FINITE SPEED OF PROPAGATION AND WAITING TIMES FOR THE STOCHASTIC POROUS MEDIUM EQUATION – A UNIFYING APPROACH

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Abstract. In this paper, we develop an energy method to study finite speed of propagation and waiting time phenomena for the stochastic porous-media equation with linear multiplicative noise in up to three spatial dimensions. Based on a novel iteration technique and on stochastic counterparts of weighted integral estimates used in the deterministic setting, we formulate a sufficient criterion on the growth of initial data which locally guarantees a waiting time phenomenon to occur almost surely. Up to a logarithmic factor, this criterion coincides with the optimal criterion known from the deterministic setting. Our technique can be modified to prove finite speed of propagation as well.

1. Introduction

Finite speed of propagation and occurrence of waiting time phenomena are well-known features of degenerate parabolic equations. To put it concisely, the former notion means that for each $x_0 \notin \operatorname{supp} u_0$ a positive time $T(x_0)$ exists such that $x_0 \notin \operatorname{supp} u(\cdot, t)$ for all $t \in [0, T(x_0))$. We say a solution u to exhibit a waiting time phenomenon if for some positive time the solution's support locally does not expand, or shrink, or both, at a point x_0 on $\partial \operatorname{supp} u_0$. In [1, 3, 9, 14, 18, 21, 24], sufficient conditions have been identified to guarantee the occurrence of waiting time phenomena for various degenerate parabolic partial differential equations. In general, these conditions impose specific requirements on the flatness of initial data in a neighbourhood of x_0.
It is an intriguing question whether similar phenomena can also be observed for stochastic degenerate parabolic partial differential equations. Barbu, Röckner [7] and Gess [13] have been the first to obtain results in that respect. They prove qualitative results on finite speed of propagation for the stochastic porous media equation

$$\begin{aligned} du = \Delta(|u|^{m-1}u)\ dt + \sigma(u)dW_t\ , \qquad & m > 1 \\ u = 0 \qquad & \text{on } \partial\mathcal{O} \\ u(\cdot,0) = u_0 \text{ in } & \mathcal{O} \subset \mathbb{R}^d. \end{aligned} \tag{1.1}$$

Barbu and Röckner confine themselves to the regime $m \in (1, 5]$ and they assume the noise to be given by

$$\sigma(u)dW_t := \sum_{k=1}^{N} \mu_k e_k u d\beta_k(t)$$

where $\{\mu_1, \cdots, \mu_N\}$ are given nonnegative numbers, $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{O})$, and $\{\beta_k\}_{k=1}^N$ are mutually independent Brownian motions. Gess considers

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Stratonovich noise given by

$$\sum_{k=1}^{N} f_k u \circ dz_t^{(k)}, \quad N \leq \infty$$

with driving terms $z^{(k)} \in C([0,T];\mathbb{R})$ and diffusion coefficients $f_k \in C^\infty(\bar{\mathcal{O}})$. Analytically both papers rely on a random change of variables to transform the stochastic porous media equation into a deterministic degenerate parabolic equation with random coefficients. Barbu and Röckner obtain qualitative results by adapting the local energy method of Antontsev [2] to the stochastic setting. Gess [13] modifies the hole-filling technique presented in [27] (in particular Lemma 14.5 therein) to formulate quantitative results on finite speed of propagation. Concerning optimality, it is worth mentioning that Lemma 14.5 in [27] in general only gives a lower bound for the radius $R(t)$ of the ball in which $u(\cdot,t)$ vanishes. The lower bound is also an upper bound in the very special case that initial data u_0 are zero on a ball contained in $\mathcal{O}$ and that they attain only one value different from zero on the rest of the domain. As soon as a waiting time phenomenon occurs, this lower bound is not optimal – see also Remark (2), page 339 in [27]. Similarly, it suggests that $\operatorname{supp} u_0 \subset B_{R_0}(x_0)$ implies that $\operatorname{supp} u(.,t) \subset B_{R(t)}(x_0)$ for any $t > 0$, where

$$R(t) \leq R_0 + C(d, m, ||u(.,.,\omega)||_{L^\infty(\mathcal{O}\times[0,t])})t^{\frac{1}{2}} \ .$$

For the deterministic porous medium equation, however, it is well known that $\operatorname{supp} u_0 \subset B_{R_0}(x_0)$ implies $\operatorname{supp} u(.,t) \subset B_{R(t)}(x_0)$ for any $t > 0$, where

$$R(t) \leq R_0 + C(d, m, ||u_0||_{L^1})t^{\frac{1}{2+d(m-1)}} \ .$$

Concerning the occurrence of waiting time phenomena for stochastic degenerate parabolic equations, to the best of our knowledge, nothing is known so far. To give a flavour of the results to come, let us formulate a toy version in one spatial dimension of our main result on waiting times, which is Theorem 3.3.

A toy result on waiting times. *For $a \in \mathbb{R}^+$, let $\mathcal{O} := (-a, a)$. Assume u to be a strong solution to the stochastic porous medium equation* (1.1) *in the sense of Definition 2.2 with linear multiplicative noise in the sense of equation* (2.5)*. Suppose initial data $u_0 \in C_0^0(\mathcal{O})$ to satisfy the growth condition*

$$|u_0(x)| \leq S(x)_+^\gamma \tag{1.2}$$

for a positive constant S and an arbitrary, but fixed parameter $\gamma > \frac{2}{m-1}$.
Then, there exists an almost surely positive stopping time $T_{S,\gamma}$ such that

$$u(\cdot, t, \omega)\,\big|_{(-a,0]} \equiv 0 \tag{1.3}$$

on $[0, T_{S,\gamma}(\omega)]$ for all $\omega \in \Omega$.

Note that the parameter γ may be chosen arbitrarily close to the corresponding parameter in the deterministic setting which is given by $\gamma_{det} := \frac{2}{m-1}$ and which is known to be optimal. In particular, this result is an easy consequence of Theorem 3.3 as the assumptions presented here imply those of Theorem 3.3 to be true.

It is the scope of the present paper to adapt ideas to the stochastic setting which are known to entail optimal results in the deterministic setting both with regard to finite speed of propagation and to the occurrence of waiting time phenomena. To describe the general strategy, it is worth having first a closer look at the deterministic case. It seems crucial to gain information on roots of the function

$$G(T,r) := \int_0^T \int_{\mathbb{R}^d \setminus B_r(x_0)} u^{2m}\, dx\, dt\ .$$

In [21], Dal Passo, Giacomelli, and Grün proposed to combine weighted energy estimates and interpolation inequalities to obtain

$$G(T,R) \leq \frac{C T^{\frac{2(m+1)}{d(m-1)+2(m+1)}}}{(R-r)^{\frac{2d(m-1)+8m}{d(m-1)+2(m+1)}}} \left(G(T,r) + (R_0 - r)^{2+d+\gamma(m+1)} \right)^{\frac{d(m-1)+4m}{d(m-1)+2(m+1)}}$$

for any $R > r > 0$ with $\operatorname{supp} u_0 \subset B_r(x_0)$. By an application of appropriate versions of an iteration lemma by Stampacchia (see [21, 25]), it is possible to identify times $T = T(r)$ such that $G(T(r), r) = 0$. In the stochastic setting, however, no growth condition can be formulated for $G(\cdot,\cdot)$ which would hold uniformly in $\omega \in \Omega$. Hence, we focus on distribution functions of free energy and dissipated energy to "filter out" events for which the growth of $G(\cdot,\cdot)$ exceeds a predefined range. In Section 4, we rigorously derive a new integral estimate which involves weight functions and which serves as the stochastic counterpart of the weighted energy estimate known from the deterministic setting. Formally, this estimate would follow from Itô's formula applied to an appropriate energy functional. Technically, we combine results on higher regularity of solutions to the stochastic porous media equation by Gess [12] with a convolution argument which states the approximability of $|u|^{m-1}u \in H^1(\mathbb{R}^d)$ by convolutions $|\rho_\delta * u|^{m-1}\rho_\delta * u$ with respect to the $H^1(\mathbb{R}^d)$-norm, see Lemma 4.1. Here, ρ_δ is a standard smooth mollifier with δ being a positive parameter supposed to tend to zero.

In Section 5, the newly derived Itô-type formula and Markov's inequality are used to bound distributions functions of appropriate versions of free energy and of dissipated energy via estimates on

$$\mathbb{P}\left(\sup_{t\in[0,T_A]} \int_{B_r(x_0)} |u|^{m+1}(.,t)\, dx > \tau \right)$$

and on

$$\mathbb{P}\left(\int_0^{T_A} \int_{\mathcal{O}} |\nabla(|u|^{m-1}u\phi_{r,R,x_0})|^2\, dx\, dt > \tau \right).$$

Finally, Section 6 is devoted to the proof of our results on finite speed of propagation and on occurrence of a waiting time phenomenon which will have been formulated and motivated already in Section 3. We define an appropriate sequence of stopping times which permits to filter out those events $\omega \in \Omega$ which come along with too strong a growth behaviour of $G(T,r)$.

It is worth mentioning that our criterion on the occurrence of waiting times coincides up to a logarithmic correction term with the corresponding criterion in the deterministic case which is known to be optimal with respect to scaling. For more details, see Remark 3.5.

In Section 2, we make our assumption on the noise precise – note that we consider linear multiplicative noise driven by appropriate Q-Wiener processes. We apply conditions on the regularity of the noise which are inherited by the assumptions Gess formulated in [12] to establish higher regularity for solutions of the stochastic porous-media equation. It is worth mentioning that these regularity properties are only needed for the proof of the Itô-type formula. The subsequent results on front propagation can be established with the weaker regularity assumptions on the noise as they were specified by Barbu and

Röckner in the existence and nonnegativity result of [4]. Section 2 also introduces the precise notions of finite speed of propagation and waiting time phenomena which will be used in the paper. Moreover, an overview on existence and nonnegativity results is given for the special version of stochastic porous media equations considered in this article.
Throughout the paper, we use standard notation for Sobolev spaces and from stochastic analysis. Sometimes, we abbreviate $I := [0, T)$ and we write $B_\delta(A)$ for the set $\{x \in \mathbb{R}^d : \operatorname{dist}(x, A) < \delta\}$ where $A \subset \mathbb{R}^d$ is given. By $a \wedge b$ and $a \vee b$ we denote the minimum and the maximum of a and b, respectively. The notation $C_0^\infty(\mathcal{O})$ is used for the class of smooth compactly supported functions on $\mathcal{O}$, and $L_2(X; Y)$ denotes the set of Hilbert-Schmidt operators from X to Y.

2. Preliminaries

We assume $\mathcal{O}$ to be a bounded domain in $\mathbb{R}^d$, $d \leq 3$, with boundary of class C^2. Denoting by $(e_k)_{k\in\mathbb{N}}$ an $L^2(\mathcal{O})$-complete orthonormal system of eigenfunctions of the negative Laplacian under homogeneous Dirichlet conditions corresponding to positive eigenvalues σ_k, $k \in \mathbb{N}$, we recall that $e_k \in W^{2,2}(\mathcal{O})$. Without loss of generality, we assume the sequence $(\sigma_k)_{k\in\mathbb{N}}$ to be monotonically increasing. By Sobolev imbedding, we have

$$\|e_k\|_{L^\infty(\mathcal{O})} \leq C \|e_k\|_{W^{2,2}(\mathcal{O})} \leq C\sigma_k. \tag{2.1}$$

For a given sequence $(\mu_k)_{k\in\mathbb{N}}$ of nonnegative numbers satisfying

$$\sum_{k\in\mathbb{N}} (\mu_k\sigma_k)^{\frac{4}{m+1}} < \infty, \tag{2.2}$$

we define the operator $Q : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ as

$$Qh := \sum_{k\in\mathbb{N}} \mu_k^2 (h, e_k)_{L^2(\mathcal{O})} e_k. \tag{2.3}$$

We consider a Wiener process in $L^2(\mathcal{O})$ of the form

$$W(t) := \sum_{k=1}^{\infty} \beta_k(t) Q^{1/2} e_k, \qquad t \geq 0, \tag{2.4}$$

where $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Recall that a normal filtration is a filtration which satisfies

- $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ and
- for all $t \geq 0$ we have $A \in \mathcal{F}_t$ for any $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$.

