varphi_i^E(v)| = 0$ for every $K \in \mathbb{R}^+$ and every j, k.

Such truncations φ_i^E satisfying (E1) to (E7) can indeed be constructed: Let $\phi \in C^\infty(\mathbb{R})$ be a smooth nonincreasing function taking values in $[0,1]$ with $\phi \equiv 1$ for $x < 0$ and $\phi \equiv 0$ for $x > 1$. Define

$$\varphi_i^E(v) := v_i\ \phi\left(\frac{\sum_{k=1}^S v_k - E}{E}\right) + 3E\ \left(1 - \phi\left(\frac{\sum_{k=1}^S v_k - E}{E}\right)\right) . \tag{10}$$

Then one verifies readily that these φ_i^E satisfy conditions (E1) to (E7). Note that we shall also use the same family of truncations in the construction of our renormalized solutions below.

Consider a renormalized solution u. We now show that even though $R_i(u) \notin L^1_{loc}(I; L^1(\Omega))$ might hold, an appropriately defined sequence of regularized versions of $R_i(u)$ in fact converges to a distribution.

Theorem 3. *Let u be a renormalized solution in the sense of Definition 1. Then, for any sequence of functions $\varphi_i^E : (\mathbb{R}_0^+)^S \rightarrow \mathbb{R}_0^+$ subject to conditions (E1) to (E7), the limits*

$$\tilde{R}_i(u) := \lim_{E\rightarrow\infty} \sum_{j=1}^S \partial_j\varphi_i^E(u) R_j(u) \tag{11}$$

exist in the sense of distributions (on $\overline{\Omega} \times [0,\infty)$). The limits are independent of the choice of the sequence φ_i^E; moreover, for any $\psi \in C^\infty_{cpt}(\overline{\Omega} \times I)$ and any $1 \leq i \leq S$ the equation

$$\begin{aligned}
&- \int_\Omega (u_0)_i \psi(.,0)\ dx - \int_0^\infty \int_\Omega u_i \frac{d}{dt}\psi\ dx\ dt \\
&= - \int_0^\infty \int_\Omega (A_i \nabla u_i) \cdot \nabla\psi\ dx\ dt + \int_0^\infty \int_\Omega u_i \vec{b}_i \cdot \nabla\psi\ dx\ dt \\
&\quad + \left\langle \tilde{R}_i(u), \psi \right\rangle_{\mathcal{D}'(\overline{\Omega}\times I)\times\mathcal{D}(\overline{\Omega}\times I)} \\
&\quad + \int_0^\infty \int_{\Gamma_{In}} g_i \psi\ d\mathcal{H}^{d-1}\ dt - \int_0^\infty \int_{\Gamma_{Out}} \vec{n} \cdot \vec{b}_i\ u_i\ \psi\ d\mathcal{H}^{d-1}\ dt
\end{aligned} \tag{12}$$

holds.

Remark 4. *The reaction terms $\tilde{R}_i(u)$ in fact have better regularity than being just a distribution: by equation (12), we see that the distributions $\tilde{R}_i(u)$ belong to the space $\left[H^1_{loc}(I;L^\infty(\Omega))\right]' + \left[L^\infty_{loc}(I;W^{1,\infty}(\Omega))\right]'$.*

The previous theorem now enables us to relate our notion of renormalized solutions to the classical notion of weak solutions:

Remark 5. *If the reaction terms $R_i(u)$ (which are defined pointwise a.e.) belong to $L^1_{loc}(I;L^1(\Omega))$ for all i, by dominated convergence (recall (E4), (E5)) we see that the distributions $\tilde{R}_i(u)$ are given by*

$$\langle \tilde{R}_i(u), \psi\rangle = \int_I \int_\Omega R_i(u)\psi \ dx \ dt$$

for all $\psi \in C^\infty_{cpt}(\overline{\Omega} \times I)$. Thus, the concept of $\tilde{R}_i(u)$ is a generalization of the concept of integrable reaction rates $R_i(u)$ and our concept of renormalized solutions generalizes the concept of weak solutions with reaction terms in $L^1_{loc}(I;L^1(\Omega))$.

Moreover, if we have some u with the property that $\tilde{R}(u)$ is a well-defined distribution and some v with the property $R_i(u)-R_i(v) \in L^1_{loc}(I;L^1(\Omega))$ for all i, then $\tilde{R}(v)$ is also well-defined as a distribution and the equality $\tilde{R}_i(v) = \tilde{R}_i(u)+(R_i(v)-R_i(u))$ holds in the sense of distributions.

3. Proof of the Main Results

3.1. **Existence of solutions for an approximate problem.** We now show existence of weak solutions for a regularized model; note that the regularization which we use preserves the entropy condition.

Lemma 6. *Assume that conditions (A1) to (A6) and (B1) are satisfied. Suppose that g_i and $\vec{b}_i$ satisfy the conditions imposed in (B2) and (B3). Let $\epsilon > 0$. Suppose that we are given nonnegative $(u_0^\epsilon)_i \in L^\infty(\Omega)$, $1 \le i \le S$. Then there exist $u_i^\epsilon \in L^2_{loc}(I;H^1(\Omega))$, $1 \le i \le S$, with $u_i^\epsilon \in H^1_{loc}(I;(H^1(\Omega))')$ such that for all $\psi \in L^2_{loc}(I;H^1(\Omega))$, a.e. $(t_1,t_2) \in I^2$, and all $i \in \{1,\ldots,S\}$ we have*

$$\begin{aligned}
&\int_{t_1}^{t_2} \left\langle \frac{d}{dt} u_i^\epsilon, \psi \right\rangle_{(H^1(\Omega))' \times H^1(\Omega)} dt \\
&= -\int_{t_1}^{t_2}\int_\Omega (A_i \nabla u_i^\epsilon)\cdot \nabla\psi \ dx \ dt + \int_{t_1}^{t_2}\int_\Omega u_i^\epsilon \vec{b}_i \cdot \nabla \psi \ dx \ dt \\
&\quad + \int_{t_1}^{t_2}\int_\Omega \frac{R_i(u^\epsilon)}{1+\epsilon|R(u^\epsilon)|}\psi \ dx \ dt \\
&\quad + \int_{t_1}^{t_2}\int_{\Gamma_{In}} g_i\psi \ d\mathcal{H}^{d-1} \ dt - \int_{t_1}^{t_2}\int_{\Gamma_{Out}} \vec{n}\cdot\vec{b}_i \ u_i^\epsilon \ \psi \ d\mathcal{H}^{d-1} \ dt \ ,
\end{aligned} \tag{13}$$

with $R(u^\epsilon)$ denoting the vector with entries $R_i(u^\epsilon)$, $1 \le i \le S$. Moreover, we have $u^\epsilon(.,0) = u_0^\epsilon$ in the sense of $(H^1(\Omega))'$.

All u_i^ϵ are nonnegative a.e..

If additionally the entropy inequality (5) holds for our reaction rates R_i, we obtain for a.e. $(t_1,t_2)\in I^2$ with $t_2>t_1$ and a.e. $t_2>0$ in case $t_1=0$

$$\int_\Omega \sum_{i=1}^S (\mu_i - 1 + \log u_i^\epsilon(.,t_2)) u_i^\epsilon(.,t_2)\ dx + c\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^S \left|\nabla\sqrt{u_i^\epsilon}\right|^2\ dx\ dt \\ \leq \left(\int_\Omega \sum_{i=1}^S (\mu_i - 1 + \log u_i^\epsilon(.,t_1)) u_i^\epsilon(.,t_1)\ dx + C\cdot(t_2-t_1)\right)\cdot e^{C\cdot(t_2-t_1)}\ , \tag{14}$$

where c and C denote some constants depending only on λ and Λ as well as $||\vec{b}_i||_{L^\infty}$ and $||g_i||_{L^\infty}$ and Ω (in particular, our c and C do not depend upon ϵ).

Proof. The proof is mostly standard material; we therefore omit details.

First of all we extend $R(.)$ to $\mathbb{R}^S$ by defining $R(v) := R(\max(v_1,0),\dots,\max(v_S,0))$. This extension obviously preserves the local Lipschitz property of R.

The proof of existence is then accomplished by standard Faedo-Galerkin approximation. Denote by $(e_j)_{j\in\mathbb{N}}$ the orthonormal Schauder basis of $L^2(\Omega)$ with $e_j\in H^1(\Omega)$ consisting of the eigenfunctions of the Laplacian with Neumann boundary conditions on Ω. We use the ansatz $u_i^{\epsilon,k}(x,t) = \sum_{j=1}^k \xi_i^{j,k}(t)e_j(x)$ with continuous functions $\xi_i^{j,k}: I\to\mathbb{R}$; the $\xi_i^{j,k}$ are determined by setting $\xi_i^{j,k}(0) = \int_\Omega (u_0^\epsilon)_i e_j\ dx$ and requiring that

$$\int_\Omega u_i^{\epsilon,k}(.,t)e_j(.)\ dx\Big|_{t_1}^{t_2} \\ = -\int_{t_1}^{t_2}\int_\Omega (A_i\nabla u_i^{\epsilon,k})\cdot\nabla e_j\ dx\ dt + \int_{t_1}^{t_2}\int_\Omega u_i^{\epsilon,k}\vec{b}_i\cdot\nabla e_j\ dx\ dt \\ + \int_{t_1}^{t_2}\int_\Omega \frac{R_i(u^{\epsilon,k})}{1+\epsilon|R(u^{\epsilon,k})|}e_j\ dx\ dt \\ + \int_{t_1}^{t_2}\int_{\Gamma_{In}} g_i e_j\ d\mathcal{H}^{d-1}\ dt - \int_{t_1}^{t_2}\int_{\Gamma_{Out}} \vec{n}\cdot\vec{b}_i\ u_i^{\epsilon,k}\ e_j\ d\mathcal{H}^{d-1}\ dt \tag{15}$$

holds for all $1\leq j\leq k$, all $1\leq i\leq S$, and all $0\leq t_1\leq t_2$. This is a system of ordinary differential equations which is (locally) solvable as (for fixed k) the right-hand side depends continuously on $(\xi_i^{j,k}(t))_{ij}$. Global solvability is a consequence of proving boundedness of the solution, which follows from the standard energy estimate below.

By first reformulating (15) to admit variable-in-time test functions (this requires an approximation argument) and then testing with $u_i^{\epsilon,k}$ and taking the sum over i, one can derive the standard L^2 energy estimate which reads

$$\frac{1}{2}\int_\Omega\sum_{i=1}^S (u_i^{\epsilon,k})^2(x,T)\ dx + \int_0^T\int_\Omega\sum_{i=1}^S (A_i\nabla u_i^{\epsilon,k})\cdot\nabla u_i^{\epsilon,k}\ dx\ dt \\ \leq \frac{1}{2}\int_\Omega\sum_{i=1}^S (u_i^{\epsilon,k})^2(x,0)\ dx \\ + \int_0^T\int_\Omega\sum_{i=1}^S u_i^{\epsilon,k}\vec{b}_i\cdot\nabla u_i^{\epsilon,k}\ dx\ dt + \int_0^T\int_\Omega\sum_{i=1}^S u_i^{\epsilon,k}\frac{R_i(u^{\epsilon,k})}{1+\epsilon|R(u^{\epsilon,k})|}\ dx\ dt \\ + \int_0^T\int_{\Gamma_{In}}\sum_{i=1}^S g_i\ u_i^{\epsilon,k}\ d\mathcal{H}^{d-1}\ dt - \int_0^T\int_{\Gamma_{Out}}\sum_{i=1}^S \vec{n}\cdot\vec{b}_i\ (u_i^{\epsilon,k})^2\ d\mathcal{H}^{d-1}\ dt\ .$$

Note that $\left|\frac{R_i(u)}{1+\epsilon|R(u)|}\right| \leq \epsilon^{-1}$. Recall (A1), (A2), (A3), (A4), (A5) and (B3). Therefore several applications of Young's inequality and a subsequent application of an interpolation-trace inequality like (16) below (see e.g. [13]) give

