A HIGHER-ORDER LARGE-SCALE REGULARITY THEORY FOR RANDOM ELLIPTIC OPERATORS

JULIAN FISCHER AND FELIX OTTO

Abstract. We develop a large-scale regularity theory of higher order for divergence-form elliptic equations with heterogeneous coefficient fields a in the context of stochastic homogenization. The large-scale regularity of a-harmonic functions is encoded by Liouville principles: The space of a-harmonic functions that grow at most like a polynomial of degree k has the same dimension as in the constant-coefficient case. This result can be seen as the qualitative side of a large-scale $C^{k,\alpha}$-regularity theory, which in the present work is developed in the form of a corresponding $C^{k,\alpha}$-"excess decay" estimate: For a given a-harmonic function u on a ball B_R, its energy distance on some ball B_r to the above space of a-harmonic functions that grow at most like a polynomial of degree k has the natural decay in the radius r above some minimal radius r_0.

Though motivated by stochastic homogenization, the contribution of this paper is of purely deterministic nature: We work under the assumption that for the given realization a of the coefficient field, the couple (ϕ, σ) of scalar and vector potentials of the harmonic coordinates, where ϕ is the usual corrector, grows sublinearly in a mildly quantified way. We then construct "kth-order correctors" and thereby the space of a-harmonic functions that grow at most like a polynomial of degree k, establish the above excess decay and then the corresponding Liouville principle.

1. Introduction

We are interested in the regularity of harmonic functions u associated with a uniformly elliptic coefficient field a in d space dimensions (by which we understand a tensor field satisfying $\lambda|\xi|^2 \leq \xi \cdot a\xi$ and $|a\xi| \leq |\xi|$ for some $\lambda > 0$ and any $\xi \in \mathbb{R}^d$) via the divergence-form equation

$$-\nabla\cdot a\nabla u = 0. \tag{1}$$

Without continuity assumptions, the local regularity of (weak finite-energy) solutions can be rather low, in particular in case of systems (see e. g. [18, Example 3] for the scalar case and [9, Section 9.1.1] for De Giorgi's celebrated counterexample in the systems case). Because of their homogeneity, the same examples show that even when the coefficients are uniformly locally smooth, the *large-scale* behavior of a-harmonic functions can be very different from the constant coefficient, that is, Euclidean case; see e. g. Proposition 21 in the appendix below. Large-scale regularity is most compactly encoded in a Liouville statement of the following form: The space of a-harmonic functions u of growth not larger than $|x|^k$ has the same dimension as in the constant-coefficient case, where the space is spanned by spherical harmonics up to order k. Because of the above-mentioned counterexamples, such

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Liouville statements may fail for uniformly elliptic coefficient fields: For example, in the case of systems, there are non-constant harmonic maps that decay to zero at infinity.

The question whether this situation generically improves for certain *ensembles* of coefficient fields, namely stationary and ergodic ensembles as in stochastic homogenization, seems to have first been phrased and partially answered by Benjamini, Duminil-Copin, Kozma, and Yadin [6, Chapter 6 and Theorem 3] in the context of random walks in random environments: Under the mere assumption of ergodicity and stationarity, sublinearly growing a-harmonic functions are almost surely constant. The argument is limited to the scalar case, but can deal with non-uniformly elliptic cases as percolation.

Motivated by error estimates in stochastic homogenization, the topic of a regularity theory for random elliptic operators was independently addressed in a more quantitative way by Marahrens and the second author [14]. In Corollary 4 of that paper, for any $\alpha < 1$, a large-scale $C^{0,\alpha}$-inner regularity estimate for a-harmonic functions has been established, with a random constant of finite algebraic moments — however under stronger assumptions on the ergodicity, namely a finite spectral gap w. r. t. Glauber dynamics in the case of a discrete medium.

A major step forward constitutes the work of Armstrong and Smart [3], where the above result was improved to a large-scale $C^{0,1}$-inner regularity estimate even in case of (symmetric) systems, by showing that the approach of Avellaneda and Lin [5] for obtaining (large-scale) regularity of a-harmonic maps, which itself is based on a Campanato-type iteration, can be extended from periodic to random coefficient fields. Under a strong assumption of ergodicity, namely that of a finite range of dependence, optimal exponential moments for the random constant are obtained.