We note, e.g. from [23], Remark 2.5.1, Prop. 2.5.2 and Section 2.5.2, that the stochastic integral $\int_0^T \phi(t) dW(t)$ is well-defined in $H^{-1}(\mathcal{O})$ for a process $\phi(t)$, $t \in [0, T]$, if ϕ takes values in $L_2(Q^{1/2}(L^2(\mathcal{O})); H^{-1}(\mathcal{O}))$, if ϕ is predictable and if

$$\mathbb{P}\left(\int_0^T \|\phi(s)\|_{L_2(Q^{1/2}(L^2(\mathcal{O})); H^{-1}(\mathcal{O}))} ds < \infty\right) = 1.$$

Introducing the operator

$$\sigma : H^{-1}(\mathcal{O}) \to L_2(Q^{1/2}(L^2(\mathcal{O})); H^{-1}(\mathcal{O}))$$

with

$$\sigma(v)(h) := vh \qquad \in H^{-1}(\mathcal{O}) \tag{2.5}$$

for all $h \in Q^{1/2}(L^2(\mathcal{O}))$, we immediately verify the following Lemma – using in particular (2.2) together with the estimate

$$\|ve_k\|_{H^{-1}} \leq C\sigma_k \|v\|_{H^{-1}} \tag{2.6}$$

which easily follows by Poincaré's inequality and integration by parts.

Lemma 2.1. *Let $d \leq 3$. Then $\sigma(v) \in L_2(Q^{1/2}(L^2(\mathcal{O})); H^{-1}(\mathcal{O}))$. In particular, there exists a positive constant C such that*

$$\|\sigma(v)\|_{L_2(Q^{1/2}(L^2(\mathcal{O}));H^{-1}(\mathcal{O}))} \leq C \|v\|_{H^{-1}(\mathcal{O})} . \tag{2.7}$$

Now, we are in the position to specify the noise term $\sigma(v)dW(t)$. It is given by

$$\sigma(v(t))dW(t) := \sum_{k=1}^{\infty} \mu_k e_k v d\beta_k. \tag{2.8}$$

¿From Lemma 2.1, we infer that the stochastic integral $\int_0^T \sigma(v(t))dW(t)$ is well-defined for $v \in L^1(\Omega; L^2((0,T); H^{-1}(\mathcal{O})))$ with a.s. continuous paths. Concerning predictability, we refer to [23], p.28.

The stochastic porous-media equation (1.1) has been studied in various publications (see [5, 12, 17] and the references therein). Let us first introduce the concept of strong solutions we will work with in the sequel.

Definition 2.2. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space with normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, assume $\mathcal{O} \subset \mathbb{R}^d$, $d \leq 3$, to be a bounded domain with boundary of class C^2 and let $m > 1$. For given $u_0 \in L^{m+1}(\mathcal{O})$, an $H^{-1}(\mathcal{O})$-valued continuous $\mathcal{F}_t$-adapted process $u \in L^2(\Omega; C((0,T); H^{-1}(\mathcal{O})))$ is called a strong solution to the stochastic porous-media equation with initial data u_0, if*

$$\sup_{t\in[0,T]} \mathbb{E} \|u(t)\|^{m+1}_{L^{m+1}(\mathcal{O})} + \mathbb{E} \int_0^T \left\||u|^{m-1}u\right\|^2_{H_0^1(\mathcal{O})} dt < \infty \tag{2.9}$$

is satisfied and if the identity

$$u(t) = u_0 + \int_0^t \Delta(|u|^{m-1}u)\ ds + \int_0^t \sigma(u(s))\ dW_s$$

holds a.s. for arbitrary $t, T > 0$.

Concerning existence and regularity of solutions, the following results hold.

Theorem 2.3. *Assume that $d \leq 3$ and that $\sum_{k\in\mathbb{N}} \mu_k^2\sigma_k^2 < \infty$. Then, for each $u_0 \in H^{-1}(\mathcal{O})$, the equation*

$$du(t) - \Delta\left(|u|^{m-1} u\right) dt = \sigma(u)dW(t) \quad , \quad t \geq 0 \tag{2.10}$$

$$u(\cdot, 0) = u_0(\cdot) \tag{2.11}$$

has a unique $\mathcal{F}_t$-adapted solution

$$u \in L^{m+1}(\Omega \times (0,T) \times \mathcal{O}) \cap L^2\left(\Omega; C([0,T]; H^{-1})\right) \tag{2.12}$$

such that

$$\int_{\mathcal{O}} u(t)e_j = \int_{\mathcal{O}} u_0 e_j + \int_0^t \int_{\mathcal{O}} |u|^{m-1} u \Delta e_j + \sum_{k=1}^{\infty} \mu_k \int_0^t \int_{\mathcal{O}} u(x,s) e_k(x) e_j(x) dx d\beta_k(s)$$

for all $j \in \mathbb{N}$, $t \in [0,T]$. *Moreover, if initial data are nonnegative, the same holds for the solution globally in space-time almost surely.*
If (2.2) *holds and if*

$$\mathbb{E}\,\|u_0\|_{L^{m+1}(\mathcal{O})}^{m+1} < \infty, \tag{2.13}$$

then u *is a strong solution in the sense of Definition 2.2.*

The theorem summarizes results to be found in [4], [6], [12], and in [23]. Concerning (2.9), we refer to [12]. More precisely, we have to verify assumptions (A5) and (A6) of that paper which are straightforward consequences of our regularity assumption (2.2).
Finally, let us explain our notion of finite speed of propagation and of occurrence of waiting time phenomena.

Definition 2.4. *Let* $v : \mathcal{O} \times [0,\infty) \to \mathbb{R}$ *be a function. We say that* v *has finite speed of propagation if for each ball* $B_R(x_0) \subset \mathcal{O}$ *with* $\operatorname{supp} v(\cdot,0) \cap B_R(x_0) = \emptyset$ *the following holds. For any* $r \in (0,R)$, *there exists a positive time* T_{r,R,x_0} *such that*

$$\int_0^{T_{r,R,x_0}} \int_{B_r(x_0)} |v(x,t)| dx dt = 0. \tag{2.14}$$

Note that Definition 2.4 provides a notion of forward finite speed of propagation. It does not exclude that the free boundary $\partial \operatorname{supp} v(.,t)$ moves backwards with infinite speed.
In a similar way, we introduce a definition for the occurrence of a waiting time phenomenon.

Definition 2.5. *Let* $v : \mathcal{O} \times [0,\infty) \to \mathbb{R}$ *be a given function and assume that* $B_R(x_0) \subset \mathcal{O}$, *that* $\operatorname{supp} v(\cdot,0) \cap B_R(x_0) = \emptyset$, *and that* $\operatorname{supp} v(\cdot,0) \cap \partial B_R(x_0) \neq \emptyset$. *We say that* v *exhibits a waiting time phenomenon in* $\partial B_R(x_0) \cap \operatorname{supp} v(\cdot,0)$ *if a positive time* T *exists such that*

$$\int_0^T \int_{B_R(x_0)} |v(x,t)| dx dt = 0. \tag{2.15}$$

Again, this is a notion of a forward waiting time phenomenon as it does not exclude the possibility that the boundary of $\operatorname{supp} v(\cdot,t)$ is retracting.

3. Main results

In this section, we present the main results of the paper. We begin with the finite speed of propagation property for strong solutions of the stochastic porous medium equation:

Theorem 3.1. *Let* u *be a strong solution to the stochastic porous medium equation in the sense of Definition 2.2. Assume that we have* $u_0 \in L^{1+m}(\mathcal{O})$ *and that* $\operatorname{supp} u_0 \cap B_R(x_0) = \emptyset$ *for some* $x_0 \in \mathcal{O}$ *and some* $R \in (0, \operatorname{dist}(x_0, \partial\mathcal{O}))$. *Then there exists a constant* $\bar{C} > 0$, *depending only on* d, m, *and* $\mathcal{S} := \sum_{k\in\mathbb{N}} \mu_k^2 \|e_k\|^2_{L^\infty(\mathcal{O})}$ *such that for any* $r \in (0, R)$, *any* $T_E > 0$, *and any* $\mu > 0$ *the following assertion holds:*

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T_E}\int_{B_r(x_0)} |u|^{2m}\ dx\ dt > 0\right) \\
\leq &\frac{C(d,m)}{(R-r)^2}\cdot \exp\left(\bar{C}T_E\right)\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot \mu^{\frac{2(m-1)}{d(m-1)+4m}} \\
&+\mathbb{P}\left(\int_0^{T_E}\int_{B_R(x_0)} |u|^{2m}(x,t)\ dx\ dt > \mu\right)
\end{aligned}$$

The following more classical formulation of the finite speed of propagation property – consistent with Definition 2.4 – is a straightforward consequence of our previous theorem:

Corollary 3.2. *Let* u *be a strong solution to the stochastic porous medium equation in the sense of Definition 2.2. Given a point* $x_0 \in \mathcal{O}$ *and some* $R > 0$ *with* $\operatorname{supp} u_0 \cap B_R(x_0) = \emptyset$, *for any* $r \in (0, R)$ *there exists an almost surely positive stopping time* T_{r,R,x_0} *such that*

$$\int_0^{T_{r,R,x_0}}\int_{B_r(x_0)} |u|^{2m}\, dxdt = 0$$

holds for all $\omega \in \Omega$.

Our second main result gives sufficient conditions for the existence of a waiting time:

Theorem 3.3. *Let* u *be a strong solution to the stochastic porous medium equation in the sense of Definition 2.2. Assume that we have* $u_0 \in L^{1+m}(\mathcal{O})$. *Let* $R > r > 0$ *and* $x_0 \in \mathcal{O}$ *be given such that* $\operatorname{supp} u_0 \cap B_r(x_0) = \emptyset$ *and* $B_R(x_0) \subset\subset \mathcal{O}$. *Let* β *be a parameter satisfying*

$$\beta > \frac{d(m-1)+4m}{2(m-1)}. \tag{3.1}$$

Assume furthermore that there exists $\bar{S} > 0$ *such that for any* $\rho \in (r, R)$ *we have*

$$\int_{B_\rho(x_0)\setminus B_r(x_0)} |u_0|^{1+m}\ dx \leq \bar{S}^{1+m}\cdot \frac{(\rho-r)^{\frac{d(m-1)+2(m+1)}{m-1}}}{\left(\log_2 \frac{2(R-r)}{\rho-r}\right)^\beta}.$$

Then there exists a constant $\bar{C} > 0$, *depending only on* d, m, *and* $\mathcal{S} := \sum_{k\in\mathbb{N}} \mu_k^2 \|e_k\|^2_{L^\infty(\mathcal{O})}$ *such that for any* $T_E > 0$ *and for any* μ *with*

$$\mu \geq \bar{S}^{m+1}\cdot (R-r)^{\frac{d(m-1)+4m}{m-1}} \tag{3.2}$$

the following assertion holds:

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T_E}\int_{B_r(x_0)} |u|^{2m} dxdt > 0\right) \\
\leq &\frac{C(d,m,\beta)}{(R-r)^2}\cdot \exp\left(\bar{C}T_E\right)\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot \mu^{\frac{2(m-1)}{d(m-1)+4m}} \\
&+\mathbb{P}\left(\int_0^{T_E}\int_{B_R(x_0)} |u|^{2m}(x,t) dxdt > \mu\right)
\end{aligned}$$

Again, we may reformulate to make the result more consistent with Definition 2.5.

Corollary 3.4. *Let u be a strong solution to the stochastic porous medium equation in the sense of Definition 2.2. Assume that we have $u_0 \in L^{1+m}(\mathcal{O})$. Let $R > r > 0$ and $x_0 \in \mathcal{O}$ be given such that $\operatorname{supp} u_0 \cap B_r(x_0) = \emptyset$ and $B_R(x_0) \subset\subset \mathcal{O}$, and let β satisfy (3.1). Assume furthermore that there exists $\bar{S} > 0$ such that for any $\rho \in (r, R)$ we have*

$$\int_{B_\rho(x_0)\setminus B_r(x_0)} |u_0|^{1+m}\, dx \le \bar{S}^{1+m} \cdot \frac{(\rho - r)^{\frac{d(m-1)+2(m+1)}{m-1}}}{\left(\log_2 \frac{2(R-r)}{\rho - r}\right)^\beta} . \tag{3.3}$$

Then there exists an almost surely positive stopping time $T_{r,R,x_0,\bar{S}}$ such that

$$\int_0^{T_{r,R,x_0,\bar{S}}} \int_{B_r(x_0)} |u|^{2m} dxdt = 0$$

for all $\omega \in \Omega$.