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \sum_{i=1}^S (u_i^{\epsilon,k})^2(x,T)\;dx + c\int_0^T\int_\Omega \sum_{i=1}^S |\nabla u_i^{\epsilon,k}|^2\;dx\;dt\\
\leq&\frac{1}{2}\int_\Omega \sum_{i=1}^S (u_i^{\epsilon,k})^2(x,0)\;dx\\
&+C\max_i ||\vec{b_i}||_{L^\infty}^2 \int_0^T\int_\Omega \sum_{i=1}^S |u_i^{\epsilon,k}|^2\;dx\;dt + \int_0^T\int_\Omega \sum_{i=1}^S \Big(|u_i^{\epsilon,k}|^2+\epsilon^{-2}\Big)\;dx\;dt\\
&+\int_0^T\int_{\Gamma_{In}} \sum_{i=1}^S \Big(|g_i|^2+|u_i^{\epsilon,k}|^2\Big)\;d\mathcal{H}^{d-1}\;dt+0\\
\leq&\frac{1}{2}\int_\Omega \sum_{i=1}^S (u_i^{\epsilon,k})^2(x,0)\;dx\\
&+C\int_0^T\int_\Omega \sum_{i=1}^S |u_i^{\epsilon,k}|^2\;dx\;dt + C\epsilon^{-2}T + \int_0^T\int_{\Gamma_{In}} \sum_{i=1}^S |g_i|^2\;d\mathcal{H}^{d-1}\;dt\\
&+\int_0^T \sum_{i=1}^S C\Big(||\nabla u_i^{\epsilon,k}(.,t)||_{L^2(\Omega)}||u_i^{\epsilon,k}(.,t)||_{L^2(\Omega)} + ||u_i^{\epsilon,k}(.,t)||_{L^2(\Omega)}^2\Big)\;dt.
\end{aligned}
$$

Making again use of Young's inequality and an absorption argument to eliminate the gradient term in the last integral, a subsequent application of the Gronwall lemma leads to an a priori estimate of the form

$$
\begin{aligned}
&\sup_{t\in[0,T]}\int_\Omega \sum_{i=1}^S (u_i^{\epsilon,k})^2(x,t)\;dx + \frac{c}{2}\int_0^T\int_\Omega \sum_{i=1}^S |\nabla u_i^{\epsilon,k}|^2\;dx\;dt\\
\leq& e^{CT}\left(C\int_\Omega \sum_{i=1}^S (P_k(u_0^\epsilon)_i)^2\;dx + C\epsilon^{-2}T + C\int_0^T\int_{\Gamma_{In}} \sum_{i=1}^S |g_i|^2\;d\mathcal{H}^{d-1}\;dt\right),
\end{aligned}
$$

where the constants C may depend on λ, Λ, β, S, Ω and where $P_k : L^2(\Omega) \to L^2(\Omega)$ denotes the projection onto $\mathrm{span}(e_1,\ldots,e_k)$.

We would now like to derive a uniform estimate on the time derivative of $u_i^{\epsilon,k}$. Note that the $u_i^{\epsilon,k}$ satisfy for any $\phi \in C^\infty(\overline{\Omega}\times[0,T])$ the equation

$$
\begin{aligned}
&\int_\Omega u_i^{\epsilon,k}\phi\;dx\Big|_0^T - \int_0^T\int_\Omega u_i^{\epsilon,k}\frac{d}{dt}\phi\;dx\;dt\\
=&-\int_0^T\int_\Omega (A_i\nabla u_i^{\epsilon,k})\cdot\nabla P_k\phi\;dx\;dt + \int_0^T\int_\Omega u_i^{\epsilon,k}\vec{b_i}\cdot\nabla P_k\phi\;dx\;dt\\
&+\int_0^T\int_\Omega \frac{R_i(u^{\epsilon,k})}{1+\epsilon|R(u^{\epsilon,k})|}P_k\phi\;dx\;dt\\
&+\int_0^T\int_{\Gamma_{In}} g_iP_k\phi\;d\mathcal{H}^{d-1}\;dt - \int_0^T\int_{\Gamma_{Out}} \vec{n}\cdot\vec{b_i}\;u_i^{\epsilon,k}\;P_k\phi\;d\mathcal{H}^{d-1}\;dt,
\end{aligned}
$$

where P_k denotes the orthogonal projection in $L^2(\Omega)$ onto $\mathrm{span}(e_1,\ldots,e_k)$. To obtain this equation, one first shows by approximation that (15) admits a variant with variable-in-time test functions; then, one tests this equation with $P_k\phi$. Now observe that P_k is also an orthogonal projection in $H^1(\Omega)$ since the e_j are eigenfunctions of the Laplacian with Neumann boundary conditions; in particular, P_k is a bounded operator from $H^1(\Omega)$ to $H^1(\Omega)$. Thus, using the above a priori estimate, we obtain a uniform (with respect to k) bound on $||u_i^{\epsilon,k}||_{H^1([0,T];(H^1(\Omega))')}$.

Passing to the limit $k \to \infty$, for a subsequence (which we do not relabel) we obtain weak convergence of $u_i^{\epsilon,k}$ to some u_i^ϵ in $L^2([0,T];H^1(\Omega))$ and in $H^1([0,T];(H^1(\Omega))')$. A subsequent application of the Aubin-Lions Lemma (see [38]) yields strong convergence of $u_i^{\epsilon,k}$ to u_i^ϵ in $L^2([0,T];L^2(\Omega))$.

These convergence properties are sufficient for passing to the limit $k \to \infty$ in equation (15) (for fixed $\epsilon > 0$). We omit the details as they can be found in many textbooks. Note that we use an interpolation-trace inequality of the form

$$||v||_{L^2(\partial\Omega)} \leq C(\Omega)||\nabla v||_{L^2(\Omega)}^{\frac{1}{2}}||v||_{L^2(\Omega)}^{\frac{1}{2}} + C(\Omega)||v||_{L^2(\Omega)} \tag{16}$$

(see e.g. [13]) to obtain strong convergence of the sequence $u_i^{\epsilon,k}$ in $L^2([0,T];L^2(\partial\Omega))$ by the uniform boundedness in $L^2([0,T];H^1(\Omega))$ and the strong convergence in $L^2([0,T];L^2(\Omega))$. Therefore the limit u_i^ϵ satisfies the equation (13) and we have $u_i^\epsilon(.,0) = (u_0^\epsilon)_i$ in $(H^1(\Omega))'$.

Nonnegativity of the u_i^ϵ is proven by testing the weak formulation of the equation for u_i^ϵ (i.e. equation (13)) with $\min(0,u_i^\epsilon)$ and using (B2), (B3) as well as the fact that we have $R_i(v) \geq 0$ in case $v_i \leq 0$ (see (A6)) and the fact that $(u_0^\epsilon)_i \geq 0$. Since $\min(u_i^\epsilon,0)^2$ is convex, it is possible to rearrange the terms involving the time derivative using a standard technique (see e.g. [3]), yielding for any $T > 0$

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\min(u_i^\epsilon(.,T),0)^2 \, dx \\
&\leq -\int_0^T \int_\Omega \lambda|\nabla \min(u_i^\epsilon,0)|^2 \, dx \, dt \\
&\quad + ||\vec{b}_i||_{L^\infty}\int_0^T \int_\Omega |\min(u_i^\epsilon,0)\nabla\min(u_i^\epsilon,0)| \, dx \, dt \; .
\end{aligned}$$

Young's inequality and the Gronwall Lemma allow to conclude.

To derive the entropy estimate (14), we first need to show boundedness of the solution u_i^ϵ. An $L^\infty([0,T];L^\infty(\Omega))$ bound however is derived easily using Moser iteration: Let $k \in [1,\infty)$ be a real number. Inserting $(u_i^\epsilon)^k$ in the equation for u_i^ϵ and taking the sum with respect to i (the required approximation argument to justify this can be found e.g. in [21]), we obtain by Young's inequality

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^S \frac{1}{k+1}|u_i^\epsilon|^{k+1} \, dx \Big|_0^T \\
&\leq -\frac{c}{k+1}\int_0^T \int_\Omega \sum_{i=1}^S |\nabla |u_i^\epsilon|^{\frac{k+1}{2}}|^2 \, dx \, dt \\
&\quad + C\max_i ||\vec{b}_i||_{L^\infty}^2 \cdot (k+1)\int_0^T \int_\Omega \sum_{i=1}^S |u_i^\epsilon|^{k+1} \, dx \, dt \\
&\quad + C(\epsilon)\int_0^T \int_\Omega \sum_{i=1}^S |u_i^\epsilon|^k \, dx \, dt + \int_0^T \int_{\Gamma_{In}} \sum_{i=1}^S g_i|u_i^\epsilon|^k \, d\mathcal{H}^{d-1} \, dt \; .
\end{aligned}$$

This yields using Young's inequality

$$\begin{aligned}
&\sup_{t\in[0,T]} \int_\Omega \sum_{i=1}^S |u_i^\epsilon(.,t)|^{k+1}\ dx + \int_0^T \int_\Omega \sum_{i=1}^S |\nabla |u_i^\epsilon|^{\frac{k+1}{2}}|^2\ dx\ dt\\
\le& C \int_\Omega \sum_{i=1}^S |(u_0^\epsilon)_i(.,t)|^{k+1}\ dx + C\int_0^T \int_{\Gamma_{In}} \sum_{i=1}^S |g_i|^{k+1}\ d\mathcal{H}^{d-1}\ dt\\
&+ C(\epsilon)\cdot T + C\cdot (k+1)^2 \int_0^T \int_\Omega \sum_{i=1}^S |u_i^\epsilon|^{k+1}\ dx\ dt\\
&+ C\cdot (k+1) \int_0^T \int_{\Gamma_{In}} \sum_{i=1}^S |u_i^\epsilon|^{k+1}\ d\mathcal{H}^{d-1}\ dt\ .
\end{aligned}$$

Thus, the interpolation-trace inequality (16) (applied to $|u_i^\epsilon|^{\frac{k+1}{2}}$, i.e. in particular the constant in the inequality is independent of k) gives in connection with a subsequent application of Young's inequality and an absorption argument

$$\begin{aligned}
&\sup_{t\in[0,T]} \int_\Omega \sum_{i=1}^S |u_i^\epsilon(.,t)|^{k+1}\ dx + \int_0^T \int_\Omega \sum_{i=1}^S |\nabla |u_i^\epsilon|^{\frac{k+1}{2}}|^2\ dx\ dt\\
\le& C \int_\Omega \sum_{i=1}^S |(u_0^\epsilon)_i(.,t)|^{k+1}\ dx + C\int_0^T \int_{\Gamma_{In}} \sum_{i=1}^S |g_i|^{k+1}\ d\mathcal{H}^{d-1}\ dt\\
&+ C(\epsilon)\cdot T + C\cdot (k+1)^2 \int_0^T \int_\Omega \sum_{i=1}^S |u_i^\epsilon|^{k+1}\ dx\ dt\ .
\end{aligned}$$

Finally, an application of the Gagliardo-Nirenberg-Sobolev interpolation inequality to the functions $|u_i^\epsilon|^{\frac{k+1}{2}}$ yields the estimate

$$\begin{aligned}
&\left(\int_0^T \int_\Omega \sum_{i=1}^S |u_i^\epsilon(.,t)|^{\beta(k+1)}\ dx\ dt\right)^{\frac{1}{\beta(k+1)}}\\
\le& \left(\sum_{i=1}^S C\int_0^T \left(\int_\Omega |u_i^\epsilon|^{k+1}\ dx\right)^{\beta-1} \left(\int_\Omega \left|\nabla |u_i^\epsilon|^{\frac{k+1}{2}}\right|^2\ dx + \int_\Omega |u_i^\epsilon|^{k+1}\ dx\right)\ dt\right)^{\frac{1}{\beta(k+1)}}\\
\le& \Bigg(C(T)\cdot(1+k)^2 \int_0^T \int_\Omega \sum_{i=1}^S |u_i^\epsilon(.,t)|^{k+1}\ dx\ dt + C(T)\int_\Omega \sum_{i=1}^S |(u_0^\epsilon)_i(.,t)|^{k+1}\ dx\\
&\quad + C(T)\int_0^T \int_{\partial\Omega} \sum_{i=1}^S |g_i|^{k+1}\ d\mathcal{H}^{d-1}\ dt + C(\epsilon,T)\Bigg)^{\frac{1}{k+1}}
\end{aligned}$$

for some $\beta > 1$. The result now follows using Moser iteration: introducing the abbreviations

$$Q := \max(1, T\mathcal{H}^{d-1}(\partial\Omega), \mathcal{L}^d(\Omega))\cdot\left(\sum_{i=1}^S ||(u_0^\epsilon)_i||_{L^\infty(\Omega)} + \sum_{i=1}^S ||g_i||_{L^\infty(\partial\Omega\times[0,T))}\right)$$

and

$$M_n := \left(\int_0^T \int_\Omega \sum_{i=1}^S |u_i^\epsilon(.,t)|^{2\beta^n}\ dx\right)^{1/2\beta^n},$$

the previous estimate implies for $n\in\mathbb{N}_0$

$$M_{n+1} \le \left(C(T)\cdot(2\beta^n)^2 M_n^{2\beta^n} + C(T)Q^{2\beta^n} + C(\epsilon,T)\right)^{1/2\beta^n}$$

which yields (for a possibly larger C_ϵ depending also on T)

$$\max(M_{n+1}, Q, 1) \leq C_\epsilon^{1/2\beta^n} (2\beta^n)^{1/\beta^n} \max(M_n, Q, 1).$$