This work motivated the paper of Gloria, Neukamm, and the second author [11], which in turn is the basis for the present paper. In that work, another tool from periodic homogenization, namely the *vector* potential σ for the harmonic coordinates (next to the well-known scalar potential ϕ, also called the corrector), was transferred to the random case, see (7) and (8) for the characterizing properties. This allowed to establish a $C^{1,\alpha}$-Liouville theorem, meaning that the space of sub-quadratically growing a-harmonic functions is almost surely spanned by the the constants and the d a-harmonic coordinates $x_i + \phi_i$. This holds even for non-symmetric systems and was shown under the mere assumptions of stationarity and ergodicity. More precisely, it relied on the almost-sure sublinear growth of the couple (ϕ, σ) of correctors in the sense of

$$\lim_{r\to\infty} \varepsilon_r = 0, \tag{2}$$

where

$$\varepsilon_r := \sup_{R\geq r} \frac{1}{R} \left(\fint_{B_R} |\phi|^2 + |\sigma|^2 \, dx \right)^{1/2}. \tag{3}$$

This sublinear growth (2) was shown to hold under the assumptions of stationarity and qualitative ergodicity. In a second step, large-scale $C^{1,\alpha}$-inner regularity estimates for a-harmonic functions were obtained, where the random constant satisfies a stretched exponential bound under mild decay assumptions on the spatial covariance of a. In a later version of [11], the optimal stochastic moments for the random constant were obtained.

In the context of non-linear elliptic systems in divergence form, the result of Armstrong and Smart [3] on the large-scale $C^{0,1}$-estimate was generalized by Armstrong and Mourrat [2] to non-symmetric coefficients and well beyond finite range, further confirming that the random large-scale regularity theory holds under just a mild quantification of ergodicity, like expressed by standard mixing conditions.

In the present work, we go beyond $C^{1,\alpha}$ and establish a large-scale $C^{k,\alpha}$-theory in form of a corresponding excess decay and Liouville result, see Theorem 3 and Corollary 4. This lifts the result of Avellaneda and Lin [5] from the periodic to the random case. To streamline presentation, we first establish the $C^{2,\alpha}$-versions of our theorems, see Theorem 7 and Corollary 8.

Let us clearly state that the contribution of this paper is exclusively on the deterministic side. The large-scale regularity is obtained under the assumption that the given realization a of the coefficient field is such that the corresponding corrector couple (ϕ, σ) satisfies the following slight quantification of (2), namely

$$\lim_{r\to\infty} \varepsilon_{2,r} = 0 \tag{4}$$

with

$$\varepsilon_{2,r} := \sum_{m=0}^{\infty} \min\{1, 2^{m+1}/r\}\varepsilon_{2^m}. \tag{5}$$

Note that (4) is equivalent to $\sum_{m=0}^{\infty} \varepsilon_{2^m} < \infty$.

In a recent preprint by the authors of the present paper [8], it is shown that (4) holds for almost every realization a in case of a stationary ensemble of coefficient fields under mild quantification of ergodicity in form of an assumption on a mild decay of correlations of a: More precisely, given a stationary centered tensor-valued Gaussian random field $\tilde{a}$ on $\mathbb{R}^d$ and a bounded Lipschitz map $\Phi : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ taking values in the set of λ-uniformly elliptic tensors, the coefficient field

$$a := \Phi(\tilde{a})$$

almost surely admits correctors with the property (4) assuming just decay of correlations in the sense

$$|\langle \tilde{a}(x)\tilde{a}(y)\rangle| \leq C|x-y|^{-\beta}$$

for some $C > 0$ and some $\beta \in (0, c(d,\lambda))$ (where $\langle\cdot\rangle$ denotes the expectation). Note that under the assumption of a spectral gap for the ensemble, as far as the corrector ϕ is concerned (but not the "vector potential" σ), an estimate like (4) could also be deduced to hold almost surely from [12, Proposition 2], modulo the passage from a discrete to a continuum medium.