Remark 3.5. *In the deterministic setting, a waiting time phenomenon is observed locally at a point $x_0 \in \partial \operatorname{supp} u_0$ if*

$$S(R) := \sup_{r \in (0,R)} r^{-\frac{d(m-1)+2(m+1)}{m-1}} \int_{\mathcal{C}(r)} |u_0|^{m+1} \tag{3.4}$$

is finite – see for instance [14]. Here, $\mathcal{C}(r)$ is a half-cone with vertex $x_0 + r\mathbf{a}$, symmetry axis parallel to $\mathbf{a}$, and an opening angle θ chosen in such a way that

$$\operatorname{supp} u_0 \cap \mathcal{C}(0) \cap B_R(x_0) = \emptyset.$$

Note that in space dimension $d = 1$, condition (3.4) coincides with (3.3) up to a logarithmic factor. In space dimensions $d > 1$, it is possible to modify (3.3) in such a way that cones are considered instead of annuli. This again would lead to a criterion for the occurrence of a waiting time phenomenon which coincides with (3.4) up to a logarithmic factor.

4. A weighted energy estimate

In this section, we prove a stochastic counterpart of the weighted energy estimate which serves in the deterministic setting as the starting point to show qualitative results on the propagation of the free boundary. The result will follow by combination of Itô's formula with the regularity properties of strong solutions (see Definition 2.2) and a subtle approximation lemma on the H^1-convergence of convolutions $|\rho_\delta * v|^{m-1}\rho_\delta * v$ towards $|v|^{m-1}v$ for functions v satisfying $|v|^m \in H^1(\mathcal{O})$. As the proof of Lemma 4.1 is technically rather involved, it might distract the reader from the core results of the paper. Note that it can easily be omitted at first reading.
Let $\rho \in C_0^\infty(\mathbb{R}^d; \mathbb{R}_0^+)$ be a rotational symmetric smoothing kernel with $\operatorname{supp} \rho = B_1(0)$. For $\delta > 0$, we introduce the scaled kernel $\rho_\delta := \delta^{-d}\rho(\frac{x}{\delta})$. The following approximation result holds true.

Lemma 4.1. *Let $m > 1$ and let v be measurable with $|v|^{m-1}v \in H^1(\mathbb{R}^d)$. Then the estimate*

$$\left\| |\rho_\delta * v|^{m-1} \rho_\delta * v \right\|_{H^1(\mathbb{R}^d)} \le C(m,d) \left\| |v|^{m-1} v \right\|_{H^1(\mathbb{R}^d)} \tag{4.1}$$

holds for any $\delta \in (0,1)$ and we have

$$\left\| |\rho_\delta * v|^{m-1} \rho_\delta * v - |v|^{m-1} v \right\|_{H^1(\mathbb{R}^d)} \to 0 \tag{4.2}$$

as $\delta \to 0$.

Proof. The convergence $\||\rho_\delta * v|^{m-1}(\rho_\delta * v) - |v|^{m-1}v\|_{L^2(\mathbb{R}^d)} \to 0$ as $\delta \to 0$ is a standard result for convolutions, as is the bound $\|(\rho_\delta * v)^m\|_{L^2(\mathbb{R}^d)} \leq C(d)\|v^m\|_{L^2(\mathbb{R}^d)}$. Thus, we only need to show boundedness and convergence of the derivatives. Without loss of generality we may assume $\operatorname{supp} \rho_\delta \subset B_\delta(0)$. We shall first establish the existence of a dominating function independent of $\delta \in (0,1)$.

Set $v_\delta := \rho_\delta * v$. Fix $x_0 \in \mathbb{R}^d$. We have

$$\nabla\left(|v_\delta|^{m-1}v_\delta\right) = m|v_\delta|^{m-1}\nabla(\rho_\delta * v)\ . \tag{4.3}$$

Fix $\nu > 0$ small with $\nu < 1$; ν will be chosen independent of x_0 and δ.

Assume that $\int_{\mathbb{R}^d} \rho_\delta(x - x_0)v(x)\, dx > 0$; the case $\int_{\mathbb{R}^d} \rho_\delta(x - x_0)v(x)\, dx < 0$ is completely analogous, since the situation is entirely symmetric with respect to a change of sign.

Set $A := \{2^i : i \in \mathbb{Z}\}$ and define $\beta := \sup\{s \in A : s \leq \nu ⨍_{B_\delta(x_0)} |v(x)|\, dx\}$ (note that we would like to set $\beta := \nu ⨍_{B_\delta(x_0)} |v(x)|\, dx$, but the numbers of possible values of β must be countable).

We introduce a cut-off function F_β given by

$$F_\beta(v) := \begin{cases} v & \text{if } |v| \geq \beta, \\ 0 & \text{if } |v| \leq \frac{\beta}{2} \\ 2v - \beta & \text{if } \frac{\beta}{2} < v < \beta \\ 2v + \beta & \text{if } -\beta < v < -\frac{\beta}{2} \end{cases} \tag{4.4}$$

We compute

$$\begin{aligned} Q :=& |v_\delta|^{m-1}\nabla(\rho_\delta * v) \\ =& |v_\delta|^{m-1}\nabla(\rho_\delta * F_\beta(v) + \rho_\delta * (v - F_\beta(v)) \\ =& |v_\delta|^{m-1}\rho_\delta * \nabla F_\beta(v) + |v_\delta|^{m-1}\nabla\rho_\delta * (v - F_\beta(v))\ . \end{aligned}$$

Now observe that

$$\begin{aligned} |v_\delta(x_0)| &\leq ⨍_{B_\delta(x_0)} |B_\delta(x_0)|\ \rho_\delta(x - x_0)\ |v(x)|\, dx \\ &\leq C ⨍_{B_\delta(x_0)} |v(x_0)| dx \leq C\frac{\beta}{\nu}. \end{aligned} \tag{4.5}$$

Combining this with $|\nabla\rho_\delta| \leq \frac{C}{\delta^{d+1}}$ and the observation $-\beta \leq v - F_\beta(v) \leq \beta$ with $v - F_\beta(v) = 0$ on $|v| \geq \beta$, we get

$$\begin{aligned} |Q(x_0)| \leq& \frac{C}{\nu^{m-1}}\beta^{m-1}(\rho_\delta * |\nabla F_\beta(v)|)(x_0) \\ &+ |v_\delta|^{m-1}\frac{C}{\delta^{d+1}}\beta\mathcal{L}(\{x : |x - x_0| \leq \delta, |v|(x) < \beta\}) \\ \leq& \frac{C}{\nu^{m-1}}\left(\rho_\delta * \left||v|^{m-1}\nabla F_\beta(v)\right|\right)(x_0) \\ &+ \frac{C}{\nu^{m-1}\delta^{d+1}}\beta^m\mathcal{L}(\{x : |x - x_0| \leq \delta, |v|(x) < \beta\})\ , \end{aligned} \tag{4.6}$$

where in the last step we have used the fact that $\nabla F_\beta(v) = 0$ for a.e. point at which $|v| \leq \frac{\beta}{2}$ and the definition of β.

Our next aim is to bound the size of the above set using Poincare's inequality in order to get a uniform bound. We have using Jensen's inequality

$$\beta^m \le \left(\nu \fint_{B_\delta(x_0)} |v|(x)\ dx\right)^m \le \nu^m \fint_{B_\delta(x_0)} |v|^m(x)\ dx.$$

By selecting ν small enough (depending only on m), we get

$$\beta^m \le \frac{1}{2}\fint_{B_\delta(x_0)} |v|^m(x) dx\ .$$

This estimate yields in connection with Poincare's inequality

$$\begin{aligned} &\mathcal{L}(\{x : |x-x_0| \le \delta, |v|(x) < \beta\}) \\ &\quad\le \frac{1}{\beta^m}\int_{B_\delta(x_0)} (2\beta^m - |v|^m(x))_+\, dx \\ &\quad\le \frac{1}{\beta^m}\int_{B_\delta(x_0)} \left(\fint_{B_\delta(x_0)} |v|^m(y)\ dy - |v|^m(x)\right)_+ dx \\ &\quad\le \frac{1}{\beta^m}\int_{B_\delta(x_0)} \left|\fint_{B_\delta(x_0)} |v|^m(y) dy - |v|^m(x)\right|\ dx \\ &\quad\le \frac{C\delta^{d+1}}{\beta^m}\fint_{B_\delta(x_0)} |\nabla |v|^m(x)| dx. \end{aligned} \tag{4.7}$$

Observe that $|F'_\beta(v)| \le 2$ with $F_\beta \equiv 0$ on $[-\frac{\beta}{2}, \frac{\beta}{2}]$ and thus $|\nabla F_\beta(v)| \le 2|\nabla\,(v \vee \epsilon)\,| + 2|\nabla\,(v \wedge -\epsilon)\,|$ for any $0 < \epsilon < \frac{\beta}{2}$. Therefore we get

$$\left||v|^{m-1}\nabla F_\beta(v)\right| \le 2\left|\frac{1}{m}\nabla\,|v\vee\epsilon|^m\right| + 2\left|\frac{1}{m}\nabla\,|v\wedge -\epsilon|^m\right|$$

and letting $\epsilon \to 0$ we obtain

$$||v|^{m-1}\nabla F_\beta(v)| \le 2\left|\frac{1}{m}\nabla |v|^{m-1} v\right|\ .$$

Combining (4.6), (4.7), and the previous estimate, we infer

$$\begin{aligned} |Q(x_0)| &\le C\left(\frac{1}{\nu^{m-1}}\rho_\delta * \left||v|^{m-1}\nabla F_\beta(v)\right|\right)(x_0) \\ &\qquad + \frac{C}{\nu^{m-1}\delta^{d+1}}\beta^m \mathcal{L}(\{x : |x-x_0| \le \delta, |v|(x) < \beta\}) \\ &\le \frac{C}{\nu^{m-1}}\left(\rho_\delta * \left|\frac{1}{m}\nabla |v|^m\right|\right)(x_0) + \frac{C}{\nu^{m-1}}\fint_{B_\delta(x_0)} |\nabla |v|^m|\, dx \\ &\le \frac{C}{\nu^{m-1}}\fint_{B_\delta(x_0)} |\nabla |v|^m|\, dx\ . \end{aligned}$$

We now have a reasonable bound independent of β. Taking the union of the countably many sets $S_i := \{x_0 : \beta(x_0) = 2^i\}$, we get a pointwise bound for Q, independent of δ, of the form

$$|\nabla|\rho_\delta * v|^m|(x) \le C\mathcal{M}\left(|\nabla |v|^m|\right)(x)\ ,$$

where $\mathcal{M}$ denotes the maximal function (see [26]). Due to the boundedness of the maximal function as a map $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, we have found the desired dominating function; moreover, we see that (4.1) holds.

It remains to prove convergence almost everywhere for $\nabla\left(|\rho_\delta * v|^{m-1}\rho_\delta * v\right)$. By modifying the method above, we argue as follows.
Lemma 7.6 in [15] implies that $\nabla(|v|^{m-1}v) = 0$ almost everywhere on the set $\{v = 0\}$. Hence, it is sufficient to consider only those points $x_0 \in \mathbb{R}^d$ for which $v(x_0) \neq 0$. For $\beta > 0$ to be chosen later on, we split

$$\begin{aligned}
&|v_\delta|^{m-1}\nabla(\rho_\delta * v) - \frac{1}{m}\nabla\left(|v|^{m-1}v\right) \\
&\qquad = |v_\delta|^{m-1}\nabla\left(\rho_\delta * v - F_\beta(v)\right) + \left(|v_\delta|^{m-1} - |v|^{m-1}\right)\nabla F_\beta(v) \\
&\qquad\quad + \left(|v|^{m-1}\nabla F_\beta(v) - \frac{1}{m}\nabla\left(|v|^{m-1}v\right)\right) =: I + II + III.
\end{aligned}$$

By Theorem 1, p. 123 in [10], II tends to zero for $\delta \to 0$. For I, we have

$$\begin{aligned}
I = {} & |v_\delta|^{m-1}\left(\rho_\delta * \nabla F_\beta(v) - \nabla F_\beta(v)\right) \\
& + |v_\delta|^{m-1}\nabla\rho_\delta * (v - F_\beta(v)) =: I_1 + I_2.
\end{aligned}$$

Applying again Theorem 1, p.123 in [10] entails that I_1 converges to zero for $\delta \to 0$. For I_2, we have

$$\begin{aligned}
&\left||v_\delta|^{m-1}\nabla\rho_\delta * (v - F_\beta(v))\right| \\
&\quad \leq C\frac{\beta}{\delta^{d+1}}|v_\delta|^{m-1}\mathcal{L}\left(\{x : |x - x_0| \leq \delta, \quad |v|(x) < \beta\}\right).
\end{aligned} \tag{4.8}$$