As the infinite product

$$\prod_{n=0}^{\infty} C_\epsilon^{1/2\beta^n} 2^{1/\beta^n} \beta^{n/\beta^n}$$

is finite, we end up with

$$\limsup_n \max\left(M_n, Q, 1\right) \leq C(\epsilon, T) \max(M_0, Q, 1).$$

Note that

$$\limsup_n M_n = \max_i ||u_i^\epsilon||_{L^\infty(\Omega\times[0,T))}.$$

In order to prove the entropy estimate (14), we test (13) with $\mu_i + \log(u_i^\epsilon + \delta)$. Note that the function $\mu_i + \log(u_i^\epsilon + \delta)$ belongs to $L^2_{loc}(I; H^1(\Omega))$ (since u_i^ϵ is nonnegative); thus it is an admissible test function. The term involving the time derivative can be rearranged using e.g. a technique in [3] since $(v_i+\delta)(\mu_i - 1 + \log(v_i+\delta))$ (which is the antiderivative of $\mu_i + \log(v_i + \delta)$) is convex on $[0, \infty)$. We thus get

$$\begin{aligned}
&\int_\Omega (\mu_i - 1 + \log(u_i^\epsilon(.,t_2)+\delta))(u_i^\epsilon(.,t_2)+\delta)\ dx \\
&+\int_{t_1}^{t_2}\int_\Omega 4(A_i\nabla\sqrt{u_i^\epsilon+\delta})\cdot\nabla\sqrt{u_i^\epsilon+\delta}\ dx\ dt \\
=&\int_\Omega (\mu_i - 1 + \log(u_i^\epsilon(.,t_1)+\delta))(u_i^\epsilon(.,t_1)+\delta)\ dx \\
&+\int_{t_1}^{t_2}\int_\Omega 2\frac{u_i^\epsilon}{\sqrt{u_i^\epsilon+\delta}}\vec{b_i}\cdot\nabla\sqrt{u_i^\epsilon+\delta}\ dx\ dt \\
&+\int_{t_1}^{t_2}\int_\Omega \frac{R_i(u^\epsilon)}{1+\epsilon|R(u^\epsilon)|}(\mu_i+\log(u_i^\epsilon+\delta))\ dx\ dt \\
&+\int_{t_1}^{t_2}\int_{\Gamma_{In}} g_i\ (\mu_i+\log(u_i^\epsilon+\delta))\ d\mathcal{H}^{d-1}\ dt \\
&-\int_{t_1}^{t_2}\int_{\Gamma_{Out}} \vec{n}\cdot\vec{b}_i\ u_i^\epsilon\ (\mu_i+\log(u_i^\epsilon+\delta))\ d\mathcal{H}^{d-1}\ dt\ .
\end{aligned}$$

Note that by (B3) we have $\vec{n}\cdot\vec{b}_i\ u_i^\epsilon\ (\mu_i+\log(u_i^\epsilon+\delta)) \geq 0$ if $\mu_i+\log(u_i^\epsilon+\delta)\geq 0$ and $\vec{n}\cdot\vec{b}_i\ u_i^\epsilon\ (\mu_i+\log(u_i^\epsilon+\delta)) \geq -C(\mu_i)(u_i^\epsilon+1)$ if $\mu_i+\log(u_i^\epsilon+\delta)\leq 0$ (the latter estimate holds since $v\log v$ is bounded from below and since (A4) is satisfied). This yields an estimate for the last term. The penultimate term is estimated using the fact that g_i is nonnegative and bounded (see (B2)) which implies that $g_i\ (\mu_i+\log(u_i^\epsilon+\delta)) \leq C(\mu_i)+Cu_i^\epsilon$ (at least for $\delta<1$). Estimating the remaining terms using (A2), (A3), (A4) and Young's inequality, we deduce

$$\begin{aligned}
&\int_\Omega (\mu_i - 1 + \log(u_i^\epsilon(.,t_2)+\delta))(u_i^\epsilon(.,t_2)+\delta)\ dx \\
&+ c\int_{t_1}^{t_2}\int_\Omega \left|\nabla\sqrt{u_i^\epsilon+\delta}\right|^2\ dx\ dt \\
\leq &\int_\Omega (\mu_i - 1 + \log(u_i^\epsilon(.,t_1)+\delta))(u_i^\epsilon(.,t_1)+\delta)\ dx \\
&+ C\int_{t_1}^{t_2}\int_\Omega u_i^\epsilon\ dx\ dt \\
&+ \int_{t_1}^{t_2}\int_\Omega \frac{R_i(u^\epsilon)}{1+\epsilon|R(u^\epsilon)|}(\mu_i + \log(u_i^\epsilon+\delta))\ dx\ dt \\
&+ \int_{t_1}^{t_2}\int_{\Gamma_{In}} C(\mu_i)\ d\mathcal{H}^{d-1}\ dt + C(\mu_i)\int_{t_1}^{t_2}\int_{\partial\Omega} u_i^\epsilon\ d\mathcal{H}^{d-1}\ dt \\
&+ \int_{t_1}^{t_2}\int_{\Gamma_{Out}} C(\mu_i)\ d\mathcal{H}^{d-1}\ dt
\end{aligned}$$

We now pass to the limit $\delta \to 0$. The only term which may cause difficulties is the reaction term. Using local Lipschitz continuity of $R_i(.)$ (see (A5)), the fact that u_i^ϵ is bounded, and the fact that $R_i(v) \geq 0$ in case $v_i = 0$ (see (A6)), we deduce that we have $R_i(u^\epsilon) \geq -Lu_i^\epsilon$, where the Lipschitz constant L depends on $||u^\epsilon||_{L^\infty(\Omega\times[0,T])}$. This estimate enables us to pass to the limit $\delta \to 0$ in the reaction term using Fatou's lemma. We estimate the boundary integral involving u_i^ϵ using an interpolation-trace inequality applied to $\sqrt{u_i^\epsilon}$ and Young's inequality. Taking the sum with respect to i and applying the entropy inequality (5), we finally end up with

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^S (\mu_i - 1 + \log u_i^\epsilon(.,t_2))u_i^\epsilon(.,t_2)\ dx + c\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^S \left|\nabla\sqrt{u_i^\epsilon}\right|^2\ dx\ dt \\
\leq &\int_\Omega \sum_{i=1}^S (\mu_i - 1 + \log u_i^\epsilon(.,t_1))u_i^\epsilon(.,t_1)\ dx + C\cdot(t_2-t_1) + C\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^S u_i^\epsilon\ dx\ dt\ .
\end{aligned}$$

Using the estimate $u_i^\epsilon \leq C + (\mu_i - 1 + \log u_i^\epsilon)u_i^\epsilon$ and the Gronwall Lemma, the entropy estimate is deduced. □

3.2. **Existence of renormalized solutions.** Having established existence of solutions for our approximate problem, we turn to our proof of existence of renormalized solutions to the original equations.

Choose $(u_0^\epsilon)_i(x) := \min((u_0)_i(x), \epsilon^{-1})$.

In the first step, we show that a subsequence of the solutions u^ϵ to the approximate problems (13) with initial data $(u_0^\epsilon)_i$ converges to some limit u as $\epsilon \to 0$.

Lemma 7. *Suppose that in addition to (A1) to (A6) and (B1) to (B3) the entropy inequality (5) holds. Let $(u_0)_i \in L^1(\Omega)$, $1 \leq i \leq S$, be nonnegative functions with $\int_\Omega \sum_{i=1}^S (u_0)_i \log(u_0)_i\ dx < \infty$. Consider a sequence u^ϵ of solutions to the approximate problems in Lemma 6, with ϵ converging to zero. Then there exists a subsequence u^ϵ (not relabeled) which converges a.e. on $\Omega \times [0,\infty)$ to some limit $u \in [L^\infty_{loc}(I; L^1(\Omega))]^S$ with $u_i \log u_i \in L^\infty_{loc}(I; L^1(\Omega))$ for all $1 \leq i \leq S$. Moreover, the convergence $\sqrt{u_i^\epsilon} \rightharpoonup \sqrt{u_i}$ weakly in $L^2([0,T]; H^1(\Omega))$ holds for all $T > 0$ and all $1 \leq i \leq S$.*

Proof. Let φ_i^E be as in (10). Noting that

$$\nabla[\varphi_i^E(u^\epsilon)] = \sum_{j=1}^S \partial_j \varphi_i^E(u^\epsilon) \nabla u_j^\epsilon$$

and that $\operatorname{supp} \partial_j \varphi_i^E$ is a compact subset of $(\mathbb{R}_0^+)^S$, by uniform (w.r.t. ϵ) boundedness of $\sqrt{u_j^\epsilon}$ in $L^2([0,T];H^1(\Omega))$ for every $T>0$ (this is a consequence of (14)) we see that $\varphi_i^E(u^\epsilon)$ is uniformly bounded w.r.t. ϵ in $L^2([0,T];H^1(\Omega))$ for every fixed $T>0$ and every fixed $E\in\mathbb{N}$.

Let $\psi \in C^\infty(\overline{\Omega}\times I)$ be a smooth function. We insert $\psi \partial_j \varphi_i^E(u^\epsilon)$ (with φ_i^E as defined in (10)) in the equation for u_j^ϵ (see (13)) and take the sum with respect to j. This yields (note that the chain rule for the time derivative can be justified by Lemma 9)

$$\begin{aligned}
&\int_\Omega \varphi_i^E(u^\epsilon(.,t_2))\psi \,dx - \int_\Omega \varphi_i^E(u^\epsilon(.,t_1))\psi \,dx - \int_{t_1}^{t_2}\int_\Omega \varphi_i^E(u^\epsilon)\frac{d}{dt}\psi \,dx\,dt \\
&= \int_{t_1}^{t_2} \sum_{j=1}^S \left\langle \frac{d}{dt}u_j^\epsilon, \psi\partial_j\varphi_i^E(u^\epsilon)\right\rangle_{(H^1(\Omega))'\times H^1(\Omega)} dt \\
&= -\int_{t_1}^{t_2}\int_\Omega \sum_{j=1}^S\sum_{k=1}^S \psi \partial_j\partial_k\varphi_i^E(u^\epsilon)(A_j\nabla u_j^\epsilon)\cdot\nabla u_k^\epsilon \,dx\,dt \qquad (17)\\
&\quad -\int_{t_1}^{t_2}\int_\Omega \sum_{j=1}^S \partial_j\varphi_i^E(u^\epsilon)(A_j\nabla u_j^\epsilon)\cdot\nabla\psi \,dx\,dt \\
&\quad +\int_{t_1}^{t_2}\int_\Omega \psi\sum_{j=1}^S\sum_{k=1}^S \partial_j\partial_k\varphi_i^E(u^\epsilon)u_j^\epsilon\vec{b}_j\cdot\nabla u_k^\epsilon \,dx\,dt \\
&\quad +\int_{t_1}^{t_2}\int_\Omega \sum_{j=1}^S \partial_j\varphi_i^E(u^\epsilon)u_j^\epsilon\vec{b}_j\cdot\nabla\psi \,dx\,dt \\
&\quad +\int_{t_1}^{t_2}\int_\Omega \sum_{j=1}^S \partial_j\varphi_i^E(u^\epsilon)\frac{R_j(u^\epsilon)}{1+\epsilon|R(u^\epsilon)|}\psi \,dx\,dt \\
&\quad +\int_{t_1}^{t_2}\int_{\Gamma_{In}} \sum_{j=1}^S g_j\ \partial_j\varphi_i^E(u^\epsilon)\ \psi \ d\mathcal{H}^{d-1}\,dt \\
&\quad -\int_{t_1}^{t_2}\int_{\Gamma_{Out}} \sum_{j=1}^S \vec{n}\cdot\vec{b}_j\ u_j^\epsilon\partial_j\varphi_i^E(u^\epsilon)\ \psi\ d\mathcal{H}^{d-1}\,dt \\
&=: I + II + III + IV + V + VI + VII
\end{aligned}$$