The key building block for this large-scale $C^{k,\alpha}$-theory is the space of a-harmonic functions that grow at most like a polynomial of degree k at infinity. Proposition 2 and Corollary 4 imply that under our assumption (4) this space has the same dimension as in the Euclidean case – e.g. for $k = 2$ the space of a-harmonic functions that grow at most quadratically is spanned by $1 + d + \frac{d(d+1)}{2} - 1$ maps – , which partially answers the question by Benjamini, Duminil-Copin, Kozma, and Yadin in [6, Chapter 6]. The kth-order excess (11), by the decay of which we encode the $C^{k,\alpha}$-theory, measures the distance to this space in terms of the averaged squared gradient. As our construction shows, there is a one-to-one correspondence between the asymptotic behavior of functions in this space and a_{hom}-harmonic polynomials of degree k. However, there is no natural one-to-one correspondence between elements of this space and kth-order a_{hom}-harmonic polynomials.

In a recent preprint by Armstrong, Kuusi, and Mourrat published after our present work, a higher-order regularity result related to our present results is obtained [1], however under a much stronger assumption on the decorrelation of coefficient fields (namely, finite range of dependence).

Before stating our results, let us recall the definition of the correctors (ϕ, σ). The corrector ϕ_i satisfies the equation

$$-\nabla\cdot a(e_i + \nabla\phi_i) = 0. \tag{6}$$

The flux correction q_{ij} is defined as

$$q_i := a(e_i + \nabla\phi_i) - a_{hom}e_i \tag{7}$$

where a_{hom} is the homogenized tensor, that is, $a_{hom}e_i$ is the expectation of $a(e_i + \nabla\phi_i)$. In our analysis, we will only use that a_{hom} is some constant elliptic coefficient. We introduce the corresponding vector potential σ_{ijk} (antisymmetric in its last two indices) by requiring that

$$\nabla\cdot \sigma_{ij} = q_{ij}. \tag{8}$$

For the actual construction of a σ with stationary gradient we refer to [11]; in this note, we just use the property (8). In the context of periodic homogenization, both the scalar and the vector potential ϕ and σ may be chosen to be periodic. In stochastic homogenization, one cannot always expect to have a stationary (ϕ, σ) (for instance in $d \leq 2$ even in case of finite range of dependence) but, as mentioned above, we expect sublinear growth in the sense of (4) under mild ergodicity assumptions.

Finally, let us give a brief historical overview on stochastic homogenization of elliptic PDEs. The qualitative theory of stochastic homogenization was initiated by Kozlov [13] and Papanicolaou and Varadhan [17]; the first (non-optimal) quantitative estimate – derived under the assumption of finite range of dependence – is due to Yurinskiĭ [19]. Naddaf and Spencer introduced spectral gap inequalities to quantify ergodicity in stochastic homogenization [16]. Gloria and the second author [12] were the first to obtain optimal estimates on the size of the homogenization error in the linear elliptic case, though with non-optimal stochastic integrability. Optimal stochastic integrability – however with non-optimal estimates on the size of the error – was obtained by Armstrong and Smart [3]. Finally, recently optimal error estimates with optimal stochastic integrability were established by Gloria and the second author [10] and Armstrong, Kuusi, and Mourrat [1]. For a more probabilistic viewpoint of stochastic homogenization of linear elliptic equations, see e. g. [15]. In the case of fully nonlinear elliptic equations, a logarithmic rate of convergence has been established by Caffarelli and Souganidis [7] under a very weak assumption on decorrelation; Armstrong and Smart [4] have obtained a power-law rate of convergence in the case of finite range of dependence.

Notation. Throughout the paper, we use the Einstein summation convention, i.e. we implicitly take the sum over an index whenever this index occurs twice. For example, $b_i\partial_i v$ is an alternative notation for $(b \cdot \nabla)v$ and $b_i\nabla v_i$ is an alternative notation for $\sum_{i=1}^d b_i\nabla v_i$.

By C we denote a generic constant whose value may be different in each appearance of the expression C; similarly, by e.g. $C(d,\lambda)$ we denote a generic constant depending only on d and λ whose value again may be different for every use of the expression $C(d,\lambda)$.

By $\mathcal{E} := \{E \in \mathbb{R}^{d\times d} : (E_{ij}+E_{ji})(a_{hom})_{ij} = 0\}$ we denote the space of matrices E_{ij} for which $E_{ij}x_ix_j$ is an a_{hom}-harmonic second-order polynomial.