Now choose $\beta := \nu \fint_{B_\delta(x_0)}|v(x)|dx$ with $\nu \in (0,1)$ to be determined later on. Observe that

$$\fint_{B_\delta(x_0)}|v(x)|dx \to |v(x_0)| \qquad \text{as } \delta \to 0$$

for almost every $x_0 \in \mathbb{R}^d$ (Lebesgue-point property). Hence, $\beta < |v(x_0)|$ for $\delta > 0$ sufficiently small. As a consequence, $F_\beta(v) = v$ for $\delta > 0$ sufficiently small. From

$$\frac{1}{m}\nabla\left(|v|^{m-1}v\right) = \frac{1}{m}\nabla\left(|F_\beta(v)|^{m-1}F_\beta(v)\right) = |F_\beta(v)|^{m-1}\nabla F_\beta(v) = |v|^{m-1}\nabla F_\beta(v)$$

for those $\delta > 0$, we infer that $III = 0$, provided $\delta > 0$ is sufficiently small.
Jensen's inequality entails

$$\beta^m \leq \nu^m \fint_{B_\delta(x_0)}|v|^m dx. \tag{4.9}$$

Observe that

$$2\frac{\nu^m}{\beta^m}\left(\frac{\beta^m}{\nu^m} - |v|^m\right)_+ \geq 1$$

if $|v| < \beta$ and if $\nu^m < \frac{1}{2}$. Hence, we may estimate the measure on the right-hand side of (4.8) as follows – using in particular (4.9).

$$\begin{aligned}
&\mathcal{L}\{x : |x - x_0| \leq \delta, |v(x)| < \beta\} \\
&\quad \leq C\delta^d\frac{\nu^m}{\beta^m}\fint_{B_\delta(x_0)}\left(\frac{\beta^m}{\nu^m} - |v(x)|^m\right)_+ dx \\
&\quad \leq C\delta^d\frac{\nu^m}{\beta^m}\fint_{B_\delta(x_0)}\left[\fint_{B_\delta(x_0)}|v|^m(y)dy - |v|^m(x)\right]_+ dx \\
&\quad \leq C\frac{\nu^m}{\beta^m}\delta^{d+1}\fint_{B_\delta(x_0)}|\nabla v^m|\,dy = C\frac{\nu^m}{\beta^m}\delta^{d+1}\fint_{B_\delta(x_0)}\left|\nabla\left(|v|^{m-1}v\right)\right|dy.
\end{aligned} \tag{4.10}$$

Note that we used the mean-value Poincaré inequality in the last step. Combining (4.8), (4.10), and (4.5) implies

$$\begin{aligned} &\left||v_\delta|^{m-1}\nabla\rho_\delta * (v - F_\beta(v))\right| \\ &\leq C|v_\delta|^{m-1}\frac{\beta}{\delta^{d+1}}\mathcal{L}\left\{x : |x-x_0| \leq \delta,\, |v|(x) < \beta\right\} \\ &\leq C\nu \fint_{B_\delta(x_0)} \left|\nabla\left(|v|^{m-1}v\right)\right| dy \leq C(x_0)\nu \end{aligned} \tag{4.11}$$

independently of $\delta > 0$ for every Lebesgue-point x_0 of $\nabla\left(|v_\delta|^{m-1}v\right) \in L^2(\mathbb{R}^d)$. Passing to the limit $\nu \to 0$ gives the result. □

Now, we are in the position to establish the main result of this section.

Theorem 4.2. *Let u be a strong solution of the stochastic porous medium equation in the sense of Definition 2.2. Assume that $\phi \in C^\infty(\mathcal{O}\times(0,T))$ such that $\phi(\cdot,t) \in C_0^\infty(\mathcal{O})$ for all $t \in [0,T]$. Then the following identity holds a.s. for a.e. $T > 0$:*

$$\begin{aligned} &\frac{1}{m+1}\int_{\mathcal{O}} |u(.,T)|^{m+1}\phi^2(.,T)dx - \frac{1}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)dx \\ &+ \int_0^T\int_{\mathcal{O}} |\nabla\left(|u|^{m-1}u\phi\right)|^2 dxdt \\ &= \int_0^T\int_{\mathcal{O}} |u|^{2m}|\nabla\phi|^2 dxdt + \frac{2}{m+1}\int_0^T\int_{\mathcal{O}} |u|^{m+1}\phi\frac{d}{dt}\phi\; dxdt \\ &+ \int_0^T \left\langle \phi^2|u|^{m-1}u, \sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}} + \frac{m}{2}\sum_{k\in\mathbb{N}}\int_0^T\int_{\mathcal{O}} \phi^2|u|^{m-1}\left(u\mu_k e_k\right)^2 dxdt \end{aligned} \tag{4.12}$$

Proof. We extend u by 0 outside of $\mathcal{O}$. For $v \in H^{-1}(\mathcal{O})$, we define $\psi_v \in H_0^1(\mathcal{O})$ by $\Delta\psi_v = v$ in $H^{-1}(\mathcal{O})$. By zero extension of ψ_v onto $\mathbb{R}^d$, we note $\rho_\delta{*}\nabla\psi_v = \nabla\left(\rho_\delta{*}\psi_v\right)$. We define the convolution $\rho_\delta{*}v$ of an element $v \in H^{-1}(\mathcal{O})$ by

$$\langle w, \rho_\delta{*}v\rangle_{H_0^1\times H^{-1}} := -\int_{\mathcal{O}} \nabla\left(\rho_\delta{*}\psi_v\right)\cdot\nabla w\; dx \tag{4.13}$$

for all $w \in H_0^1(\mathcal{O})$. We approximate the energy $F[v](t) := \frac{1}{m+1}\int_{\mathcal{O}} |v|^{m+1}\phi^2(\cdot,t)$ by $F_\delta[v](t) := \frac{1}{m+1}\int_{\mathcal{O}} |\Delta\left(\rho_\delta{*}\psi_v\right)|^{m+1}\phi^2(\cdot,t)$. We observe that

$$F_\delta[v](t) = \frac{1}{m+1}\int_{\mathcal{O}} |\rho_\delta{*}v|^{m+1}\phi^2(\cdot,t)dx \tag{4.14}$$

for all $v \in L^2(\mathcal{O})$ provided $\operatorname{dist}(\operatorname{supp}\phi, \partial\mathcal{O}) > \delta$. We wish to apply the Itô-formula of [22]. By Lemma 8.1, the first and second variations of F_δ are uniformly continuous on bounded subsets of $H^{-1}(\mathcal{O})\times(0,T)$. The same holds for the time derivative of F_δ by a similar reasoning. We get

$$\begin{aligned} F_\delta[u](t) =& F_\delta[u](0) + \int_0^t \left\langle \partial F_\delta[u](s), \sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}} \\ &+ \int_0^t \partial_t F_\delta[u](s) + \left\langle \partial F_\delta[u](s), \Delta\left(|u|^{m-1}u\right)\right\rangle_{H_0^1\times H^{-1}} ds \\ &+ \frac{1}{2}\int_0^t tr\left[\partial^2 F_\delta[u](s)\left(\sigma(u)\circ Q^{1/2}\right)\left(\sigma(u)\circ Q^{1/2}\right)^*\right] ds \end{aligned} \tag{4.15}$$

with Q defined in (2.3). By (4.14), (8.3), (8.6), (8.4) and the L^2-regularity of $\nabla\left(|u|^{m-1}u\right)$ (see Definition 2.2), we infer

$$\begin{aligned}
&\frac{1}{m+1}\int_{\mathcal{O}}\phi^2(\cdot,t)\left|(\rho_\delta * u)(.,t)\right|^{m+1}\,dx\Big|_0^T \\
&\quad+\int_0^T\int_{\mathcal{O}}\phi^2\nabla\left(|\rho_\delta{*}u|^{m-1}\rho_\delta{*}u\right)\cdot\nabla\left(\rho_\delta{*}\left(|u|^{m-1}u\right)\right)\,dx\,dt \\
&\quad=-2\int_0^T\int_{\mathcal{O}}\phi\,|\rho_\delta{*}u|^{m-1}\rho_\delta{*}u\,\nabla\left(\rho_\delta{*}u\left(|u|^{m-1}u\right)\right)\cdot\nabla\phi\,dx\,dt \\
&\qquad+\frac{2}{m+1}\int_0^T\int_{\mathcal{O}}|\rho_\delta{*}u|^{m+1}\phi\partial_t\phi\,dx\,dt \\
&\qquad+\int_0^T\left\langle\rho_\delta{*}\left(|\rho_\delta{*}u|^{m-1}(\rho_\delta{*}u)\,\phi^2\right),\sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}} \\
&\qquad+\frac{m}{2}\sum_{k\in\mathbb{N}}\int_0^T\int_{\mathcal{O}}\phi^2\,|\rho_\delta{*}u|^{m-1}\left(\rho_\delta{*}(\mu_k u e_k)\right)^2\,dx\,dt\,.
\end{aligned}\tag{4.16}$$

Let us pass to the limit $\delta\to 0$ in equation (4.16). First, we discuss the penultimate term on the right-hand side. We observe that the term $\rho_\delta{*}\left(|\rho_\delta{*}u|^{m-1}(\rho_\delta{*}u)\phi^2\right)$ strongly converges in $L^2(\Omega\times(0,T);H_0^1(\mathcal{O}))$ to $|u|^{m-1}u\phi^2$ by Lemma 4.1 and the uniform continuity of the mollification by ρ_δ on $H_0^1(\mathcal{O})$. Invoking Lemma 8.3 and using the regularity results (2.12) and (2.9) implies the convergence of the stochastic integral. Convergence of all the terms which do not contain derivatives in u is immediate. For the remainder terms, convergence towards the corresponding terms in (4.12) follows by application of Lemma 4.1. □

5. Estimates on the distribution functions for energy and dissipation

Both the Theorems (3.1) and (3.3) require an estimate of $\mathbb{P}\left(\int_0^T\int_A|u|^{2m}(x,t)dxdt>\tau\right)$. Since the Gagliardo-Nirenberg inequality allows to control $\int_0^T\int_A|u|^{2m}(x,t)dxdt$ by combining the two integrals $\int_0^T\int_A\left|\nabla\left(|u|^{m-1}u\right)\right|^2dxdt$ and $\sup_t\int_A|u(x,t)|^{m+1}dx$ in an appropriate way, we start with two lemmas which estimate the probability of the event that localized version of those integrals attain values above a given threshold τ. Our arguments are based on a combination of the energy estimate (4.12) with Markov's inequality. First, we have to introduce a class of cut-off functions – denoted by ϕ_{r,R,x_0} where $x_0\in\mathcal{O}$ and $0<r<R<\operatorname{dist}(x_0,\partial\mathcal{O})$. They are imposed to have the following properties:

- (A_1) $\phi_{r,R,x_0}\in C_0^\infty(\mathcal{O})$ and $0\le\phi_{r,R,x_0}\le 1$.
- (A_2) $\phi_{r,R,x_0}\equiv 1$ on $B_r(x_0)$.
- (A_3) $\phi_{r,R,x_0}\equiv 0$ on $\mathcal{O}\setminus B_R(x_0)$.
- (A_4) The inequalities $|\nabla\phi_{r,R,x_0}|\le\frac{C(d)}{R-r}$ and $|D^2\phi_{r,R,x_0}|\le\frac{C(d)}{(R-r)^2}$ are satisfied for some constant $C(d)$ depending only on d.

Our first result reads as follows.

Lemma 5.1. *Assume that u is a strong solution to the stochastic porous medium equation in the sense of Definition 2.2. Let $R\in(0,\operatorname{dist}(x_0,\partial\Omega))$ and $T_E>0$. Let T_A be a stopping time taking values in $[0,T_E]$ almost surely. Assume that there exists $\lambda\in\mathbb{R}_0^+$ with the* *property*

$$\int_0^{T_A} \int_{B_R(x_0)} |u|^{2m}(x,t)\ dx\ dt \leq \lambda\ . \tag{5.1}$$

Then there exists a positive constant $C = C(d)$ such that for any $\tau > 0$ and any $r \in (0,R)$ the estimate

$$\mathbb{P}\left(\int_0^{T_A} \int_{\mathcal{O}} |\nabla(|u|^{m-1} u \phi_{r,R,x_0})|^2\ dx\ dt > \tau\right) \tag{5.2}$$
$$\leq \exp(C_1 T_E) \cdot \frac{\frac{1}{m+1}\int_{B_R(x_0)} |u(.,0)|^{m+1}\ dx + \frac{C(d)}{(R-r)^2}\lambda}{\tau}$$

holds. Here, C_1 is given by the formula $C_1 := \frac{m(m+1)}{2}\sum_k \mu_k^2 \|e_k\|^2_{L^\infty(\mathcal{O})}$.