for a.e. $(t_1,t_2)\in[0,\infty)^2$ and a.e. $t_2>0$ in case $t_1=0$. Using the fact that $\operatorname{supp} D\varphi_i^E$ is a compact subset of $(\mathbb{R}_0^+)^S$ and the fact that $\sqrt{u_j^\epsilon}$ is uniformly (w.r.t. ϵ) bounded in $L^2([0,T];H^1(\Omega))$ for every $T>0$, we see that $\varphi_i^E(u^\epsilon)$ ist bounded uniformly in $W^{1,1}([0,T];(W^{1,\infty}(\Omega))')$ for every $T>0$ and every fixed E (note that functions in $H^1(\Omega)$ have an L^2 trace on $\partial\Omega$, thus the boundary terms VI and VII are uniformly bounded in $L^2([0,T];(H^1(\Omega))'))$.

An application of the Aubin-Lions Lemma now yields compactness of the sequence $\varphi_i^E(u^\epsilon)$ in $L^2([0,T];L^2(\Omega))$ for every fixed $T>0$ and every fixed $E\in\mathbb{N}$. By a diagonal sequence argument (we do not relabel the subsequence), we may assume that for every i and every $E\in\mathbb{N}$ the sequence $(\varphi_i^E(u^\epsilon))_\epsilon$ converges a.e. to some measurable limit w_i^E. From the uniform boundedness of $\sum_{j=1}^S u_j^\epsilon \log u_j^\epsilon$ in $L^\infty([0,T];L^1(\Omega))$

which holds for every fixed $T > 0$ (this is also a consequence of (14)) we deduce using (E5) that $\varphi_i^E(u^\epsilon)\log\varphi_i^E(u^\epsilon)$ is also bounded uniformly in $L^\infty([0,T];L^1(\Omega))$ for every fixed $T > 0$; moreover, using (E5) and (E6) we see that the boundedness is also uniform w.r.t. E. Thus, by Fatou's lemma we know that w_i^E is a.e. finite and $w_i^E \log w_i^E$ is bounded uniformly (w.r.t. E) in $L^\infty([0,T];L^1(\Omega))$.

We now prove that the pointwise limit $\lim_{E\to\infty} w_i^E$ exists a.e. and defines a measurable function u_i with $u_i \log u_i \in L^\infty([0,T];L^1(\Omega))$. If for some (x,t) and some E we have $\sum_{j=1}^S w_j^E(x,t) = \lim_{\epsilon\to 0}\sum_{j=1}^S \varphi_j^E(u^\epsilon(x,t)) < E$, then $w_j^{\tilde{E}}(x,t) = w_j^E(x,t)$ holds for all $\tilde{E} > E$ and all j: by our choice of φ_i^E (see (10)) we know that $\sum_{j=1}^S \varphi_j^E(v) < E$ implies $\varphi_j^E(v) = v_j = \varphi_j^{\tilde{E}}(v)$ for all j. If we have $\sum_{j=1}^S w_j^E(x,t) < E$, then for ϵ small enough it holds that $\sum_{j=1}^S \varphi_j^E(u^\epsilon(x,t)) < E$ and therefore we get $w_i^E(x,t) = w_i^{\tilde{E}}(x,t)$ for $\tilde{E} > E$. Since $\sum_{j=1}^S w_j^E$ is bounded uniformly in $L^\infty([0,T];L^1(\Omega))$, the measure of the set of points (x,t) for which $\sum_{j=1}^S w_j^E(x,t) \geq E$ holds tends to zero as $E \to \infty$; thus, the limit $\lim_{E\to\infty} w_i^E(x,t)$ exists for a.e. $(x,t) \in \Omega\times[0,T]$ and defines a measurable function u_i. The estimate $u_i \log u_i \in L^\infty([0,T];L^1(\Omega))$ is a consequence of Fatou's Lemma.

The functions u_i are now the natural candidate for being a renormalized solution of (1).

First we notice that (after possibly passing to another subsequence) u_i^ϵ converges a.e. to u_i: By uniform boundedness of $\sum_{j=1}^S u_j^\epsilon$ in $L^1(\Omega\times[0,T])$, the measure of the set of points (x,t) with $\sum_{j=1}^S u_j^\epsilon(x,t) \geq E$ tends to zero as $E\to\infty$, uniformly in ϵ; thus the measure of the set of points (x,t) for which $\varphi_i^E(u^\epsilon(x,t)) \neq u_i^\epsilon(x,t)$ holds tends to zero as $E \to \infty$, uniformly in ϵ. We have for any $\delta > 0$

$$
\begin{aligned}
&\mathcal{L}^{d+1}\left(\left\{(x,t)\in\Omega\times[0,T] : |u_i^\epsilon(x,t) - u_i(x,t)| > \delta\right\}\right)\\
\leq&\mathcal{L}^{d+1}\left(\left\{(x,t)\in\Omega\times[0,T] : u_i^\epsilon(x,t) \neq \varphi_i^E(u^\epsilon)(x,t)\right\}\right)\\
&+\mathcal{L}^{d+1}\left(\left\{(x,t)\in\Omega\times[0,T] : |\varphi_i^E(u^\epsilon)(x,t) - w_i^E(x,t)| > \frac{\delta}{2}\right\}\right)\\
&+\mathcal{L}^{d+1}\left(\left\{(x,t)\in\Omega\times[0,T] : |w_i^E(x,t) - u_i(x,t)| > \frac{\delta}{2}\right\}\right),
\end{aligned}
$$

where by the previous considerations the first term on the right-hand side converges to zero as $E\to\infty$, uniformly in $\epsilon > 0$. The last term tends to zero as $E\to\infty$ by definition of u_i; it is independent of ϵ. The penultimate term converges to zero as $\epsilon\to 0$ for fixed E. Summing up, we have shown that u_i^ϵ converges to u_i in measure which implies convergence a.e. for a subsequence.

As $u_i^\epsilon \log u_i^\epsilon$ is bounded uniformly in $L^\infty([0,T];L^1(\Omega))$ for every $T>0$, we deduce that u_i^ϵ converges to u_i strongly in $L^p([0,T];L^1(\Omega))$ for every $T > 0$ and all $p \geq 1$. This in particular implies convergence of $\sqrt{u_i^\epsilon}$ to $\sqrt{u_i}$ in the sense of distributions, and we obtain that $\sqrt{u_i} \in L^2([0,T];H^1(\Omega))$ with

$$
\int_0^T\int_\Omega |\nabla\sqrt{u_i}|^2\,dx\,dt \leq \liminf_{\epsilon\to 0}\int_0^T\int_\Omega |\nabla\sqrt{u_i^\epsilon}|^2\,dx\,dt
$$

(the latter lim inf being finite due to (14)). In particular, $\sqrt{u_i^\epsilon}$ converges to $\sqrt{u_i}$ weakly in $L^2([0,T];H^1(\Omega))$ for every $T > 0$. □

In the second step of our proof of existence of renormalized solutions, we show that the "truncations" $\varphi_i^E(u)$ of the limit u which has been constructed in the previous step satisfy a certain PDE. However, this PDE may differ from the desired PDE by a defect measure.

Lemma 8. *Let u_j, $1 \leq j \leq S$, be the functions constructed in the previous lemma. Let φ_i^E, $1 \leq i \leq S$, be the functions defined in (10). Let $\psi \in C^\infty_{cpt}(\overline{\Omega} \times I)$. Then $\varphi_i^E(u)$ satisfies*

$$\begin{aligned}
&-\int_0^\infty \int_\Omega \varphi_i^E(u) \frac{d}{dt}\psi \; dx \; dt - \int_\Omega \varphi_i^E(u_0)\psi(.,0) \; dx \\
&= - \int_{\overline{\Omega}\times[0,\infty)} \psi \; d\mu_i^E(x,t) \\
&- \int_0^\infty \int_\Omega \sum_{j=1}^S \partial_j \varphi_i^E(u)(A_j \nabla u_j) \cdot \nabla \psi \; dx \; dt \\
&+ \int_0^\infty \int_\Omega \psi \sum_{j=1}^S \sum_{k=1}^S \partial_j \partial_k \varphi_i^E(u) u_j \vec{b}_j \cdot \nabla u_k \; dx \; dt \\
&+ \int_0^\infty \int_\Omega \sum_{j=1}^S \partial_j \varphi_i^E(u) u_j \vec{b}_j \cdot \nabla \psi \; dx \; dt \\
&+ \int_0^\infty \int_\Omega \sum_{j=1}^S \partial_j \varphi_i^E(u) R_j(u) \psi \; dx \; dt \\
&+ \int_0^\infty \int_{\Gamma_{In}} \sum_{j=1}^S g_j \; \partial_j \varphi_i^E(u) \; \psi \; d\mathcal{H}^{d-1} \; dt \\
&- \int_0^\infty \int_{\Gamma_{Out}} \sum_{j=1}^S \vec{n} \cdot \vec{b}_j \; u_j \partial_j \varphi_i^E(u) \; \psi \; d\mathcal{H}^{d-1} \; dt \; ,
\end{aligned} \tag{18}$$

where μ_i^E denotes a sequence of signed Radon measures satisfying

$$\lim_{E\to\infty} |\mu_i^E|(\overline{\Omega} \times [0,T)) = 0 \tag{19}$$

for all $T > 0$ and all i.

Note that the measure

$$\mu_i^E - \sum_{j,k=1}^S \partial_j \partial_k \varphi_i^E(u)(A_j \nabla u_j) \cdot \nabla u_k \; dx \; dt$$

is called the defect measure (since it is the convergence defect in the sequence of equations (17) caused by the lack of strong convergence of the spatial derivatives of the sequence u^ϵ). We shall later show implicitly that it in fact vanishes: in the proof of Theorem 2, it is shown that (9) holds for our solution candidate u. Inserting φ_i^E in place of ξ in equation (9) and subtracting the resulting equation from (18), we see that we actually have

$$\mu_i^E = \sum_{j,k=1}^S \partial_j \partial_k \varphi_i^E(u)(A_j \nabla u_j) \cdot \nabla u_k \; dx \; dt.$$

Note that however the proof of formula (9) for our u essentially relies on the present lemma, which just contains the weaker statement that the defect measure vanishes in the limit $E \to \infty$.

Proof. Let $T > 0$ and $\psi \in C^\infty_{cpt}(\overline{\Omega} \times [0,T))$. For fixed $E \in \mathbb{N}$ we pass to the limit $\epsilon \to 0$ in (17) for $t_1 := 0$, $t_2 := T$. Convergence of the left-hand side and of the terms II, III, IV, V is immediate by the convergence properties proven in the previous lemma and by the fact that $\operatorname{supp} D\varphi_i^E$ is compact (i.e. in case u^ϵ becomes to large, all terms of the form $\partial_j \varphi_i^E(u^\epsilon)$ and $\partial_j\partial_k\varphi_i^E(u^\epsilon)$ vanish; see (E3)). Convergence of terms VI and VII is an easy consequence of uniform boundedness of $\sqrt{u_i^\epsilon}$ in $L^2([0,T];H^1(\Omega))$, strong convergence of $\sqrt{u_i^\epsilon}$ in $L^2([0,T];L^2(\Omega))$, and the interpolation-trace inequality (16) (this inequality can be found e.g. in [13]), which together imply strong convergence of the boundary trace of $\sqrt{u_i^\epsilon}$ to the boundary trace of $\sqrt{u_i}$ in $L^2([0,T];L^2(\partial\Omega))$.