The notation P (or $P(x)$) generally refers to a polynomial. By $\mathcal{P}^k$, we denote the space of homogeneous polynomials of degree k. By $\mathcal{P}^k_{a_{hom}}$, we denote the space of homogeneous polynomials of degree k which are a_{hom}-harmonic. On the space $\mathcal{P}^k$, we introduce the norm $||P|| := \sup_{x\in B_1} |P(x)|$; note that any other norm on this finite-dimensional space would do as well, since we do not care for $C(k)$-constants.

2. Main Results

The proof of our large-scale $C^{k,\alpha}$ regularity theory relies in an essential way on the existence of kth-order correctors for the homogenization problem, which enable us to correct a_{hom}-harmonic polynomials of degree k by adding a small (in the L^2-sense) perturbation.

The ansatz for the deformation of an a_{hom}-harmonic polynomial P, homogeneous of degree k (i.e. $P\in\mathcal{P}^k_{a_{hom}}$), into an a-harmonic function u with the same growth behavior is motivated by homogenization: We consider P as the "homogenized solution of the problem solved by u", so that we think in terms of the two-scale expansion $u \approx P + \phi_k\partial_k P$ and have that the error $\psi_P := u - (P+\phi_k\partial_k P)$ satisfies $-\nabla\cdot a\nabla\psi_P = \nabla\cdot((\phi_k a - \sigma_k)\nabla\partial_k P)$. In order to construct u, we reverse the logic and first construct a solution ψ_P to the above elliptic equation and then set $u := P + \phi_k\partial_k P + \psi_P$.

Theorem 1 (Existence of higher-order "correctors for polynomials")**.** *Let $d\geq 2$, $k\geq 2$, and suppose that the corrector ϕ and the flux-correction potential σ satisfy the growth assumption (4). Let r_0 be large enough so that $\varepsilon_{2,r_0}\leq\varepsilon_0$ holds (the existence of such r_0 is ensured by (4)), where $\varepsilon_0=\varepsilon_0(d,k,\lambda)>0$ is a constant defined in the proof below. Given any $P\in\mathcal{P}^k$, there exists a "corrector for polynomials" ψ_P satisfying*

$$-\nabla\cdot a\nabla\psi_P = \nabla\cdot((\phi_i a - \sigma_i)\nabla\partial_i P) \tag{9}$$

as well as

$$\sup_{R\geq r}\frac{1}{R^{k-1}}\left(\fint_{B_R}|\nabla\psi_P|^2\,dx\right)^{1/2} \leq C(d,k,\lambda)||P||\varepsilon_{2,r} \tag{10}$$

for any $r\geq r_0$. Moreover, ψ_P depends linearly on P.

Our ψ_P indeed enable us – in conjunction with the first-order correctors ϕ_i – to correct a_{hom}-harmonic kth-order polynomials.

Proposition 2. *Let $d\geq 2$, $k\geq 2$, and let $P\in\mathcal{P}^k_{a_{hom}}$. Suppose that ψ_P satisfies* (9)*. We then have*

$$-\nabla\cdot a\nabla(P+\phi_i\partial_i P+\psi_P) = 0.$$

Let us now state our $C^{k,\alpha}$ large-scale regularity result.

Theorem 3 ($C^{k,\alpha}$ large-scale excess-decay estimate)**.** *Let $d \geq 2$, $k \geq 2$, and suppose that (4) holds. Let u be an a-harmonic function. Let $\psi_P \equiv 0$ for linear polynomials P (in order to simplify notation) and let ψ_P be the functions constructed in Theorem 1 for higher-order polynomials. Consider the kth-order excess*

$$\mathrm{Exc}_k(r) := \inf_{P_\kappa \in \mathcal{P}^\kappa_{a_{hom}}} \qquad \fint_{B_r} \left| \nabla u - \nabla \sum_{\kappa=1}^{k} (P_\kappa + \phi_i \partial_i P_\kappa + \psi_{P_\kappa}) \right|^2 \, dx. \tag{11}$$

Let $0 < \alpha < 1$ and let r_0 be large enough so that $\varepsilon_{2,r_0} \leq \varepsilon_0$ holds (the existence of such r_0 is ensured by (4)), where $\varepsilon_0 = \varepsilon_0(d,k,\lambda,\alpha) > 0$ is a constant defined in the proof below. Then for all $r, R \geq r_0$ with $r < R$ the $C^{k,\alpha}$ excess-decay estimate

$$\mathrm{Exc}_k(r) \leq C(d,k,\lambda,\alpha) \left(\frac{r}{R}\right)^{2(k-1)+2\alpha} \mathrm{Exc}_k(R) \tag{12}$$

is satisfied.