Remark 5.2. *Note that C_1 is finite due to* (2.2)*, Sobolev's embedding result and the assumption $d \leq 3$.*

Proof. We set $\phi(x,t) := \phi_{r,R,x_0}(x)\exp(-\frac{1}{2}C_1 t)$, and we choose C_1 as stated in the Lemma. Observe that the stochastic integral on the right-hand side of (4.12) is in fact a martingale. Indeed, for $\psi \in C^\infty(\mathcal{O})$ and for $v \in H^{-1}(\mathcal{O})$ satisfying $|v|^{m-1}v \in H_0^1(\mathcal{O})$, we introduce the operator $R(v,\psi) : Q^{1/2}L^2(\mathcal{O}) \to \mathbb{R}$ defined by

$$R(v,\psi)h := \sum_{k=1}^{\infty} (h, e_k)_{L^2(\mathcal{O})} \left\langle |v|^{m-1} v\psi^2, v e_k \right\rangle_{H_0^1 \times H^{-1}}.$$

We find its Hilbert-Schmidt norm to be bounded by

$$\|R(v,\psi)\|^2_{L_2(Q^{1/2}L^2(\mathcal{O});\mathbb{R})} \leq C \left\| |v|^{m-1} v \psi^2 \right\|^2_{H_0^1} \cdot \|v\|^2_{H^{-1}}\,, \tag{5.3}$$

where we have used (2.6) and (2.2) once more. Related to u and ϕ as above, we introduce the quadratic variation

$$\langle M \rangle_t := \int_0^t \|R(u(\cdot,s),\phi(\cdot,s))\|^2_{L_2(Q^{1/2}L^2(\mathcal{O});\mathbb{R})}\, ds.$$

We infer from Theorem 3.28 in [16] that it is sufficient to prove the boundedness of $\mathbb{E}\left(\sqrt{\langle M \rangle_t}\right)$ for each $t \in \mathbb{R}^+$. Using (5.3), we get

$$\begin{aligned}
\mathbb{E}\left(\sqrt{\langle M \rangle_t}\right) &\leq C\mathbb{E}\left(\sup_{s\in[0,t]} \|u\|_{H^{-1}} \sqrt{\int_0^t \left\| |u|^{m-1} u(\cdot,s)\phi^2(\cdot,s)\right\|^2_{H_0^1} ds}\right) \\
&\leq C\left(\mathbb{E}\left(\sup_{s\in[0,t]} \|u\|^2_{H^{-1}}\right)\right)^{1/2} \left(\mathbb{E}\left(\int_0^t \left\| |u|^{m-1} u(\cdot,s)\phi^2(\cdot,s)\right\|^2_{H_0^1} ds\right)\right)^{1/2} \\
&< \infty
\end{aligned}$$

by (2.12) and (2.9).
Using Doob's optional sampling theorem, we may take the expectation in (4.12) to obtain

$$\begin{aligned}
&\frac{1}{m+1}\ \mathbb{E}\left[\int_{\mathcal{O}} |u(.,T\wedge T_A)|^{m+1}\phi^2(.,T\wedge T_A)\ dx\right] \\
&+\mathbb{E}\left[\int_0^{T\wedge T_A}\int_{\mathcal{O}} |\nabla(|u|^{m-1}u\phi)|^2\ dx\ dt\right] \\
\le&\frac{1}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\ dx \\
&+\mathbb{E}\left[\int_0^{T\wedge T_A}\int_{\mathcal{O}} |u|^{2m}|\nabla\phi|^2\ dx\ dt\right] \\
&-\frac{C_1}{m+1}\ \mathbb{E}\left[\int_0^{T\wedge T_A}\int_{\mathcal{O}} |u|^{m+1}\phi^2\ dx\ dt\right] \\
&+\frac{m}{2}\sum_{k\in\mathbb{N}}\mu_k^2\,\|e_k\|^2_{L^\infty(\mathcal{O})}\ \mathbb{E}\left[\int_0^{T\wedge T_A}\int_{\mathcal{O}} |u|^{m+1}\phi^2\ dx\ dt\right]\ .
\end{aligned}$$

Using (5.1) as well as (A_4) and our special choice for C_1, we get

$$\begin{aligned}
&\frac{1}{m+1}\ \mathbb{E}\left[\int_{\mathcal{O}} |u(.,T\wedge T_A)|^{m+1}\phi^2(.,T\wedge T_A)\ dx\right] \\
&+\ \mathbb{E}\left[\int_0^{T\wedge T_A}\int_{\mathcal{O}} |\nabla(|u|^{m-1}u\phi)|^2\ dx\ dt\right] \\
\le&\frac{1}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\ dx+\frac{C(d)}{(R-r)^2}\lambda\ .
\end{aligned}$$

Choosing $T := T_E$ and using the Markov inequality (see, e.g., [8]), we end up with

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T_A}\int_{\mathcal{O}} |\nabla(|u|^{m-1}u\phi_{r,R,x_0})|^2\ dx\ dt>\tau\right) \\
\le&\exp(C_1T_E)\cdot\frac{\frac{1}{m+1}\int_{B_R(x_0)}|u(.,0)|^{m+1}\ dx+\frac{C(d)}{(R-r)^2}\lambda}{\tau}\ .
\end{aligned}$$

□

Here is our second lemma.

Lemma 5.3. *Assume that u is a strong solution to the stochastic porous medium equation in the sense of Definition 2.2. Let* $R\in(0,\operatorname{dist}(x_0,\partial\Omega))$ *and* $T_E>0$. *Let* T_A *be a stopping time taking values in* $[0,T_E]$ *almost surely. Assume that there exists* $\lambda\in\mathbb{R}_0^+$ *with the property*

$$\int_0^{T_A}\int_{B_R(x_0)} |u|^{2m}(x,t)\ dx\ dt\le\lambda\ . \tag{5.4}$$

Then there exists a constant $C_2 > 0$ depending only on d, m, and $\mathcal{S} := \sum_k \mu_k^2 \|e_k\|^2_{L^\infty(\mathcal{O})}$, such that for any $\tau > 0$ and any $r \in (0, R)$ the estimate

$$\mathbb{P}\left(\sup_{t\in[0,T_A]} \int_{B_r(x_0)} |u|^{m+1}(.,t)\; dx > \tau\right) \tag{5.5}$$
$$\leq C(m)\cdot \exp(C_2 T_E) \cdot \frac{\frac{1}{m+1}\int_{B_R(x_0)} |u(.,0)|^{m+1}\; dx + \frac{C(d,m)}{(R-r)^2}\lambda}{\tau}$$

holds.

Proof. We set $\phi(x,t) := \phi_{r,R,x_0}(x)\exp(-\frac{1}{2}C_2 t)$ where C_2 will be chosen later on depending only on d, m, and $\mathcal{S}$. The identity (4.12) then yields

$$\begin{aligned}
&\frac{1}{m+1}\int_{\mathcal{O}} |u(.,T)|^{m+1}\phi^2(.,T)\; dx\\
\leq&\frac{1}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\; dx\\
&+\int_0^T\int_{\mathcal{O}} |u|^{2m}|\nabla\phi|^2\; dx\; dt\\
&-\frac{C_2}{m+1}\int_0^T\int_{\mathcal{O}} |u|^{m+1}\phi^2\; dx\; dt\\
&+\frac{m}{2}\sum_k \mu_k^2 ||e_k||^2_{L^\infty}\int_0^T\int_{\mathcal{O}} \phi^2|u|^{m+1}\; dx\; dt\\
&+\int_0^T \left\langle \phi^2|u|^{m-1}u, \sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}}.
\end{aligned}$$

Taking the supremum with respect to T, we obtain for any stopping time $\tilde{T}$ with $\tilde{T}\leq T_A$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\frac{1}{m+1}\int_{\mathcal{O}} |u(.,T)|^{m+1}\phi^2(.,T)\; dx\right]\\
\leq&\frac{1}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\; dx\\
&+\mathbb{E}\left[\int_0^{\tilde{T}}\int_{\mathcal{O}} |u|^{2m}|\nabla\phi|^2\; dx\; dt\right]\\
&+\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\left|\int_0^T \left\langle \phi^2|u|^{m-1}u, \sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}}\right|\right].
\end{aligned} \tag{5.6}$$

Similarly we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\frac{C_2-C_1}{m+1}\int_0^{\tilde{T}}\int_{\mathcal{O}}|u(.,t)|^{m+1}\phi^2(.,t)\ dx\ dt\right]\\
\leq&\frac{1}{m+1}\int_{\mathcal{O}}|u(.,0)|^{m+1}\phi^2(.,0)\ dx \qquad (5.7)\\
&+\mathbb{E}\left[\int_0^{\tilde{T}}\int_{\mathcal{O}}|u|^{2m}|\nabla\phi|^2\ dx\ dt\right]\\
&+\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\left|\int_0^T\left\langle\phi^2|u|^{m-1}u,\sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}}\right|\right].
\end{aligned}
$$

For $v\in L^2((0,T);H_0^1(\mathcal{O}))$, we may estimate the Hilbert-Schmidt norm of the operator

$$
T(v(.,s))h:=\sum_{k=1}^{\infty}\mu_k(h,e_k)_{L^2(\mathcal{O})}\langle v(.,s),u(.,s)e_k\rangle_{H_0^1\times H^{-1}}
$$

by

$$
\|T(v)\|^2_{L_2(L^2(\mathcal{O});\mathbb{R})}\leq\sum_{k\in\mathbb{N}}\mu_k^2\,\|e_k\|^2_{L^\infty(\mathcal{O})}\left(\int_{\mathcal{O}}|v||u|dx\right)^2.
$$

By the regularity result (2.9) and the Burkholder-Davis-Gundy inequality (see [16]), this implies that for any stopping time $\tilde{T}$ we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\left|\int_0^T\left\langle\phi^2|u|^{m-1}u,\sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}}\right|\right]\\
\leq&C\mathbb{E}\left[\left(\sum_{k\in\mathbb{N}}\mu_k^2\,\|e_k\|^2_{L^\infty(\mathcal{O})}\right)^{1/2}\left(\int_0^{\tilde{T}}\left(\int_{\mathcal{O}}|u|^{m+1}\phi^2\ dx\right)^2dt\right)^{1/2}\right]\\
\leq&C\left(\sum_{k\in\mathbb{N}}\mu_k^2\,\|e_k\|^2_{L^\infty(\mathcal{O})}\right)^{1/2}\mathbb{E}\Bigg[\left(\int_0^{\tilde{T}}\int_{\mathcal{O}}|u|^{m+1}\phi^2\ dx\ dt\right)^{\frac{1}{2}}\\
&\qquad\cdot\left(\sup_{T\in[0,\tilde{T}]}\int_{\mathcal{O}}|u|^{m+1}(.,T)\phi^2(.,T)\ dx\right)^{\frac{1}{2}}\Bigg].
\end{aligned}
$$

Thus, for any $\epsilon>0$ we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\left|\int_0^T\left\langle\phi^2|u|^{m-1}u,\sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}}\right|\right]\\
\leq&C\epsilon\left(\sum_{k\in\mathbb{N}}\mu_k^2\,\|e_k\|^2_{L^\infty(\mathcal{O})}\right)^{1/2}\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\int_{\mathcal{O}}|u|^{m+1}(.,T)\phi^2(.,T)\ dx\right]\\
&+\frac{C}{\epsilon}\left(\sum_{k\in\mathbb{N}}\mu_k^2\,\|e_k\|^2_{L^\infty(\mathcal{O})}\right)^{1/2}\mathbb{E}\left[\int_0^{\tilde{T}}\int_{\mathcal{O}}|u|^{m+1}\phi^2\ dx\ dt\right].
\end{aligned}
$$

Combining this inequality with (5.6) and (5.7), we finally obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]} \frac{1}{m+1}\int_{\mathcal{O}} |u(.,T)|^{m+1}\phi^2(.,T)\;dx\right] \\
&\quad+\mathbb{E}\left[\frac{C_2-C_1}{m+1}\int_0^{\tilde{T}}\int_{\mathcal{O}} |u(.,t)|^{m+1}\phi^2(.,t)\;dx\;dt\right] \\
\leq&\frac{2}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\;dx \\
&\quad+2\mathbb{E}\left[\int_0^{\tilde{T}}\int_{\mathcal{O}} |u|^{2m}|\nabla\phi|^2\;dx\;dt\right] \\
&\quad+2\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\left|\int_0^T \left\langle \phi^2|u|^{m-1}u,\sigma(u)dW(s)\right\rangle_{H_0^1\times H^{-1}}\right|\right] \\
\leq&\frac{2}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\;dx \\
&\quad+2\mathbb{E}\left[\int_0^{\tilde{T}}\int_{\mathcal{O}} |u|^{2m}|\nabla\phi|^2\;dx\;dt\right] \\
&\quad+2\left(\sum_{k\in\mathbb{N}}\mu_k^2\left\|e_k\right\|_{L^\infty(\mathcal{O})}^2\right)^{1/2}\epsilon C\;\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\int_{\mathcal{O}} |u|^{m+1}(.,T)\phi^2(.,T)\;dx\right] \\
&\quad+2\left(\sum_{k\in\mathbb{N}}\mu_k^2\left\|e_k\right\|_{L^\infty(\mathcal{O})}^2\right)^{1/2}\frac{C}{\epsilon}\;\mathbb{E}\left[\int_0^{\tilde{T}}\int_{\mathcal{O}} |u|^{m+1}\phi^2\;dx\;dt\right].
\end{aligned}$$