The only term whose convergence cannot be ensured is term I, as this term is quadratic in ∇u^ϵ. In order to deal with this term, we intend to show that it vanishes in the limit $E \to \infty$. Consider the signed measures

$$
\begin{aligned}
\mu_{i,\epsilon}^E :=& \sum_{j,k=1}^S \partial_j\partial_k\varphi_i^E(u^\epsilon)(A_j\nabla u_j^\epsilon)\cdot\nabla u_k^\epsilon \; dx \; dt \\
=&4\sum_{j,k=1}^S \sqrt{u_j^\epsilon}\sqrt{u_k^\epsilon}\partial_j\partial_k\varphi_i^E(u^\epsilon)\left(A_j\nabla\sqrt{u_j^\epsilon}\right)\cdot\nabla\sqrt{u_k^\epsilon}\; dx\; dt\; . \qquad (20)
\end{aligned}
$$

Note that we have

$$
|\mu_{i,\epsilon}^E|(\overline{\Omega}\times[0,T)) \leq C\Lambda \sum_{j=1}^S \int_0^T\int_\Omega \left|\nabla\sqrt{u_j^\epsilon}\right|^2 \; dx\; dt
$$

which follows from the definition of $\mu_{i,\epsilon}^E$ using (A2) and (E2) as well as Young's inequality. The uniform boundedness of $\sqrt{u_j^\epsilon}$ in $L^2([0,T];H^1(\Omega))$ for any $T > 0$ implies that after passing to a subsequence we may assume that $\mu_{i,\epsilon}^E$ weak-$*$ converges on $\overline{\Omega}\times[0,\infty)$ to some limit μ_i^E as ϵ tends to 0.

It remains to prove (19). We now consider the measures

$$
\nu_{j,\epsilon}^K := \chi_{\{|u^\epsilon|\in[K-1,K)\}}\left|\nabla\sqrt{u_j^\epsilon}\right|^2\; dx\; dt
$$

on $\overline{\Omega}\times[0,\infty)$. Using (A2) and (E2) we deduce from (20) that

$$
\begin{aligned}
&|\mu_{i,\epsilon}^E|(\overline{\Omega}\times[0,T)) \\
\leq& C\Lambda\sum_{j=1}^S\sum_{k=1}^S\sum_{K=1}^\infty\int_0^T\int_\Omega \chi_{\{|u^\epsilon|\in[K-1,K)\}}\sqrt{u_j^\epsilon}\sqrt{u_k^\epsilon}\left|\partial_j\partial_k\varphi_i^E(u^\epsilon)\right|\left|\nabla\sqrt{u_j^\epsilon}\right|^2\; dx\; dt \\
\leq& C\Lambda\sum_{l=1}^S\sum_{K=1}^\infty \nu_{l,\epsilon}^K\left(\overline{\Omega}\times[0,T)\right)\cdot \sup_{|v|\in[K-1,K);1\leq j,k\leq S}\sqrt{v_j}\sqrt{v_k}\left|\partial_j\partial_k\varphi_i^E(v)\right|\; .
\end{aligned}
$$

By (E3), for fixed $E \in \mathbb{N}$ only finitely many terms in the series do not vanish. We may therefore pass to the limit $\epsilon \to 0$; using the fact that the measure of open sets is lower semicontinuous w.r.t. weak-$*$ convergence of measures, we obtain after passing to a subsequence (the passage to a subsequence in particular ensuring that the limits in the last line of the next formula exist)

$$
\begin{aligned}
&|\mu_i^E|(\overline{\Omega}\times[0,T)) \\
\leq& \liminf_{\epsilon\to 0}|\mu_{i,\epsilon}^E|(\overline{\Omega}\times[0,T)) \qquad (21)\\
\leq& C\Lambda\sum_{l=1}^S\sum_{K=1}^\infty \lim_{\epsilon\to 0}\nu_{l,\epsilon}^K\left(\overline{\Omega}\times[0,T)\right)\cdot \sup_{|v|\in[K-1,K);1\leq j,k\leq S}\sqrt{v_j}\sqrt{v_k}\left|\partial_j\partial_k\varphi_i^E(v)\right|\; .
\end{aligned}
$$

However, we have

$$\sum_{K=1}^{\infty} \nu_{j,\epsilon}^K \left(\overline{\Omega} \times [0,T)\right) = \int_0^T \int_\Omega \left|\nabla \sqrt{u_j^\epsilon}\right|^2 \, dx \, dt \ .$$

As the latter quantity is bounded uniformly w.r.t. ϵ (recall (14)), we obtain using Fatou's lemma (for the counting measure on $\mathbb{N}$; recall that the limits in the next formula actually exist since we have passed to an appropriate subsequence)

$$\sum_{K=1}^{\infty} \lim_{\epsilon\to 0} \nu_{j,\epsilon}^K \left(\overline{\Omega} \times [0,T)\right) < \infty \ .$$

By dominated convergence applied to the counting measure on $\mathbb{N}$ (which is possible by (E2) and (E7) as well as the previous estimate), we deduce from (21)

$$\begin{aligned}
&\limsup_{E\to\infty} |\mu_i^E|(\overline{\Omega} \times [0,T)) \\
\leq & C\Lambda \sum_{l=1}^{S} \sum_{K=1}^{\infty} \lim_{\epsilon\to 0} \nu_{l,\epsilon}^K \left(\overline{\Omega} \times [0,T)\right) \cdot \lim_{E\to\infty} \sup_{|v|\in[K-1,K);1\leq j,k\leq S} \sqrt{v_j}\sqrt{v_k} \left|\partial_j\partial_k \varphi_i^E(v)\right| \\
= & C\Lambda \sum_{l=1}^{S} \sum_{K=1}^{\infty} \lim_{\epsilon\to 0} \nu_{l,\epsilon}^K \left(\overline{\Omega} \times [0,T)\right) \cdot 0 \\
= & 0 \ .
\end{aligned}$$

This finishes the proof of the lemma. □

Finally we can prove our main theorem.

Proof of Theorem 2. In order to show that u is a renormalized solution, we apply Lemma 9 below – which is some kind of chain rule for the time derivative in a regime of weak regularity – to the map $v := \varphi^E(u)$ in order to approximately identify the weak time derivative of $\xi(\varphi^E(u))$; then we pass to the limit $E \to \infty$ to deduce the equation for $\xi(u)$.

More precisely, choose some $T > 0$ arbitrary but fixed; we then prove that u is a renormalized solution on $[0,T)$. Let $\xi : \mathbb{R}^S \to \mathbb{R}$ be a smooth function with compactly supported derivatives. Recalling that $\varphi^E(u)$ satisfies (18), we see that in Lemma 9 we need to choose

$$\begin{aligned}
(v_0)_i &:= \varphi_i^E(u_0), \\
\nu_i &:= -\mu_i^E, \\
z_i &:= -\sum_{j=1}^{S} \partial_j \varphi_i^E(u)(A_j \nabla u_j) + \sum_{j=1}^{S} \partial_j \varphi_i^E(u) u_j \vec{b}_j, \\
w_i &:= \sum_{j,k=1}^{S} \partial_j\partial_k \varphi_i^E(u) u_j \vec{b}_j \cdot \nabla u_k + \sum_{j=1}^{S} \partial_j \varphi_i^E(u) R_j(u), \\
q_i &:= \sum_{j=1}^{S} g_j \partial_j \varphi_i^E(u) \chi_{\Gamma_{In}\times[0,T)} - \sum_{j=1}^{S} \vec{n}\cdot\vec{b}_j u_j \partial_j \varphi_i^E(u) \chi_{\Gamma_{Out}\times[0,T)}.
\end{aligned}$$

Obviously, ν_i is a Radon measure. It is also easily seen that we have $w_i \in L^1(\Omega \times [0,T))$ and that $q_i \in L^1([0,T); L^1(\partial\Omega))$. Furthermore, we notice that $z_i \in L^2([0,T); [L^2(\Omega)]^d)$. By the chain rule, we have $\varphi_i^E(u) \in L^2([0,T); H^1(\Omega))$. Thus, the assumptions of Lemma 9 hold and we infer that for any $\psi \in C^\infty_{cpt}(\overline{\Omega} \times [0,T))$ the function $\xi(\varphi^E(u))$ must satisfy the estimate

$$\begin{aligned}
&\Bigg| -\int_0^T \int_\Omega \xi(\varphi^E(u)) \, \frac{d}{dt}\psi \; dx \; dt - \int_\Omega \xi(\varphi^E(u_0)) \; \psi(.,0) \; dx \\
&+ \int_0^T \int_\Omega \sum_{i=1}^S \sum_{j=1}^S \partial_i \xi(\varphi^E(u)) \; \partial_j \varphi_i^E(u) \; A_j \nabla u_j \cdot \nabla \psi \; dx \; dt \\
&+ \int_0^T \int_\Omega \sum_{i,j,k,l=1}^S \psi \; \partial_j \varphi_i^E(u) \; A_j \nabla u_j \cdot \partial_i \partial_k \xi(\varphi^E(u)) \partial_l \varphi_k^E(u) \nabla u_l \; dx \; dt \\
&- \int_0^T \int_\Omega \sum_{i=1}^S \sum_{j=1}^S \sum_{k=1}^S \psi \; \partial_j \varphi_i^E(u) u_j \vec{b}_j \cdot \partial_i \partial_k \xi(\varphi^E(u)) \nabla(\varphi_k^E(u)) \; dx \; dt \\
&- \int_0^T \int_\Omega \sum_{i=1}^S \sum_{j=1}^S \partial_i \xi(\varphi^E(u)) \; \partial_j \varphi_i^E(u) u_j \vec{b}_j \cdot \nabla \psi \; dx \; dt \\
&- \int_0^T \int_\Omega \sum_{i=1}^S \sum_{j=1}^S \sum_{k=1}^S \psi \; \partial_i \xi(\varphi^E(u)) \; \partial_j \partial_k \varphi_i^E(u) u_j \vec{b}_j \cdot \nabla u_k \; dx \; dt \\
&- \int_0^T \int_\Omega \sum_{i=1}^S \sum_{j=1}^S \partial_i \xi(\varphi^E(u)) \; \psi \; \partial_j \varphi_i^E(u) R_j(u) \; dx \; dt \\
&- \int_0^T \int_{\Gamma_{In}} \sum_{i=1}^S \sum_{j=1}^S \partial_i \xi(\varphi^E(u)) \; g_j \; \partial_j \varphi_i^E(u) \; \psi \; d\mathcal{H}^{d-1} \; dt \\
&+ \int_0^T \int_{\Gamma_{Out}} \sum_{i=1}^S \sum_{j=1}^S \partial_i \xi(\varphi^E(u)) \; \vec{n} \cdot \vec{b}_j \; u_j \partial_j \varphi_i^E(u) \; \psi \; d\mathcal{H}^{d-1} \; dt \Bigg| \\
&\leq C(\Omega) ||\psi||_{L^\infty} \sup_v |D\xi(v)| \sum_{i=1}^S |\mu_i^E|(\overline{\Omega} \times [0,T)) \; .
\end{aligned}$$

To obtain the desired equation for $\xi(u)$, we now pass to the limit $E \to \infty$. To do so, we use (10) (which implies (E1) to (E7)) as well as (19); note that due to (19), the left-hand side must be zero in the limit, i.e. we obtain an exact equation (and not an estimate) in the limit. Convergence of the terms in the first line is immediate, as is convergence of the terms in the second, the fifth, and the last two lines of the left-hand side (observe that $\varphi^E(u)$ converges pointwise a.e. to u and that the $\partial_j \varphi_i^E$ are bounded by a constant by (E5)).