Our large-scale $C^{k+1,\alpha}$ excess-decay estimate entails the following kth-order Liouville principle.

Corollary 4 (kth-order Liouville principle)**.** *Let $d \geq 2$, $k \geq 2$, and suppose that the assumption (4) is satisfied. Then the following property holds: Any a-harmonic function u satisfying the growth condition*

$$\liminf_{r\to\infty} \frac{1}{r^{k+1}} \left(\fint_{B_r} |u|^2 \, dx \right)^{1/2} = 0 \tag{13}$$

is of the form

$$u = a + b_i(x_i + \phi_i) + \sum_{\kappa=2}^{k} (P_\kappa + \phi_i \partial_i P_\kappa + \psi_{P_\kappa})$$

with some $a \in \mathbb{R}$, $b \in \mathbb{R}^d$, and $P_\kappa \in \mathcal{P}^\kappa_{a_{hom}}$ for $2 \leq \kappa \leq k$ (i.e. P_κ is a homogeneous a_{hom}-harmonic polynomial of degree κ). Here, the ψ_P denote the higher-order correctors whose existence is guaranteed by Theorem 1.

In particular, the space of all a-harmonic functions satisfying (13) *has the same dimension as if a was replaced by a constant coefficient, say a_{hom}.*

Note that the defining equation (9) and the growth condition

$$\lim_{r\to\infty} \frac{1}{r^{k+1}} \left(\fint_{B_r} |\psi_P|^2 \, dx \right)^{1/2} = 0$$

together determine the corrector of order k only up to a-harmonic "polynomials" of order $k-1$: The first-order corrector ϕ_i is determined only up to an additive constant; the second-order corrector ψ_P (for a quadratic polynomial P) is determined only up to corrected affine functions of the form $x \mapsto \xi \cdot (x + \phi) + c$ with $\xi \in \mathbb{R}^d$ and $c \in \mathbb{R}$, and so on. Let us denote by $\mathcal{P}^k_a$ the space of solutions to the problem $-\nabla \cdot a \nabla v = 0$ which satisfy the growth condition

$$\lim_{r\to\infty} \frac{1}{r^{k+1}} \left(\fint_{B_r} |v|^2 \, dx \right)^{1/2} = 0.$$

With this notation, our higher-order correctors yield a canonical isomorphism of the quotient spaces

$$\mathcal{P}^k_{a_{hom}} / \mathcal{P}^{k-1}_{a_{hom}} \cong \mathcal{P}^k_a / \mathcal{P}^{k-1}_a$$

defined by

$$[P] \mapsto [P + \phi \cdot \nabla P + \psi_P]$$

for any $P \in \mathcal{P}^k_{a_{hom}}$. Note that this isomorphism is independent of the particular choice of the correctors ϕ and ψ_P.

The basic strategy of the proof of Theorem 1 and Theorem 3 is as follows:

- First, under the assumption that we already have constructed an appropriate kth-order corrector on a ball B_R, we show a $C^{k,\alpha}$ excess-decay estimate on large scales within this ball for a-harmonic functions (Lemma 14). This result directly implies Theorem 3 as soon as we have proven the existence of a corrector on $\mathbb{R}^d$ (i. e. as soon as we have established Theorem 1). The basic idea for this first part of the proof is a standard approach from regularity theory: We transfer the regularity properties of the constant-coefficient equation $-\nabla\cdot a_{hom}\nabla u_{hom} = 0$ to the equation $-\nabla\cdot a\nabla u = 0$. To accomplish this, we employ an error estimate for the homogenization error.
- Our $C^{k,\alpha}$ estimate implies a $C^{k-1,1}$ theory for a-harmonic functions on balls B_R, provided that we have already constructed an appropriate kth-order corrector on B_R. This is done in Lemma 17.
- At last, we are able to build our corrector, starting from small balls and iteratively doubling the size of our balls: We decompose the right-hand side of equation (9) into contributions from dyadic annulli. In each step, we add the contribution from the next larger scale $\xi_P^{2^m r_0}$, determined as the Lax-Milgram solution to the problem