Choosing ϵ small enough depending only on d, m, and $\sum_{k\in\mathbb{N}}\mu_k^2\left\|e_k\right\|_{L^\infty(\mathcal{O})}^2$ and then C_2 large enough depending only on d, m, and $\sum_{k\in\mathbb{N}}\mu_k^2\left\|e_k\right\|_{L^\infty(\mathcal{O})}^2$, we see that the last two terms terms on the right-hand side can be absorbed. This finally yields

$$\begin{aligned}
&\frac{1}{2(m+1)}\mathbb{E}\left[\sup_{T\in[0,\tilde{T}]}\int_{\mathcal{O}} |u(.,T)|^{m+1}\phi^2(.,T)\;dx\right] \\
&\qquad\leq\frac{2}{m+1}\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\;dx+2\mathbb{E}\left[\int_0^{\tilde{T}}\int_{\mathcal{O}} |u|^{2m}|\nabla\phi|^2\;dx\;dt\right].
\end{aligned}$$

Choosing $\tilde{T} := T_A$ and using (5.4), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T\in[0,T_A]}\int_{\mathcal{O}} |u(.,T)|^{m+1}\phi^2(.,T)\;dx\right] \\
&\qquad\leq C(m)\int_{\mathcal{O}} |u(.,0)|^{m+1}\phi^2(.,0)\;dx+C(d,m)\frac{\lambda}{(R-r)^2}.
\end{aligned}$$

A standard estimate and the definition of our test function ϕ now yield the result. □

6. A novel iteration lemma and proof of the main results

We begin this section with an iteration lemma which is technically perhaps the key result of this paper.

Lemma 6.1. *Assume that u is a strong solution to the stochastic porous medium equation. Let $R \in (0, \operatorname{dist}(x_0, \partial\mathcal{O}))$ and $T_E > 0$. Let T_A be a stopping time taking values in $[0, T_E]$ almost surely. Assume that there exists $\lambda_A \in \mathbb{R}_0^+$ with the property*

$$\int_0^{T_A} \int_{B_R(x_0)} |u|^{2m}(x,t)\ dx\ dt \leq \lambda_A\ . \tag{6.1}$$

Let $r \in (0, R)$ and assume that

$$\int_{B_R(x_0)} |u_0|^{m+1}(x)\ dx \leq \frac{\lambda_A}{(R-r)^2}\ . \tag{6.2}$$

Then for any $\lambda_B > 0$, the random time T_B defined by

$$T_B := T_A \wedge \inf\left\{T \in [0,\infty) :\ \int_0^T \int_{B_r(x_0)} |u|^{2m}(x,t)\ dx\ dt > \lambda_B\right\} \tag{6.3}$$

is a stopping time and satisfies

$$\mathbb{P}(T_B < T_A) \leq \frac{C(d,m)\cdot(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\lambda_A}{(R-r)^2\lambda_B^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}},$$

where the constants C_1, C_2 depend only on d, m and on $\mathcal{S} := \sum_k \mu_k^2 \|e_k\|^2_{L^\infty(\mathcal{O})}$ and are similar to those obtained in Lemma 5.1 and in Lemma 5.3.

Proof. The Gagliardo-Nirenberg inequality (see Theorem 8.5) applied to the function $v := |u|^{m-1}u\phi_{r,\frac{r+R}{2},x_0}$ with $s = r = 2$ and $q = \frac{m+1}{m}$ implies that

$$\begin{aligned}
&\int_0^T \int_{\mathcal{O}} |u|^{2m}\phi^2_{r,\frac{r+R}{2},x_0}\ dx\ dt\\
\leq &C(d,m)\int_0^T \left(\int_{\mathcal{O}} |\nabla(|u|^{m-1}u\phi_{r,\frac{r+R}{2},x_0})|^2\ dx\right)^\theta \cdot \left(\int_{\mathcal{O}} |u|^{m+1}\phi^{\frac{m+1}{m}}_{r,\frac{r+R}{2},x_0}\ dx\right)^{\frac{2(1-\theta)m}{m+1}} dt\\
\leq &C(d,m)T^{1-\theta}\left(\int_0^T \int_{\mathcal{O}} |\nabla(|u|^{m-1}u\phi_{r,\frac{r+R}{2},x_0})|^2\ dx\ dt\right)^\theta\\
&\qquad\cdot\left(\sup_{t\in[0,T]}\int_{B_{\frac{r+R}{2}}(x_0)} |u|^{m+1}(.,t)\ dx\right)^{\frac{2(1-\theta)m}{m+1}},
\end{aligned}$$

where in the last step we have used the fact that $\operatorname{supp}\phi_{r,\frac{r+R}{2},x_0} \subset B_{\frac{r+R}{2}}(x_0)$ and that $0 \leq \phi_{r,\frac{r+R}{2},x_0} \leq 1$. Noting that $\theta = \frac{d(m-1)}{d(m-1)+2(m+1)}$, we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathcal{O}} |u|^{2m}\phi^2_{r,\frac{r+R}{2},x_0}\ dx\ dt\\
\leq &C(d,m)T^{1-\theta}\left(\int_0^T \int_{\mathcal{O}} |\nabla(|u|^{m-1}u\phi_{r,\frac{r+R}{2},x_0})|^2\ dx\ dt\right)^{\frac{d(m-1)}{d(m-1)+2(m+1)}}\\
&\qquad\cdot\left(\sup_{t\in[0,T]}\int_{B_{\frac{r+R}{2}}(x_0)} |u|^{m+1}(.,t)\ dx\right)^{\frac{4m}{d(m-1)+2(m+1)}}.
\end{aligned} \tag{6.4}$$

To estimate

$$\mathbb{P}\left(\int_0^{T_A}\int_{B_r(x_0)}|u|^{2m}dxdt > C(d,m)T_E^{1-\theta}\tau^{\frac{d(m-1)+4m}{d(m-1)+2(m+1)}}\right),$$

we argue as follows. By (6.4),

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T_A}\int_{B_r(x_0)}|u|^{2m}dxdt > C(d,m)T_E^{1-\theta}\tau^{\frac{d(m-1)+4m}{d(m-1)+2(m+1)}}\right)\\
&\quad\le \mathbb{P}\Bigg[C(d,m)T_A^{1-\theta}\left(\int_0^{T_A}\int_{\mathcal{O}}\left|\nabla\left(|u|^{m-1}u\phi_{r,\frac{r+R}{2},x_0}\right)\right|^2 dxdt\right)^{\frac{d(m-1)}{d(m-1)+2(m+1)}}\\
&\qquad\quad\cdot\left(\sup_{t\in[0,T_A]}\int_{B_{\frac{r+R}{2}}(x_0)}|u|^{m+1}(\cdot,t)dx\right)^{\frac{4m}{d(m-1)+2(m+1)}} > C(d,m)T_E^{1-\theta}\tau^{\frac{d(m-1)+4m}{d(m-1)+2(m+1)}}\Bigg]\\
&\quad\le \mathbb{P}\Bigg[\int_0^{T_A}\int_{\mathcal{O}}\left|\nabla\left(|u|^{m-1}u\phi_{r,\frac{r+R}{2},x_0}\right)\right|^2 dxdt > \tau \quad\vee\\
&\qquad\quad \sup_{t\in[0,T_A]}\int_{B_{\frac{r+R}{2}}(x_0)}|u|^{m+1}(\cdot,t)dx > \tau\Bigg]\\
&\quad\le \mathbb{P}\left[\int_0^{T_A}\int_{\mathcal{O}}\left|\nabla\left(|u|^{m-1}u\phi_{r,\frac{r+R}{2},x_0}\right)\right|^2 dxdt > \tau\right]\\
&\qquad\quad + \mathbb{P}\left[\sup_{t\in[0,T_A]}\int_{B_{\frac{r+R}{2}}(x_0)}|u|^{m+1}(\cdot,t)dx > \tau\right].
\end{aligned} \tag{6.5}$$

Using formula (5.2) with r and $\frac{r+R}{2}$ in place of r and R and formula (5.5) with $\frac{r+R}{2}$ and R in place of r and R, we obtain for any $\tau > 0$

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T_A}\int_{B_r(x_0)}|u|^{2m}\,dx\,dt > C(d,m)T_E^{1-\theta}\tau^{\frac{d(m-1)+4m}{d(m-1)+2(m+1)}}\right)\\
\le& C(m)\cdot(\exp(C_1T_E)+\exp(C_2T_E))\cdot\frac{\frac{1}{m+1}\int_{B_R(x_0)}|u(.,0)|^{m+1}\,dx + \frac{C(d,m)}{(R-r)^2}\lambda_A}{\tau}\,.
\end{aligned}$$

Choosing $\tau := \left(\frac{\lambda_B}{C(d,m)T_E^{1-\theta}}\right)^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}$ and taking (6.2) into account, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T_A}\int_{B_r(x_0)}|u|^{2m}\,dx\,dt > \lambda_B\right)\\
\le&\frac{C(d,m)\cdot(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\lambda_A}{(R-r)^2\lambda_B^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}}\,.
\end{aligned}$$

We infer from Lemma 8.4 that T_B as defined in (6.3) is a stopping time. Hence, our result is established. □

We now have the necessary estimates for proving our result by an iteration technique which is reminiscent of iteration methods by Stampacchia and which is inspired from the ideas presented in [21].

Remark 6.2. *Note that equivalently we may write*

$$T_B = \sup\left\{T \in [0,T_A] : \int_0^T \int_{B_r(x_0)} |u|^{2m}(x,t)dxdt \le \lambda_B\right\}. \tag{6.6}$$

Proof of Theorem 3.1. For given $0 < r < R$, $x_0 \in \mathcal{O}$ with $B_R(x_0) \subset \mathcal{O}$, and for given $T_E > 0$, we introduce radii $r_k := r + \frac{R-r}{2^k}$, $k \in \mathbb{N}_0$.
The strategy is to apply Lemma 6.1 to estimate $\mathbb{P}\left(\int_0^{T_E}\int_{B_r(x_0)} |u|^{2m}(x,t)dxdt > 0\right)$. To this purpose, we introduce a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of nonnegative real numbers, monotonically decreasing to zero, to be specified later on, and an associated sequence of stopping times $(T_k)_{k\in\mathbb{N}_0}$ by

$$\begin{aligned} T_0 &:= T_E, \\ T_{k+1} &:= \sup\left\{T \in [0,T_k] : \int_0^T \int_{B_{r_k}(x_0)} |u|^{2m}(x,t)dxdt \le \lambda_k\right\}. \end{aligned} \tag{6.7}$$

Obviously, $(T_k)_{k\in\mathbb{N}}$ is monotonically decreasing. We refer to Remark 6.2 and to Lemma 8.4 for the proof that T_k, $k \in \mathbb{N}_0$, are indeed stopping times. Without loss of generality, we may choose $\lambda_0 := \mu$. Let us prove that

$$\begin{aligned} &\left\{\int_0^{T_E}\int_{B_r(x_0)} |u|^{2m}(x,t)dxdt > 0\right\} \\ &\quad\subset \left\{\int_0^{T_E}\int_{B_{r_0}(x_0)} |u|^{2m}(x,t)dxdt > \lambda_0\right\} \cup \bigcup_{k=1}^{\infty}\left\{\int_0^{T_k}\int_{B_{r_k}(x_0)} |u|^{2m}(x,t)dxdt > \lambda_k\right\} \\ &\quad=: \mathcal{A}. \end{aligned} \tag{6.8}$$