It remains to deal with the terms in the third, the fourth, the sixth, and the seventh line. To show convergence of these terms, besides $\nabla\sqrt{u_i} \in L^2([0,T);L^2(\Omega))$ we need the following assertion: there exists a constant r such that for all $E > r$ the estimate $\sum_{i=1}^S u_i(x,t) \geq r$ implies $\partial_i \xi(\varphi^E(u(x,t))) = \partial_i \xi(u(x,t)) = 0$ and $\partial_i \partial_k \xi(\varphi^E(u(x,t))) = \partial_i \partial_k \xi(u(x,t)) = 0$. Given this assertion, convergence of the remaining terms in the previous formula as $E \to \infty$ is also immediate since one factor in the integrals will be zero as soon as $\max_i u_i(x,t)$ becomes too large.

To show this assertion, choose r so large that $\operatorname{supp} D\xi \subset B_{S^{-1}r}(0)$. Let $E > r$. Then $\sum_{i=1}^S u_i(x,t) \geq r$ implies $\sum_{i=1}^S \varphi_i^E(u(x,t)) \geq r$ (by definition (10)) and therefore $\partial_i \xi(\varphi^E(u(x,t))) = 0$ as well as $\partial_i \partial_k \xi(\varphi^E(u(x,t))) = 0$.

In total, this establishes the result for $\psi \in C^\infty_{cpt}(\overline{\Omega}\times[0,T))$. Recalling that $T>0$ was arbitrary and using an approximation argument to allow for $\psi \in C^\infty(\overline{\Omega}\times[0,T])$ (i.e. to allow for ψ to be nonzero at $t=T$), we see that u is indeed a global renormalized solution. □

Proof of Theorem 3. Choose $T>0$ and let $\psi \in C^\infty_{cpt}(\overline{\Omega}\times[0,T))$. Setting $\xi := \varphi_i^E$ in equation (9) and taking the limit $E\to\infty$, we obtain by dominated convergence (which can be applied due to (E2), (E4), (E5), (E6), (E7))

$$
\begin{aligned}
&-\int_\Omega (u_0)_i(.)\psi(.,0)\,dx - \int_0^\infty\int_\Omega u_i \frac{d}{dt}\psi\,dx\,dt\\
&+0\\
&+\int_0^\infty\int_\Omega (A_i\nabla u_i)\cdot\nabla\psi\,dx\,dt\\
&-0\\
&-\int_0^\infty\int_\Omega u_i\vec{b}_i\cdot\nabla\psi\,dx\,dt\\
&-\int_0^\infty\int_{\Gamma_{In}} g_i\ \psi\ d\mathcal{H}^{d-1}\,dt\\
&+\int_0^\infty\int_{\Gamma_{Out}} \vec{n}\cdot\vec{b}_i\ u_i\ \psi\ d\mathcal{H}^{d-1}\,dt\\
=&\lim_{E\to\infty}\int_0^\infty\int_\Omega\sum_{j=1}^S \partial_j\varphi_i^E(u)R_j(u)\psi\,dx\,dt\ .
\end{aligned}
\tag{22}
$$

In particular, the limit $\tilde{R}_i(u)=\lim_{E\to\infty}\sum_{j=1}^S\partial_j\varphi_i^E(u)R_j(u)$ exists in the sense of distributions (on $\overline{\Omega}\times I$). Moreover, it is independent of the precise sequence φ_i^E, as the left-hand side of the previous equation is independent of the precise sequence φ_i^E. □

In the proof of the existence of renormalized solutions, we have used the following approximate chain rule for the time derivative in a regime of weak regularity.

Lemma 9. *Let Ω be a bounded domain with Lipschitz boundary. Assume that $T>0$ and that $v\in L^1([0,T);[L^1(\Omega)]^S)\cap L^2([0,T);[H^1(\Omega)]^S)$; suppose that $v_0\in(L^1(\Omega))^S$. Let $\nu_i\in RM(\overline{\Omega}\times[0,T))$, $w_i\in L^1([0,T);L^1(\Omega))$, $z_i\in L^2([0,T);[L^2(\Omega)]^d)$, $q_i\in L^1([0,T);L^1(\partial\Omega))$ for $1\le i\le S$.*

Assume that we have for any $\psi\in C^\infty_{cpt}(\overline{\Omega}\times[0,T))$

$$
\begin{aligned}
&-\int_0^T\int_\Omega v_i(.,t)\frac{d}{dt}\psi(.,t)\,dx\,dt - \int_\Omega (v_0)_i(.)\psi(.,0)\,dx\\
=&\int_{\overline{\Omega}\times[0,T)}\psi\,d\nu_i + \int_0^T\int_\Omega w_i\psi\,dx\,dt\\
&+\int_0^T\int_{\partial\Omega} q_i\psi\,d\mathcal{H}^{d-1}\,dt + \int_0^T\int_\Omega z_i\cdot\nabla\psi\,dx\,dt.
\end{aligned}
$$

Let $\xi : \mathbb{R}^S \to \mathbb{R}$ be a smooth function with compactly supported first derivatives. Then we have for all $\psi \in C^\infty_{cpt}(\overline{\Omega} \times [0,T))$

$$\begin{aligned}
\Bigg| &- \int_0^T \int_\Omega \xi(v) \frac{d}{dt}\psi \; dx \; dt - \int_\Omega \xi(v_0(.))\psi(.,0) \; dx \\
&- \sum_{i=1}^S \int_0^T \int_\Omega \psi \; \partial_i \xi(v) w_i \; dx \; dt - \sum_{i=1}^S \int_0^T \int_{\partial\Omega} \psi \; \partial_i \xi(v) q_i \; d\mathcal{H}^{d-1} \; dt \\
&- \sum_{i=1}^S \int_0^T \int_\Omega \partial_i \xi(v) z_i \cdot \nabla\psi \; dx \; dt - \sum_{i=1}^S \sum_{k=1}^S \int_0^T \int_\Omega \psi \; \partial_i \partial_k \xi(v) z_i \cdot \nabla v_k \; dx \; dt \Bigg| \\
\leq & C(\Omega) ||\psi||_{L^\infty} (\sup_u |D\xi(u)|) \cdot \sum_{i=1}^S ||\nu_i||_{RM(\overline{\Omega}\times[0,T))}.
\end{aligned}$$

Proof. Making use of a partition of unity on $\overline{\Omega}$, we may assume that either ψ is compactly supported in $\Omega \times [0,T)$ or that ψ is compactly supported in $U \times [0,T)$ for some coordinate chart U containing a part of $\partial\Omega$.

We first treat the case $\psi \in C^\infty_{cpt}(\Omega \times [0,T))$. Let ρ_δ denote a standard mollifier with respect to space (in particular, ρ_δ is assumed to be symmetric). Set $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$. Our assumption implies that for any smooth ψ with $\operatorname{supp}\psi \subset\subset \Omega_\delta \times [0,\infty)$ the equation

$$\begin{aligned}
&- \int_0^T \int_\Omega (\rho_\delta * v_i) \; \frac{d}{dt}\psi \; dx \; dt - \int_\Omega (\rho_\delta * (v_0)_i) \; \psi(.,0) \; dx \\
= &\int_{\overline{\Omega}\times[0,T)} \psi \; d(\rho_\delta * \nu_i) \\
&+ \int_0^T \int_\Omega \psi \; (\rho_\delta * w_i) \; dx \; dt \\
&- \int_0^T \int_\Omega \psi \; \nabla \cdot (\rho_\delta * z_i) \; dx \; dt
\end{aligned}$$

holds; indeed, to see this it is sufficient to use $\rho_\delta * \psi$ as a test function in the assumption of the present lemma, use the symmetry of ρ_δ and Fubini's theorem, and finally integrate by parts in the last term. As a consequence of the previous equation, Lemma 10 may be applied to $\hat{v} := \rho_\delta * v$ with

$$\begin{aligned}
\hat{v}_0 &:= \rho_\delta * v_0, \\
\hat{\nu}_i &:= \rho_\delta * \nu_i, \\
\hat{w}_i &:= \rho_\delta * w_i - \nabla \cdot (\rho_\delta * z_i).
\end{aligned}$$

Note that obviously $\hat{v}_0 \in [L^1(\Omega_\delta)]^S$ and that $\hat{\nu}_i \in RM([0,T); L^1(\Omega_\delta))$; furthermore, we have $\hat{w}_i \in L^1([0,T); L^1(\Omega_\delta))$. Thus, Lemma 10 is indeed applicable. Therefore we get for any smooth ψ with $\psi \in C^\infty_{cpt}(\Omega_\delta \times [0,T))$

$$\begin{aligned}\Bigg| &-\int_0^T \int_\Omega \xi(\rho_\delta * v)\ \frac{d}{dt}\psi\ dx\ dt - \int_\Omega \xi(\rho_\delta * v_0)\ \psi(.,0)\ dx \\ &-\int_0^T \int_\Omega \sum_{i=1}^S \psi\ \partial_i \xi(\rho_\delta * v)\ (\rho_\delta * w_i)\ dx\ dt \\ &+\int_0^T \int_\Omega \sum_{i=1}^S \psi\ \partial_i \xi(\rho_\delta * v)\ \nabla\cdot(\rho_\delta * z_i)\ dx\ dt \Bigg| \\ &\leq ||\psi||_{L^\infty} \sup_u |D\xi(u)| \sum_{i=1}^S ||\nu_i||_{RM(\overline{\Omega}\times[0,T])}.\end{aligned}$$

Integrating by parts in the last integral, we deduce

$$\begin{aligned}\Bigg| &-\int_0^T \int_\Omega \xi(\rho_\delta * v)\ \frac{d}{dt}\psi\ dx\ dt - \int_\Omega \xi(\rho_\delta * v_0)\ \psi(.,0)\ dx \\ &-\int_0^T \int_\Omega \sum_{i=1}^S \psi\ \partial_i \xi(\rho_\delta * v)\ (\rho_\delta * w_i)\ dx\ dt \\ &-\int_0^T \int_\Omega \sum_{i=1}^S \partial_i \xi(\rho_\delta * v)\ (\rho_\delta * z_i)\cdot\nabla\psi\ dx\ dt \\ &-\int_0^T \int_\Omega \sum_{i=1}^S \sum_{k=1}^S \psi\ \partial_i\partial_k \xi(\rho_\delta * v)\ (\rho_\delta * z_i)\cdot\nabla(\rho_\delta * v_k)\ dx\ dt \Bigg| \\ &\leq ||\psi||_{L^\infty} \sup_u |D\xi(u)| \sum_{i=1}^S ||\nu_i||_{RM(\overline{\Omega}\times[0,T])}.\end{aligned}$$

Passing to the limit $\delta \to 0$, we infer the assertion of the lemma in case of $\psi \in C^\infty_{cpt}(\Omega \times [0,T))$.

It remains to consider the case when ψ is supported in some coordinate patch at the boundary. Let $x_0 \in \partial\Omega$. Since Ω has a Lipschitz boundary, we now have to treat the case that ψ is supported in the image U of an open subset V of $\mathbb{R}^+_0 \times \mathbb{R}^{d-1}$ under a bi-Lipschitz homeomorphism $\theta : V \to U$, where U satisfies $x_0 \in U$ and where after rotation and translation θ takes the form $\theta(x) = (x_1 + \eta(x_2,\ldots,x_d), x_2,\ldots,x_d)$; here, $\eta : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function. Thus, in particular we have $\det D\theta \equiv 1$ a.e..