$$-\nabla\cdot a\nabla\xi_P^{2^m r_0} = \nabla\cdot(\chi_{B_{2^{m+1}r_0}-B_{2^m r_0}}(\phi_i a - \sigma_i)\nabla\partial_i P),$$

  to the corrector on the old scale $\psi_P^{2^m r_0}$. At this point, we make use of the $C^{k-1,1}$ theory to show that after possibly subtracting an appropriate $k-1$-th order a-harmonic "polynomial", the new contribution $\xi_P^{2^m r_0}$ displays kth-order decay in the interior $\{|x| < 2^m r_0\}$, down to the ball $\{|x| < r_0\}$. This ensures that on a ball of a given fixed size r with $r < 2^m r_0$, the contribution from the next larger scale does not destroy the smallness of the corrector. We are therefore able to construct the corrector on the next larger scale $\psi_P^{2^{m+1}r_0}$ as the sum of the corrector on the old scale $\psi_P^{2^m r_0}$ and the new contribution $\xi_P^{2^m r_0}$ minus the aforementioned a-harmonic "polynomial". This iterative enlargement is carried out in Lemma 18 and finally enables us to prove Theorem 1.
- The kth-order Liouville principle stated in Corollary 4 is an easy consequence of our $C^{k+1,\alpha}$ large-scale excess-decay estimate.

3. A $C^{2,\alpha}$ Large-Scale Regularity Theory for Homogeneous Elliptic Equations with Random Coefficients

For the reader's convenience, we shall first provide a proof for the $C^{2,\alpha}$ case of our theorems, as in this case the proofs are less technical while already containing the key ideas. In particular, the overall structure of our proofs is the same as in the $C^{k,\alpha}$ case. Since we shall use a somewhat simplified notation in the $C^{2,\alpha}$ case, let us reformulate the $C^{2,\alpha}$ case of our theorems using this notation.

Theorem 5 (Existence of second-order correctors)**.** *Let $d \geq 2$ and suppose that the corrector ϕ and the flux-correction potential σ satisfy the growth assumption (4). Let r_0 be large enough so that $\varepsilon_{2,r_0} \leq \varepsilon_0$ holds (the existence of such r_0 is ensured by (4)), where $\varepsilon_0 = \varepsilon_0(d,\lambda) > 0$ is a constant defined in the proof below. Given any $E \in \mathbb{R}^{d\times d}$, there exists a second-order corrector ψ_E satisfying*

$$-\nabla\cdot a\nabla\psi_E = E_{ij}\,\nabla\cdot[\sigma_{ij}+\sigma_{ji}+a(\phi_i e_j+\phi_j e_i)] \tag{14}$$

as well as

$$\sup_{R\geq r}\frac{1}{R}\left(\fint_{B_R}|\nabla\psi_E|^2\,dx\right)^{1/2} \leq C(d,\lambda)|E|\varepsilon_{2,r}$$

for any $r \geq r_0$. Moreover, the corrector $\nabla\psi_E$ depends linearly on E.

Due to the linear dependence of ψ_E on E, below we shall also write $E_{ij}\psi_{ij}$ in place of ψ_E.

Note that our second-order correctors indeed enable us – in conjunction with the first-order correctors ϕ_i – to correct a_{hom}-harmonic second-order polynomials.

Proposition 6. *Let $d \geq 2$ and let $E \in \mathcal{E}$ (i.e. assume that the polynomial $E_{ij}x_ix_j$ is a_{hom}-harmonic). Suppose that ψ_E satisfies* (14)*. We then have*

$$-\nabla\cdot a\nabla E_{ij}(x_ix_j + x_i\phi_j + \phi_i x_j + \psi_{ij}) = 0.$$

Our $C^{2,\alpha}$ large-scale regularity theorem reads as follows.