For this, it is sufficient to verify that for $\omega \notin \mathcal{A}$, we have $\int_0^{T_E}\int_{B_r(x_0)} |u|^{2m}(x,t)dxdt = 0$.
Since $r_0 = R > r$ and $\omega \notin \mathcal{A}$, we find

$$\int_0^{T_E}\int_{B_{r_0}(x_0)} |u|^{2m}(x,t)dxdt \le \lambda_0.$$

Hence, $T_1(\omega) = T_0(\omega) = T_E$ for such ω. Moreover, $\omega \notin \left\{\int_0^{T_1(\omega)}\int_{B_{r_1}(x_0)} |u|^{2m}(x,t)dxdt > \lambda_1\right\}$.
Hence, $T_2(\omega) = T_1(\omega) = T_E$. By induction, we have $T_k(\omega) = T_E$ and

$$\int_0^{T_E}\int_{B_{r_k}(x_0)} |u|^{2m}(x,t)dxdt \le \lambda_k$$

for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty}\lambda_k = 0$ and $\lim_{k\to\infty} r_k = r$, the inclusion (6.8) is proven.
Observe that

$$\left\{\int_0^{T_k}\int_{B_{r_k}(x_0)} |u|^{2m}(x,t)dxdt > \lambda_k\right\} = \{T_{k+1} < T_k\}. \tag{6.9}$$

Hence,

$$\begin{aligned}&\mathbb{P}\left(\int_0^{T_E}\int_{B_r(x_0)}|u|^{2m}(x,t)\,dxdt>0\right)\\&\leq\mathbb{P}\left(\int_0^{T_E}\int_{B_{r_0}(x_0)}|u|^{2m}(x,t)\,dxdt>\lambda_0\right)+\sum_{k=1}^{\infty}\mathbb{P}(T_{k+1}<T_k).\end{aligned}\tag{6.10}$$

Observing that due to (6.7)

$$\int_0^{T_{k+1}}\int_{B_{r_k}(x_0)}|u|^{2m}(x,t)\,dxdt\leq\lambda_k,\qquad k\in\mathbb{N}_0,$$

we may apply Lemma 6.1 with $T_A:=T_{k+1}$, $T_B:=T_{k+2}$, radii r_k and r_{k+1}, $\lambda_A:=\lambda_k$, and $\lambda_B:=\lambda_{k+1}$ to get the estimate

$$\mathbb{P}(T_{k+2}<T_{k+1})\leq\frac{C(d,m)\cdot(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\lambda_k}{(R-r)^2 2^{-2(k+1)}\lambda_{k+1}^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}},$$

$k\in\mathbb{N}_0$.

Since $\operatorname{supp} u_0\cap B_R(x_0)=\emptyset$, condition (6.2) is always satisfied. The next step is to choose the λ_k in such a way that the sum on the right-hand side in (6.10) becomes sufficiently small. Let $\mu\in\mathbb{R}^+$ and $\nu\in(0,1)$. We define $\lambda_k:=\mu\cdot\nu^k$; note that the sequence converges to zero. For our choice of $(\lambda_k)_{k\in\mathbb{N}_0}$, we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathbb{P}(T_{k+1}<T_k)\\&\leq\sum_{k=0}^{\infty}\frac{C(d,m)\cdot(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\lambda_k}{(R-r)^2 2^{-2(k+1)}\lambda_{k+1}^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}}\\&\leq\frac{C(d,m)\,(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}}{(R-r)^2}\\&\quad\cdot\sum_{k=0}^{\infty}\frac{\mu\cdot\nu^k}{2^{-2(k+1)}\cdot(\mu\cdot\nu^{k+1})^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}}\\&\leq\frac{C(d,m,\nu)\,(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\mu^{\frac{2(m-1)}{d(m-1)+4m}}}{(R-r)^2}\\&\quad\cdot\sum_{k=0}^{\infty}\left(2^2\nu^{\frac{2(m-1)}{d(m-1)+4m}}\right)^k.\end{aligned}$$

Thus choosing $\nu>0$ small enough depending only on d and m, we obtain[1]

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathbb{P}(T_{k+1}<T_k)\\&\leq\frac{C(d,m)\,(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\mu^{\frac{2(m-1)}{d(m-1)+4m}}}{(R-r)^2}.\end{aligned}$$

[1] Recall that $m>1$, therefore the geometric series converges if ν is sufficiently small.

Recalling $r_0 = R$ as well as (6.8), the result is established. □

Proof of Corollary 3.2. By Lemma 8.4, the time defined by

$$T_{r,R,x_0} := \inf\left\{T \in [0,\infty) : \int_0^T \int_{B_r(x_0)} |u|^{2m} dxdt > 0\right\}$$

is a stopping time. By Theorem 3.1, it is positive almost surely (since the probability in Theorem 3.1 can be made arbitrarily small by first choosing $\mu > 0$ large enough and then choosing $T_E > 0$ small enough). □

This argument can be modified to establish a sufficient criterion for the occurrence of a waiting time phenomenon.

Proof of Theorem 3.3. Similarly as in Theorem 3.1, we have to estimate

$$\mathbb{P}\left(\int_0^{T_E} \int_{B_r(x_0)} |u|^{2m}(x,t)dxdt > 0\right)$$

for r, T_E as in the formulation of the theorem. Following the argument of the proof of Theorem 3.1, we have

$$\begin{aligned}\mathbb{P}&\left(\int_0^{T_E} \int_{B_r(x_0)} |u|^{2m}(x,t)dxdt > 0\right)\\ &\leq \mathbb{P}\left(\int_0^{T_E} \int_{B_R(x_0)} |u|^{2m}(x,t)dxdt > \lambda_0\right) + \sum_{k=1}^{\infty} \mathbb{P}(T_{k+1} < T_k)\end{aligned} \tag{6.11}$$

with the sequences $(r_k)_{k\in\mathbb{N}_0}$, and $(T_k)_{k\in\mathbb{N}_0}$ defined in the same way – again corresponding to a decreasing sequence $(\lambda_k)_{k\in\mathbb{N}_0}$ of nonnegative real numbers which will be specified below. Since by construction (see (6.7))

$$\int_0^{T_{k+1}} \int_{B_{r_k}(x_0)} |u|^{2m}(x,t)dxdt \leq \lambda_k, \qquad k \in \mathbb{N}_0,$$

we may apply Lemma 6.1 with $T_A := T_{k+1}$, $T_B := T_{k+2}$, radii r_k and r_{k+1}, $\lambda_A := \lambda_k$ and $\lambda_B := \lambda_{k+1}$ to get the estimate

$$\mathbb{P}(T_{k+2} < T_{k+1}) \leq \frac{C(d,m) \cdot (\exp(C_1 T_E) + \exp(C_2 T_E)) \cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}} \cdot \lambda_k}{(R-r)^2 2^{-2(k+1)} \lambda_{k+1}^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}},$$

$k \in \mathbb{N}_0$, provided that

$$\int_{B_{r_k}(x_0)} u_0^{m+1} \leq \frac{\lambda_k}{(r_k - r_{k+1})^2} = \frac{\lambda_k}{(R-r)^2 2^{-2(k+1)}}\ . \tag{6.12}$$

Postponing the verification of (6.12), we continue in the spirit of the proof of Theorem 3.1. For μ and β satisfying (3.2) and (3.1), we take $\nu \in (0,1)$ to be made precise later on and we define $\lambda_k := \frac{\mu\nu^k}{(k+1)^\beta}$. For this choice of $(\lambda_k)_{k\in\mathbb{N}_0}$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{P}(T_{k+1} < T_k) \\
\leq &\sum_{k=0}^{\infty} \frac{C(d,m)\cdot(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\lambda_k}{(R-r)^2 2^{-2(k+1)}\lambda_{k+1}^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}} \\
\leq &\frac{C(d,m)\,(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}}{(R-r)^2} \\
&\cdot\sum_{k=0}^{\infty} \frac{\mu\cdot\nu^k\cdot(k+1)^{-\beta}}{2^{-2(k+1)}\cdot(\mu\cdot\nu^{k+1})^{\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}\cdot(k+2)^{-\beta\cdot\frac{d(m-1)+2(m+1)}{d(m-1)+4m}}} \\
\leq &\frac{C(d,m,\nu,\beta)\,(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\mu^{\frac{2(m-1)}{d(m-1)+4m}}}{(R-r)^2} \\
&\cdot\sum_{k=0}^{\infty}\left(2^2\nu^{\frac{2(m-1)}{d(m-1)+4m}}\right)^k\cdot(k+1)^{-\beta\cdot\frac{2(m-1)}{d(m-1)+4m}}.
\end{aligned}$$

We set $\nu := 2^{\frac{-d(m-1)-4m}{(m-1)}}$. For β as in our theorem, we see that the series converges. Thus we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{P}(T_{k+1} < T_k) \\
&\quad\leq \frac{C(d,m,\beta)\,(\exp(C_1T_E)+\exp(C_2T_E))\cdot T_E^{\frac{2(m+1)}{d(m-1)+4m}}\cdot\mu^{\frac{2(m-1)}{d(m-1)+4m}}}{(R-r)^2}\ .
\end{aligned}$$

Finally, we show that condition (6.12) is satisfied. This condition is seen to be equivalent to

$$\int_{B_{r_k}(x_0)} u_0^{m+1}(x,t)\ dx \leq \frac{\mu\cdot\nu^k}{(k+1)^\beta\cdot(R-r)^2\cdot 2^{-2k-2}}\ .$$

Inserting our choice of ν, we see that it is equivalent to

$$\int_{B_{r_k}(x_0)} u_0^{m+1}(x,t)\ dx \leq \frac{4\mu}{(k+1)^\beta\cdot(R-r)^2}\left(2^{2+\frac{-d(m-1)-4m}{m-1}}\right)^k$$

and therefore

$$\begin{aligned}
&\int_{B_{r_k}(x_0)} u_0^{m+1}(x,t)\ dx \\
&\qquad\leq \frac{4\mu}{\left(\log_2\frac{2(R-r)}{r_k-r}\right)^\beta\cdot(R-r)^{\frac{d(m-1)+4m}{m-1}}}(r_k-r)^{\frac{(d-2)(m-1)+4m}{m-1}}\ .
\end{aligned}$$

The proof is finished. ☐

Finally, we note that Corollary 3.4 can be established in a similar way as Corollary 3.2.

7. Concluding remarks

We have proposed a criterion for the occurrence of waiting time phenomena for the stochastic porous media equation. It differs by a logarithmic factor from the corresponding condition in the deterministic setting. Our method can easily be modified to obtain results on finite speed of propagation without confining ourselves to the case of noise given by a sum of finitely many driving terms. Nor is our technique confined to the case of at most three spatial dimensions. However, it may be necessary to alter the regularity condition of the noise – see the exposition in [12]. Finally, let us emphasize that we have considered the porous medium equation only as a first application of our method. We expect the ideas to carry over to other degenerate parabolic stochastics PDEs like e.g. the stochastic parabolic p-Laplacian.

8. Appendix

In the first part of the Appendix, we collect three lemmas which are used in the proof of Theorem 4.2.

Lemma 8.1. *For a given element $v \in H^{-1}(\mathcal{O})$, let $\psi_v \in H_0^1(\mathcal{O})$ be the unique solution of*

$$\Delta\psi_v = v \qquad \text{in } H^{-1}(\mathcal{O}), \tag{8.1}$$

$$\psi_v = 0 \qquad \text{on } \partial\mathcal{O} \tag{8.2}$$

and assume $g \in L^1(\mathcal{O};\mathbb{R}_0^+)$.
For given $\delta > 0$ and $2 \le p < \infty$, the functional $F_\delta(v) := \int_{\mathcal{O}} |\Delta(\rho_\delta{*}\psi_v)|^p g(\cdot)dx$ defines a real-valued mapping of class C^2 on $H^{-1}(\mathcal{O})$.
In particular, the first and second variations are given by the following formulas:

$$\partial_w F_\delta(v) = p\int_{\mathcal{O}} |\Delta(\rho_\delta{*}\psi_v)|^{p-2}\Delta(\rho_\delta{*}\psi_v)\Delta(\rho_\delta{*}\psi_w)g(\cdot)dx \qquad \forall w \in H^{-1}(\mathcal{O}), \tag{8.3}$$

$$\partial^2_{w,z} F_\delta(v) = p(p-1)\int_{\mathcal{O}} |\Delta(\rho_\delta{*}\psi_v)|^{p-2}\Delta(\rho_\delta{*}\psi_w)\Delta(\rho_\delta{*}\psi_z)g(\cdot)dx \qquad \forall w,z \in H^{-1}(\mathcal{O}). \tag{8.4}$$

In the special case that $g \in C_0^2(\mathcal{O})$ with $\operatorname{dist}(\operatorname{supp} g, \partial\mathcal{O}) > \delta$ and that $v \in L^1(\mathcal{O}) \cap H^{-1}(\mathcal{O})$, the following is true:

$$\begin{aligned}\partial_w F_\delta(v) &= p\int_{\mathcal{O}} |\rho_\delta{*}v|^{p-2}(\rho_\delta{*}v)\Delta(\rho_\delta{*}\psi_w)g(\cdot)dx \\ &= p\left\langle \rho_\delta{*}\left(|\rho_\delta{*}v|^{p-2}(\rho_\delta{*}v)g(\cdot)\right), w\right\rangle_{H_0^1(\mathcal{O})\times H^{-1}(\mathcal{O})}.\end{aligned} \tag{8.5}$$

Proof. Let us prove first that F_δ is continuous. For a sequence $(v_n)_{n\in\mathbb{N}}$ in $H^{-1}(\mathcal{O})$, which converges to v in $H^{-1}(\mathcal{O})$, we find that $\psi_{v_n} - \psi_v$ solves the homogeneous Dirichlet problem related to $\Delta(\psi_{v_n} - \psi_v) = v_n - v$. Hence, $\psi_{v_n} \to \psi_v$ in $H_0^1(\mathcal{O})$ satisfies

$$\|\psi_{v_n} - \psi_v\|_{H_0^1(\mathcal{O})} \le C\|v_n - v\|_{H^{-1}(\mathcal{O})}.$$

Note that $\rho_\delta * \psi_v \in C_0^\infty(\mathbb{R}^d)$ satisfies

$$\|\rho_\delta * \psi_v\|_{H^{k,2}(\mathbb{R}^d)} \le C_\delta(k,d)\|\psi_v\|_{H_0^1(\mathbb{R}^d)}$$

for every $k \in \mathbb{N}$. The assertion follows immediately by localization and embedding. Formulas (8.3) and (8.4) are computed in a straightforward way. The continuity w.r.t. v is proven in the same way as before.