Define $\Theta(x,t) := (\theta(x),t)$. Applying the change of variables θ to the equation in the assumptions of our lemma, we get for any $\psi \in C^\infty_{cpt}(\overline{\Omega} \times [0,T))$ with $\operatorname{supp}\psi \subset U \times [0,T)$

$$\begin{aligned}&-\int_0^T \int_V \tilde{v}_i \frac{d}{dt}\tilde{\psi}\ dx\ dt - \int_V (\tilde{v}_0)_i \tilde{\psi}(.,0)\ dx \\ =&\int_{V\times[0,T)} \tilde{\psi}\ d\tilde{\nu}_i + \int_0^T \int_V \tilde{w}_i \tilde{\psi}\ dx\ dt \\ &+\int_0^T \int_V \tilde{z}_i \cdot \nabla\tilde{\psi}\ dx\ dt + \int_0^T \int_{\theta^{-1}(\partial\Omega)} \tilde{q}_i\ \tilde{\psi}\ d\mathcal{H}^{d-1}\ dt,\end{aligned}$$

where we have defined

$$\begin{aligned}
\tilde{\psi} &:= \psi \circ \Theta,\\
\tilde{v} &:= v \circ \Theta,\\
\tilde{v}_0 &:= v_0 \circ \theta,\\
\tilde{w}_i &:= w_i \circ \Theta,\\
\tilde{q}_i(.,t) &:= (q_i \circ \Theta(.,t)) \cdot \sqrt{1+|D\eta|^2(.)},\\
\tilde{z}_i(.,t) &:= (D\theta(.))^{-1} z_i(\theta(.),t),\\
\tilde{\nu_i}(A) &:= \nu_i(\Theta(A)) \quad \text{for all Borel sets } A \subset V \times [0,T).
\end{aligned}$$

We set $V_m := \{(-x_1, x_2, \ldots, x_d) : x \in V\}$ and $V_a := V \cup V_m$. Then we extend the quantities $\tilde{v}_i$, $\tilde{v}_0$, $\tilde{w}_i$, and $\tilde{z}_i$ to V_a by mirroring on the plane $\operatorname{span}(\vec{e}_2, \ldots, \vec{e}_d)$ (note that the vector $\tilde{z}_i$ needs to be transformed accordingly). Define $\hat{\nu}_i(A) := \tilde{\nu}_i\big(A \cap (V \times [0,T))\big) + \tilde{\nu}_i\big(\{(-x_1, x_2, \ldots, x_d, t) : (x,t) \in A\} \cap (V \times [0,T))\big)$ for any $A \subset V_a \times [0,T)$.

As the previous equation is invariant with respect to mirroring, an analogous equation holds for the extended quantities and any smooth $\tilde{\psi}$ supported in $V_a \times [0,T)$ (note that however the integrals over the hyperplane $\operatorname{span}(e_2, \ldots, e_d)$ now appear twice and that $\tilde{\nu}_i$ is replaced by $\hat{\nu}_i$).

Let ρ_δ denote a mollifier with respect to space; set $V_a^\delta := \{x \in V_a : \operatorname{dist}(x, \partial V_a) > \delta\}$. For any $\tilde{\psi} \in C^\infty_{cpt}(V_a^\delta \times [0,T))$, we may test the equation for the extended $\tilde{v}_i$ with $\rho_\delta * \tilde{\psi}$, which yields using the symmetry of ρ_δ and integration by parts in the term involving $\tilde{z}_i$

$$\begin{aligned}
&-\int_0^T \int_{V_a} (\rho_\delta * \tilde{v}_i) \frac{d}{dt}\tilde{\psi}\, dx\, dt - \int_{V_a} (\rho_\delta * \tilde{v}_0)_i \tilde{\psi}(.,0)\, dx\\
=&\int_{V_a \times [0,T)} \tilde{\psi}\, d(\rho_\delta * \hat{\nu}_i) + \int_0^T \int_{V_a} \tilde{\psi}\, (\rho_\delta * \tilde{w}_i)\, dx\, dt\\
&-\int_0^T \int_{V_a} \tilde{\psi} \operatorname{div}(\rho_\delta * \tilde{z}_i)\, dx\, dt + 2\int_0^T \int_{V_a} \tilde{\psi}\, \big(\rho_\delta * (\tilde{q}_i \mathcal{H}^{d-1}|_{\theta^{-1}(\partial\Omega)})\big)\, dx\, dt.
\end{aligned}$$

We now intend to apply Lemma 10 below. To do so, we choose V_a^δ as our domain and

$$\begin{aligned}
\hat{v}_i &:= \rho_\delta * \tilde{v}_i,\\
\hat{v}_0 &:= \rho_\delta * \tilde{v}_0,\\
\hat{w}_i &:= \rho_\delta * \tilde{w}_i - \operatorname{div}(\rho_\delta * \tilde{z}_i) + 2\rho_\delta * (\tilde{q}_i \mathcal{H}^{d-1}|_{\theta^{-1}(\partial\Omega)}),\\
\hat{\hat{\nu}}_i &:= \rho_\delta * \hat{\nu}_i.
\end{aligned}$$

We immediately see that $\hat{v}_i \in L^1([0,T); L^1(V_a^\delta))$, that $\hat{v}_0 \in [L^1(V_a^\delta)]^S$, that $\hat{w}_i \in L^1([0,T); L^1(V_a^\delta))$ and that $\hat{\hat{\nu}}_i \in RM([0,T); L^1(V_a^\delta))$. Thus, Lemma 10 is indeed applicable. We thus infer for any $\tilde\psi \in C^\infty_{cpt}(V_a^\delta \times [0,T))$

$$\begin{aligned}
&\bigg| - \int_0^T \int_{V_a} \xi(\rho_\delta * \tilde v) \frac{d}{dt}\tilde\psi \; dx \; dt - \int_{V_a} \xi(\rho_\delta * \tilde v_0)\tilde\psi(.,0) \; dx \\
&\quad - \int_0^T \int_{V_a} \sum_{i=1}^S \tilde\psi \; \partial_i\xi(\rho_\delta * \tilde v) \; (\rho_\delta * \tilde w_i) \; dx \; dt \\
&\quad + \int_0^T \int_{V_a} \sum_{i=1}^S \tilde\psi \; \partial_i\xi(\rho_\delta * \tilde v) \; \operatorname{div}(\rho_\delta * \tilde z_i) \; dx \; dt \\
&\quad - 2\int_0^T \int_{V_a} \sum_{i=1}^S \tilde\psi \; \partial_i\xi(\rho_\delta * \tilde v) \; \big(\rho_\delta * (\tilde q_i \mathcal{H}^{d-1}|_{\theta^{-1}(\partial\Omega)})\big) \; dx \; dt \bigg| \\
&\leq 2||\tilde\psi||_{L^\infty} \sup_u |D\xi(u)| \sum_{i=1}^S \; |\tilde\nu_i|(V \times [0,T)).
\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}
&\bigg| - \int_0^T \int_{V_a} \xi(\rho_\delta * \tilde v) \frac{d}{dt}\tilde\psi \; dx \; dt - \int_{V_a} \xi(\rho_\delta * \tilde v_0)\tilde\psi(.,0) \; dx \\
&\quad - \int_0^T \int_{V_a} \sum_{i=1}^S \tilde\psi \; \partial_i\xi(\rho_\delta * \tilde v) \; (\rho_\delta * \tilde w_i) \; dx \; dt \\
&\quad - \int_0^T \int_{V_a} \sum_{i=1}^S \partial_i\xi(\rho_\delta * \tilde v) \; (\rho_\delta * \tilde z_i) \cdot \nabla\tilde\psi \; dx \; dt \\
&\quad - \int_0^T \int_{V_a} \sum_{i,k=1}^S \tilde\psi \; \partial_i\partial_k\xi(\rho_\delta * \tilde v) \; (\rho_\delta * \tilde z_i) \cdot \nabla(\rho_\delta * \tilde v_k) \; dx \; dt \\
&\quad - 2\int_0^T \int_{\theta^{-1}(\partial\Omega)} \sum_{i=1}^S \tilde q_i \; \Big(\rho_\delta * \Big(\partial_i\xi(\rho_\delta * \tilde v) \; \tilde\psi\Big)\Big) \; d\mathcal{H}^{d-1} \; dt \bigg| \\
&\leq 2||\tilde\psi||_{L^\infty} \sup_u |D\xi(u)| \sum_{i=1}^S \; |\nu_i|(\overline{\Omega} \times [0,T)).
\end{aligned}$$

Let $\tilde\psi \in C^\infty_{cpt}(V_a^\delta \times [0,T))$ be a test function which is symmetric with respect to the hyperplane $\operatorname{span}(\vec{e_2}, \ldots, \vec{e_d})$. Passing to the limit $\delta \to 0$ and noting that the resulting integrals over V_a are equal to twice the integral over V, we end up with

$$\begin{aligned}
&\Bigg| -\int_0^T \int_V \xi(\tilde v) \frac{d}{dt}\tilde\psi \; dx\; dt - \int_V \xi(\tilde v_0)\tilde\psi(.,0)\; dx \\
&\quad -\int_0^T \int_V \sum_{i=1}^S \tilde\psi\; \partial_i \xi(\tilde v)\; \tilde w_i \; dx\; dt \\
&\quad -\int_0^T \int_V \sum_{i=1}^S \partial_i \xi(\tilde v)\; \tilde z_i \cdot \nabla\tilde\psi \; dx\; dt \\
&\quad -\int_0^T \int_V \sum_{i,k=1}^S \tilde\psi\; \partial_i\partial_k \xi(\tilde v)\; \tilde z_i \cdot \nabla \tilde v_k \; dx\; dt \\
&\quad -\int_0^T \int_{\theta^{-1}(\partial\Omega)} \sum_{i=1}^S \tilde q_i\; \partial_i \xi(\tilde v)\; \tilde\psi\; d\mathcal{H}^{d-1}\; dt \Bigg| \\
\le & ||\tilde\psi||_{L^\infty} \sup_u |D\xi(u)| \sum_{i=1}^S \; |\nu_i|(\overline\Omega\times[0,T)).
\end{aligned}$$

Note that for any $W \subset\subset V_a$ one can show that $\tilde\psi\partial_i\xi(\rho_\delta * \tilde v)$ strongly converges in $L^2([0,T];H^1(W))$ as $\delta\to 0$; thus, $\rho_\delta * (\tilde\psi\partial_i\xi(\rho_\delta * \tilde v))$ also converges strongly in $L^2([0,T];H^1(W))$ for any $W\subset\subset V_a$. By an interpolation-trace inequality like (16), strong convergence of $\rho_\delta * (\partial_i\xi(\rho_\delta * \tilde v)\tilde\psi)$ in $L^2([0,T];L^2(\theta^{-1}(\partial\Omega)\cap W))$ follows for any open set $W\subset\subset V_a$; by boundedness of $\partial_i\xi$ and $\tilde\psi$ and compact support of $\tilde\psi$, this implies convergence of the integral over $\theta^{-1}(\partial\Omega)$ as $\delta\to 0$ using dominated convergence.

By a standard approximation argument (extension by reflection and subsequent mollification), this estimate also holds for general test functions $\tilde\psi = \psi\circ\Theta$ with $\psi\in C^{0,1}_{cpt}(U\times[0,T))$. Reversing the change of variables, our lemma is proven. □

Lemma 10. *Let Ω be a bounded domain. Assume that $T>0$ and that $v\in L^1([0,T);[L^1(\Omega)]^S)$; suppose that $v_0\in(L^1(\Omega))^S$. Let $\nu_i\in RM([0,T);L^1(\Omega))$, $1\le i\le S$, be L^1-valued Radon measures (which may be considered as Radon measures on $\Omega\times[0,T)$) and let $w_i\in L^1([0,T);L^1(\Omega))$ be functions such that we have for any $\psi\in C^\infty_{cpt}(\Omega\times[0,T))$*

$$\begin{aligned}
&-\int_0^T\int_\Omega v_i(.,t)\frac{d}{dt}\psi(.,t)\; dx\; dt - \int_\Omega (v_0)_i(.)\psi(.,0)\; dx \\
=&\int_{\Omega\times[0,T)} \psi\; d\nu_i + \int_0^T\int_\Omega w_i\psi\; dx\; dt\; .
\end{aligned}$$

Let $\xi:\mathbb{R}^S\to\mathbb{R}$ be a bounded continuous function with continuous bounded derivatives. Then we have for all $\psi\in C^\infty_{cpt}(\Omega\times[0,T))$

$$\begin{aligned}
&\Bigg| -\int_0^T\int_\Omega \xi(v(.,t))\frac{d}{dt}\psi(.,t)\; dx\; dt - \int_\Omega \xi(v_0(.))\psi(.,0)\; dx \\
&\quad -\sum_{i=1}^S\int_0^T\int_\Omega \psi(.,t)\; \partial_i\xi(v(.,t))w_i\; dx\; dt\Bigg| \\
\le & ||\psi||_{L^\infty}(\sup_u |D\xi(u)|)\cdot\sum_{i=1}^S ||\nu_i||_{RM(\Omega\times[0,T))}\; .
\end{aligned}$$

Proof. We extend v, w_i, ν_i to $(-T,T)$ by defining $w_i(x,t)=0$ for $t<0$, $\nu_i(A):=\nu_i(A\cap[0,T))$ for $A\subset(-T,T)$, $v(x,t)=v_0(x)$ for $t<0$. We then have

$$-\int_{-T}^{T}\int_\Omega v_i(.,t)\frac{d}{dt}\psi(.,t)\;dx\;dt$$
$$=\int_{\Omega\times(-T,T)}\psi\;d\nu_i+\int_{-T}^{T}\int_\Omega w_i\psi\;dx\;dt$$

for any $\psi\in C^\infty_{cpt}(\Omega\times(-T,T))$ since $\int_{-T}^0\int_\Omega v_i(.,t)\frac{d}{dt}\psi(.,t)\;dx\;dt=\int_\Omega(v_0)_i(.)\psi(.,0)\;dx$ (as $v(.,t)=v_0(.)$ for $t\leq 0$).