Theorem 7 ($C^{2,\alpha}$ large-scale excess-decay estimate)**.** *Let $d \geq 2$ and suppose that (4) holds. Let u be an a-harmonic function. Let ψ_E be the second-order corrector constructed in Theorem 5. Consider the second-order excess*

(15)

$$\mathrm{Exc}_2(r) := \inf_{b\in\mathbb{R}^d, E\in\mathcal{E}}\fint_{B_r}\left|\nabla u - \nabla\big(b_i(x_i+\phi_i) + E_{ij}(x_ix_j + x_i\phi_j + \phi_i x_j + \psi_{ij})\big)\right|^2\,dx.$$

Let $0 < \alpha < 1$ and let r_0 be large enough so that $\varepsilon_{2,r_0} \leq \varepsilon_0$ holds (the existence of such r_0 is ensured by (4)), where $\varepsilon_0 = \varepsilon_0(d,\lambda,\alpha) > 0$ is a constant defined in the proof below. Then for all $r, R \geq r_0$ with $r < R$ the $C^{2,\alpha}$ excess-decay estimate

$$\mathrm{Exc}_2(r) \leq C(d,\lambda,\alpha)\left(\frac{r}{R}\right)^{2+2\alpha}\mathrm{Exc}_2(R) \tag{16}$$

is satisfied.

Our large-scale excess-decay estimate entails the following $C^{2,\alpha}$ Liouville principle.

Corollary 8 ($C^{2,\alpha}$ Liouville principle)**.** *Let $d \geq 2$ and suppose that the assumption (4) is satisfied. Then the following property holds: Any a-harmonic function u satisfying the growth condition*

$$\liminf_{r\to\infty}\frac{1}{r^{2+\alpha}}\left(\fint_{B_r}|u|^2\,dx\right)^{1/2} = 0$$

for some $\alpha \in (0,1)$ is of the form

$$u = a + b_i(x_i+\phi_i) + E_{ij}(x_ix_j + x_i\phi_j + \phi_i x_j + \psi_{ij})$$

with some $a \in \mathbb{R}$, $b \in \mathbb{R}^d$, and $E \in \mathcal{E}$ (i.e. some $E \in \mathbb{R}^{d\times d}$ for which $E_{ij}x_ix_j$ is an a_{hom}-harmonic polynomial).

Let us start with the proof of Proposition 6, which only requires a simple computation.

Proof of Proposition 6. Making use of the fact that $E_{ij}((a_{hom})_{ij} + (a_{hom})_{ji}) = 0$ (in the third step below), we compute

$$\begin{aligned}
&E_{ij}\,\nabla\cdot(\sigma_{ij}+\sigma_{ji}) + E_{ij}\,\nabla\cdot a(\phi_i e_j + \phi_j e_i)\\
&\overset{(8)}{=} E_{ij}q_{ij} + E_{ij}q_{ji} + E_{ij}\,\nabla\cdot a(\phi_i e_j + \phi_j e_i)\\
&\overset{(7)}{=} E_{ij}(a_{jk}((\mathrm{Id})_{ik} + \partial_k\phi_i) - (a_{hom})_{ji}) + E_{ij}(a_{ik}((\mathrm{Id})_{jk} + \partial_k\phi_j) - (a_{hom})_{ij})\\
&\quad + E_{ij}\,\nabla\cdot a(\phi_i e_j + \phi_j e_i)\\
&= E_{ij}(a_{jk}(\partial_k x_i + \partial_k\phi_i) + a_{ik}(\partial_k x_j + \partial_k\phi_j))\\
&\quad + E_{ij}\,\nabla\cdot a(\phi_i\nabla x_j + \phi_j\nabla x_i)\\
&= E_{ij}(a\nabla(x_i+\phi_i)\cdot\nabla x_j + a\nabla(x_j+\phi_j)\cdot\nabla x_i) + E_{ij}\,\nabla\cdot a(\phi_i\nabla x_j + \phi_j\nabla x_i)\\
&\overset{(6)}{=} E_{ij}\,\nabla\cdot(x_j a\nabla(x_i+\phi_i) + x_i a\nabla(x_j+\phi_j)) + E_{ij}\,\nabla\cdot a(\phi_i\nabla x_j + \phi_j\nabla x_i).
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
&E_{ij}\,\nabla\cdot(\sigma_{ij}+\sigma_{ji}) + E_{ij}\,\nabla\cdot a(\phi_i e_j + e_i\phi_j)\\
&= E_{ij}\,\nabla\cdot a\nabla(x_ix_j + x_i\phi_j + \phi_i x_j),
\end{aligned}$$

which together with (14) implies our proposition. □

3.1. **The $C^{2,\alpha}$ excess-decay estimate.** To establish our $C^{2,\alpha}$ excess-decay estimate, we make use of the following lemma, which essentially generalizes Theorem 7 to correctors which are only available on balls B_R.