It remains to prove that (8.5) is true. If $v \in L^1(\mathcal{O}) \cap H^{-1}(\mathcal{O})$, the first identity in (8.5) follows by use of the fact that $B_\delta(\operatorname{supp} g) \subset \mathcal{O}$. Note that $|\rho_\delta * v|^{p-2} (\rho_\delta * v)\, g \in H_0^1(\mathcal{O})$. Then, the second identity follows by the subsequent Lemma 8.2. □

Lemma 8.2. *Let $G \in H_0^1(\mathcal{O})$ and δ, ψ_w defined as in Lemma 8.1. If $\operatorname{dist}(\operatorname{supp} G, \partial\mathcal{O}) > \delta$, then*

$$\int_{\mathcal{O}} G\Delta\,(\rho_\delta * \psi_w)\, dx = \langle \rho_\delta * G, w\rangle_{H_0^1 \times H^{-1}(\mathcal{O})} \tag{8.6}$$

for all $w \in H^{-1}(\mathcal{O})$.

Proof. We apply a density argument. For this, we assume first that $w \in L^2(\mathcal{O}) \cap H^{-1}(\mathcal{O})$. By rotational symmetry of ρ_δ and $B_\delta(\operatorname{supp} G) \subset \mathcal{O}$,

$$\int_{\mathcal{O}} G\Delta\,(\rho_\delta * \psi_w)\, dx = \int_{\mathcal{O}} G\,(\rho_\delta * w)\, dx = \int_{\mathcal{O}} (\rho_\delta * G)\, w\ dx = \langle \rho_\delta * G, w\rangle_{H_0^1 \times H^{-1}} \tag{8.7}$$

To conclude, use the density of $L^2(\mathcal{O})$ in $H^{-1}(\mathcal{O})$ together with the fact that $\Delta\,(\rho_\delta * \psi_w)$ defines a continuous mapping from $H^{-1}(\mathcal{O})$ onto itself. □

Lemma 8.3. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence that converges in $L^2(\Omega \times (0,T); H_0^1(\mathcal{O}))$ to a function v and assume u to be contained in $L^2(\Omega; L^\infty((0,T); H^{-1}(\mathcal{O})))$. Then,*

$$\mathbb{E}\left\{ \sup_{t\in[0,T]} \left| \int_0^T \langle v_n - v, \sigma(u) dW(s)\rangle_{H_0^1(\mathcal{O})\times H^{-1}(\mathcal{O})} \right| \right\} \to 0 \tag{8.8}$$

as n tends to ∞.

Proof. Without loss of generality assume $v = 0$. Consider for $s \in (0,T)$ the operator $T(v_n(.,s)) : Q^{1/2}L^2(\mathcal{O}) \to \mathbb{R}$ given by

$$T(v_n(.,s))h := \sum_{k=1}^\infty (h, e_k)_{L^2(\mathcal{O})} \langle v_n(.,s), u(.,s)e_k\rangle_{H_0^1\times H^{-1}}. \tag{8.9}$$

For the Hilbert-Schmidt norm of $T(v_n(.,s))$, we obtain

$$\begin{aligned} \|T(v_n(.,s))\|^2_{L_2(Q^{1/2}L^2(\mathcal{O});\mathbb{R})} &= \sum_{k\in\mathbb{N}} \mu_k^2 \left| \langle v_n, ue_k\rangle_{H_0^1\times H^{-1}} \right|^2 \\ &\le \|v_n\|^2_{H_0^1} \sum_{k\in\mathbb{N}} \mu_k^2 \|ue_k\|^2_{H^{-1}} \\ &\le C \sum_{k\in\mathbb{N}} \mu_k^2 \sigma_k^2 \|u\|^2_{H^{-1}} \|v_n\|^2_{H_0^1(\mathcal{O})} \end{aligned} \tag{8.10}$$

where inequality (2.6) has been used once more. Hence, the quadratic variation of the stochastic integral

$$N_t(v_n) := \int_0^t \langle v_n, \sigma(u) dW(s)\rangle_{H_0^1\times H^{-1}} := \int_0^t T(v_n) dW_s$$

is estimated by

$$\langle N_t(v_n)\rangle = \int_0^t \|T(v_n)\|^2_{L_2(Q^{1/2}L^2(\mathcal{O});\mathbb{R})}\, ds \le C \sup_{s\in(0,t)} \|u\|^2_{H^{-1}} \int_0^t \|v_n\|^2_{H_0^1}\ ds.$$

By the Burkholder-Davis-Gundy inequality, see [16], we conclude

$$\mathbb{E}\left\{\sup_t \left|\int_0^t \langle v_n, \sigma(u)dW(s)\rangle_{H_0^1\times H^{-1}}\right|\right\} \leq C\mathbb{E}\left(\sup_t \|u\|_{H^{-1}}^2\right)^{1/2} \mathbb{E}\left(\|v_n\|^2_{L^2((0,t);H_0^1)}\right)^{1/2} \tag{8.11}$$

which gives the result by our assumption on the sequence $(v_n)_{n\in N}$. □

The following lemma helps to justify the filtering mechanism applied in Lemma 6.1.

Lemma 8.4. *Let $T_E > 0$ and let u be a strong solution of the stochastic porous media equation* (1.1)*. Let T_S be a stopping time with respect to the normal filtration $\{\mathcal{F}_t\}$ associated with u attaining almost surely values in $[0, T_E]$. Let $x_0 \in \mathcal{O}$ and suppose that $0 < R < \operatorname{dist}(x_0, \partial\mathcal{O})$. Then,*

$$T_0 := T_S \wedge \inf\left\{T \in [0, T_E] : \int_0^T \int_{B_R(x_0)} |u|^{2m}(x,t)dxdt > \lambda\right\} \tag{8.12}$$

is a stopping time for any $\lambda \geq 0$.

Proof. Since minimum and maximum of two stopping times are stopping times themselves, it is sufficient to show that

$$\tau := \inf\left\{T \in [0, T_E] : \int_0^T \int_{B_R(x_0)} |u|^{2m}(x,t)dxdt > \lambda\right\}$$

is a stopping time. For this, we define the random variable

$$X(t) := \int_0^t \int_{B_R(x_0)} |u|^{2m}(x,s)dxds.$$

Let us first prove that $X(\omega,\cdot) : [0, T_E] \to \mathbb{R}$ is continuous almost surely in Ω. By Definition 2.2, we know that

$$\mathbb{E}\int_0^{T_E} \int_{\mathcal{O}} |\nabla u^m|^2 \, dxdt < \infty,$$

hence

$$\int_0^{T_E} \int_{\mathcal{O}} |\nabla u^m(\omega)| \, dxdt < \infty$$

almost surely. Using $u^m \in H_0^1(\Omega)$ and Poincare's inequality, we deduce that

$$\int_0^{T_E} \int_{\mathcal{O}} |u|^{2m}(\omega)dxdt < \infty$$

almost surely. The absolute continuity of the integral implies that $X(\omega, t)$ is continuous with respect to time almost surely in Ω.

Now observe that $\tau = \inf\{T \in [0, T_E] : X(T) \in (\lambda, \infty)\}$. From [16], Chapter 1.2, we infer that in the case of a normal (hence right-continuous) filtration such a hitting time $H_E(\omega) := \inf\{t \geq 0 : X_t(\omega) \in E\}$ is a stopping time provided E is open (or closed), X has continuous paths almost surely, and X is adapted to the filtration $\{\mathcal{F}_t\}$. Hence, it remains to prove that $X(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$ associated with u. For this, let us prove that $X(t)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$. We proceed by a convolution argument, introducing for $0 < \delta < R$ the random variable

$$Y_\delta(t) := \int_{B_R(x_0)} \left|\Delta\left(\rho_\delta * \psi_{u(\cdot,t)}\right)\right|^{2m} dx,$$

where $\psi_{u(\cdot,t)}$ is the unique solution to the homogeneous Dirichlet problem

$$\Delta\psi_{u(\cdot,t)} = u(\cdot,t) \qquad \text{in } \mathcal{O}$$
$$\psi_{u(\cdot,t)} = 0 \qquad \text{on } \partial\mathcal{O}.$$

For $\delta < \operatorname{dist}(x_0, \partial\mathcal{O}) - R$, we have

$$\Delta\left(\rho_\delta * \psi_{u(\cdot,t)}\right)\big|_{B_R(x_0)} = \left(\rho_\delta * u(\cdot,t)\right)\big|_{B_R(x_0)} \tag{8.13}$$

Applying Lemma 8.1 with the choices $g := \chi_{B_R(x_0)}$ and $p = 2m$, we find that

$$H_\delta : v \in H^{-1}(\mathcal{O}) \to H_\delta(v) := \int_{B_R(x_0)} \left|\Delta\left(\rho_\delta * \psi_v\right)\right|^{2m} dx$$

defines a continuous mapping from $H^{-1}(\mathcal{O})$ to $\mathbb{R}$. Observing that

$$Y_\delta(\cdot,t) := H_\delta(u(\cdot,t))$$

is a composition of a $\mathcal{F}_t$-measurable mapping $u(\cdot,t) : \Omega \to H^{-1}(\mathcal{O})$ with a continuous mapping from $H^{-1}(\mathcal{O})$ to $\mathbb{R}$, the measurability of Y_δ is immediate. By Definition 2.2, we have $u(\cdot,t) \in L^{m+1}(\mathcal{O}) \cap L^{2m}(\mathcal{O})$ almost surely for almost all time. By (8.13), we may write

$$Y_\delta(\cdot,t) = \int_{B_R(x_0)} \left|\rho_\delta * u(\cdot,t)\right|^{2m} dx$$

provided $0 < \delta < \operatorname{dist}(x_0, \partial\mathcal{O}) - R$. Lemma 4.1 implies that

$$Y(t) := \int_{B_R(x_0)} |u(\cdot,t)|^{2m} = \lim_{\delta\to 0} Y_\delta(\cdot,t)$$

almost surely. Hence, $Y(\cdot,t)$ is $\mathcal{F}_t$-measurable, too.
Finally, the $\mathcal{F}_t$-measurability of $X(t) := \int_0^t Y(s)ds$ follows using $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ as well as the fact that this integral can be written as the limit of integrals of step-functions with respect to time. □

For the sake of completeness, we note the Gagliardo-Nirenberg interpolation inequality (see [11], [19], [20]).

Theorem 8.5. *Suppose* $0 < q < s$, $1 \le r \le \infty$. *For* $v \in L^q(\mathbb{R}^d)$ *with* $Dv \in L^r(\mathbb{R}^d)$ *we have the estimate*

$$||v||_{L^s} \le C||Dv||_{L^r}^{\theta} \cdot ||v||_{L^q}^{1-\theta}$$

with

$$\frac{1}{s} = \theta\left(\frac{1}{r} - \frac{1}{d}\right) + (1-\theta)\frac{1}{q}\ .$$

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