Let ρ_δ denote a usual mollifier *with respect to time*. Using $\rho_\delta*\psi$ as a test function, due to the symmetry of ρ_δ we get for any $\psi\in C^\infty_{cpt}(\Omega\times(-T+2\delta,T-2\delta))$

$$-\int_{-T}^{T}\int_\Omega(\rho_\delta*v_i(.,t))\;\frac{d}{dt}\psi(.,t)\;dx\;dt$$
$$=\int_{\Omega\times(-T,T)}\rho_\delta*\psi\;d\nu_i+\int_{-T}^{T}\int_\Omega(\rho_\delta*w_i)\;\psi\;dx\;dt\;.$$

This implies $\rho_\delta*v\in C^1((-T+\delta,T-\delta);L^1(\Omega))$; moreover, we obtain the representation $\frac{d}{dt}(\rho_\delta*v_i)(.,t)=(\rho_\delta*w_i)(.,t)+(\rho_\delta*\nu_i)(.,t)$ (the equality holding in the $L^1(\Omega)$ sense). By the chain rule, we get for any $\psi\in C^\infty_{cpt}(\Omega\times(-T+2\delta,T-2\delta))$

$$-\int_{-T}^{T}\int_\Omega\xi((\rho_\delta*v)(.,t))\;\frac{d}{dt}\psi(.,t)\;dx\;dt$$
$$=\sum_{i=1}^S\int_{\Omega\times(-T,T)}\partial_i\xi(\rho_\delta*v)\;\psi\;d(\rho_\delta*\nu_i)+\sum_{i=1}^S\int_{-T}^{T}\int_\Omega\partial_i\xi(\rho_\delta*v)\;\psi\;(\rho_\delta*w_i)\;dx\;dt\;.$$

Passing to the limit $\delta\to 0$, we deduce for any $\psi\in C^\infty_{cpt}(\Omega\times(-T,T))$

$$\left|-\int_{-T}^{T}\int_\Omega\xi(v(.,t))\frac{d}{dt}\psi(.,t)\;dx\;dt-\sum_{i=1}^S\int_{-T}^{T}\int_\Omega\partial_i\xi(v)\;\psi\;w_i\;dx\;dt\right|$$
$$\leq||\psi||_{L^\infty}\cdot\sup_{i,u}|\partial_i\xi(u)|\;\cdot\sum_{i=1}^S||\nu_i||_{RM(\Omega\times(-T,T))}\;.$$

This implies the lemma since $\int_{-T}^0\int_\Omega\xi(v)\frac{d}{dt}\psi(.,t)\;dx\;dt=\int\xi(v_0)\psi(.,0)\;dx$ (recall that $v(.,t)=v_0(.)$ for $t\leq 0$) and since $w_i(.,t)=0$ for $t<0$. □

4. Conclusion and Open Problems

We have introduced a notion of renormalized solutions for reaction-diffusion-advection equations with entropy-dissipating reaction rates and proven existence of such renormalized solutions for initial data in the $L\log L$ Orlicz class.

However, there are numerous related problems which remain open. In particular, the following questions remain unsolved:

- Does our notion of solution guarantee uniqueness? Can a stronger notion of solution be defined sufficient for proving uniqueness? This is probably a difficult problem, as already in the case of solutions which satisfy $R(u)\in L^1$, but $u\notin L^\infty$, the question seems to be largely open (see the overview by Pierre [34]).

- In case that the equation may be regarded formally as a gradient flow of the entropy, is it possible to construct renormalized solutions using some kind of (Otto calculus-like) steepest-descent scheme, i.e. is it possible to make the formal considerations by Mielke [28] rigorous?
- Can we prove convergence of solutions to a steady state? Does $R(u)$ belong to some L^p space for large times, at least in case of no-flux boundary conditions?
- Is there a similar notion of solution in case we do not have the entropy inequality, but just the dissipation of mass condition? In this case, it seems to be difficult to give a meaning to the term $(A_i \nabla u_i) \cdot \nabla u_j$ in (9) as the dissipation of mass condition does not yield any control on the spatial derivatives.
- Is it possible to construct solutions with weak initial trace, i.e. measure-valued initial data?
- Can the existence result be extended to Dirichlet boundary conditions? To provide a positive answer to this question, one would first need to find a way to control the evolution of the total entropy in case of Dirichlet boundary conditions. Apart from Dirichlet boundary conditions corresponding to an equilibrium state of the reactions or certain special reaction terms, in general this issue seems to be an open problem.

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References

[1] Radjesvarane Alexandre. A definition of renormalized solutions for Boltzmann equation without cutoff. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(11):987–991, 1999.

[2] Radjesvarane Alexandre and Cedric Villani. On the Landau approximation in plasma physics. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(1):61–95, 2004.

[3] Hans Wilhelm Alt and Stephan Luckhaus. Quasilinear Elliptic-Parabolic Differential Equations. *Math. Z.*, 183:311–341, 1983.

[4] Dieter Bothe and Michel Pierre. Quasi-steady-state approximation for a reaction-diffusion system with fast intermediate. *J. Math. Anal. Appl.*, 368(1):120–132, 2010.

[5] Jose A. Canizo, Laurent Desvillettes, and Klemens Fellner. Improved duality estimates and applications to reaction-diffusion equations. *Comm. Partial Differential Equations*, 39:1185–1204, 2014.

[6] M. Cristina Caputo and Alexis Vasseur. Global regularity of solutions to systems of reaction-diffusion with sub-quadratic growth in any dimension. *Comm. Partial Differential Equations*, 34:1228–1250, 2009.

[7] Gianni Dal Maso, Francois Murat, Luigi Orsina, and Alain Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28:741–808, 1999.

[8] Laurent Desvillettes and Klemens Fellner. Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations. *J. Math. Anal. Appl.*, 319:157–176, 2006.

[9] Laurent Desvillettes and Klemens Fellner. Entropy methods for reaction-diffusion equations with degenerate diffusion arising in reversible chemistry. *accepted for the Proceedings of the Equadiff*, 2007.

[10] Laurent Desvillettes and Klemens Fellner. Entropy methods for reaction-diffusion systems. *Discrete Contin. Dyn. Syst. Suppl.*, 24:304–312, 2007.

[11] Laurent Desvillettes and Klemens Fellner. Entropy methods for reaction-diffusion equations: slowly growing a-priori bounds. *Rev. Mat. Iberoamericana*, 24:407–431, 2008.

[12] Laurent Desvillettes, Klemens Fellner, Michel Pierre, and Julien Vovelle. About Global Existence for Quadratic Systems of Reaction-Diffusion. *Adv. Nonlinear Stud.*, 7:491–511, 2007.

[13] J. Ildefonso Diaz and Laurent Veron. Local vanishing properties of solutions of elliptic and parabolic quasilinear equations. *Trans. Amer. Math. Soc.*, 290(2):787–814, 1985.

[14] R.J. DiPerna and P.-L. Lions. On the Fokker-Planck-Boltzmann equation. *Commun. Math. Phys.*, 120:1–23, 1988.

[15] R.J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math.*, 130(2):321–366, 1989.

[16] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98:511–517, 1989.

[17] Martin Feinberg and F.J.M. Horn. Chemical mechanism structure and the coincidence of the stoichiometric and kinetic subspaces. *Arch. Ration. Mech. Anal*, 66:83–97, 1977.

[18] André Fiebach, Annegret Glitzky, and Alexander Linke. Uniform global bounds for solutions of an implicit voronoi finite volume method for reaction-diffusion problems. *Numer. Math.*, 128(1):31–72, 2014.

[19] H. Gajewski and K. Gröger. Reaction-diffusion processes of electrically charged species. *Math. Nachr.*, 177(1):109–130, 1996.

[20] H. Gajewski and I.V. Skrypnik. Existence and uniqueness results for reaction-diffusion processes of electrically charged species. In *Nonlinear elliptic and parabolic problems*, volume 64 of *Prog. Nonlinear Differential Equations Appl.*, pages 151–188. Birkhäuser, Basel, 2005.

[21] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Springer, 2001.

[22] Annegret Glitzky and R. Hünlich. Global estimates and asymptotics for electro-reaction-diffusion systems in heterostructures. *Applicable Analysis*, 66(3-4):205–225, 1997.

[23] Annegret Glitzky and Alexander Mielke. A gradient structure for systems coupling reaction-diffusion effects in bulk and interfaces. *Z. Angew. Math. Phys.*, 64(1):29–52, 2013.

[24] Thierry Goudon and Alexis Vasseur. Regularity analysis for systems of reaction-diffusion equations. *Ann. Sci. Éc. Norm. Supér.*, 368:120–132, 2010.

[25] R. Jordan, D. Kinderlehrer, and F. Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Anal.*, 29(1):1–17, 1998.

[26] Serge Kräutle. Existence of Global Solutions of Multicomponent Reactive Transport Problems with Mass Action Kinetics in Porous Media. *Journal of Applied Analysis and Computation*, 1(4):497–515, 2011.

[27] Matthias Liero and Alexander Mielke. Gradient structures and geodesic convexity for reaction-diffusion systems. *Phil. Trans. R. Soc. A*, 371:20120346, 2013.

[28] Alexander Mielke. A gradient structure for reaction-diffusion systems and for energy-drift-diffusion systems. *Nonlinearity*, 24:1329–1346, 2011.

[29] Alexander Mielke, Jan Haskovec, and Peter Markowich. On uniform decay of the entropy for reaction-diffusion systems. *J. Dynam. Differential Equations*, in press, DOI:10.1007/s10884-014-9394-x, 2014.

[30] Maya Mincheva and Didier Siegel. Stability of Mass Action Reaction-Diffusion Systems. *Nonlinear Anal.*, 56(8):1105–1131, 2004.

[31] F. Murat. Solutiones renormalizadas de EDP elipticas non lineares. *Technical report R93023*, 1993.

[32] F. Otto. Dynamics of labyrinthine pattern formation in magnetic fluids: A mean-field theory. *Arch. Ration. Mech. Anal.*, 141(1):63–103, 1998.

[33] Michel Pierre. Weak Solutions and Supersolutions in L^1 for Reaction-Diffusion Systems. *J. Evol. Equ.*, 3:153–168, 2003.

[34] Michel Pierre. Global Existence in Reaction-Diffusion Systems with Control of Mass: a Survey. *Milan J. Math.*, 78(2):417–455, 2010.

[35] Michel Pierre and Didier Schmitt. Blow-up in Reaction-Diffusion Systems with Dissipation of Mass. *SIAM J. Math. Anal.*, 28:259–269, 1997.

[36] Michel Pierre and Didier Schmitt. Blow-up in Reaction-Diffusion Systems with Dissipation of Mass. *SIAM Review*, 42:93–106, 2000.

[37] Stefan Schuster and Ronny Schuster. A generalization of Wegscheiders condition. Implications for properties of steady states and for quasi-steady-state approximation. *J. Math. Chem.*, 3:25–42, 1989.

[38] Jacques Simon. Compact Sets in the Space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146(1):65–96, 1986.

[39] Cedric Villani. On the Cauchy problem for the Landau equation: sequential stability, global existence. *Adv. Differential Equations*, 1(5):793–816, 1996.

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