Lemma 9. *Let $d \geq 2$. For any $E \in \mathcal{E}$, denote by $\tilde\psi_E$ a solution to the equation of the second-order corrector (14) on the ball B_R (without boundary conditions); assume that $\tilde\psi_E$ depends linearly on E. Set*

$$\varepsilon_{\tilde\psi,r,R} := \sup_{r\leq\rho\leq R} \rho^{-1}\left(\max_{E\in\mathcal{E},|E|=1} \fint_{B_\rho} |\nabla\tilde\psi_E|^2\,dx\right)^{1/2}. \tag{17}$$

For an a-harmonic function u in B_R, consider the second-order excess

$$\widetilde{\mathrm{Exc}}_2(r) := \inf_{b\in\mathbb{R}^d, E\in\mathcal{E}} \fint_{B_r} \left|\nabla u - \nabla\big(b_i(x_i+\phi_i) + E_{ij}(x_ix_j + x_i\phi_j + \phi_i x_j + \tilde\psi_{ij})\big)\right|^2\,dx. \tag{18}$$

For any $0<\alpha<1$ there exists a constant $\varepsilon_{min}>0$ depending only on d, λ, and α such that the following assertion holds:

Suppose that $r_0>0$ satisfies $\varepsilon_{r_0} + \varepsilon_{\tilde\psi,r_0,R} \leq \varepsilon_{min}$. Then for all $r\in[r_0,R]$ the $C^{2,\alpha}$ excess-decay estimate

$$\widetilde{\mathrm{Exc}}_2(r) \leq C(d,\lambda,\alpha)\left(\frac{r}{R}\right)^{2+2\alpha}\widetilde{\mathrm{Exc}}_2(R) \tag{19}$$

is satisfied.

Note that the infimum in (18) *is actually attained, as the average integral in the definition of* $\widetilde{\mathrm{Exc}}_2(\rho)$ *is a quadratic functional of* b *and* E. *Denote by* $b^{\rho,min}$ *and* $E^{\rho,min}$ *a corresponding optimal choice of* b *and* E *in* (18). *We then have the estimates*

$$R^2|E^{r,min}-E^{R,min}|^2+|b^{r,min}-b^{R,min}|^2\leq C(d,\lambda,\alpha)\widetilde{\mathrm{Exc}}_2(R) \tag{20}$$

and

$$R^2|E^{r,min}|^2+|b^{r,min}|^2\leq C(d,\lambda,\alpha)\fint_{B_R}|\nabla u|^2\,dx. \tag{21}$$

Proof of Theorem 7. Theorem 7 obviously follows from Lemma 9 by setting $\tilde\psi_E := \psi_E$, with ψ_E being the second-order corrector whose existence is guaranteed by Theorem 5. □

The following lemma is essentially a special case of our $C^{2,\alpha}$ large-scale excess-decay estimate Lemma 9; it entails the general case of Lemma 9 (see below).

Lemma 10. *Let* $d\geq 2$ *and let* $R,r>0$ *satisfy* $r<R/4$ *and* $\varepsilon_R\leq 1$. *For any* $E\in\mathcal{E}$, *denote by* $\tilde\psi_E$ *a solution to the equation of the second-order corrector* (14) *on the ball* B_R *(without boundary conditions); assume that* $\tilde\psi_E$ *depends linearly on* E. *For an* a*-harmonic function* u *in* B_R, *consider again the second-order excess* (18). *Then the excess on the smaller ball* B_r *is estimated in terms of the excess on the larger ball* B_R *and our quantities* ε_R *and* $\nabla\tilde\psi_E$*: We have*

$$\widetilde{\mathrm{Exc}}_2(r)\leq C(d,\lambda)\left[\left(\frac{r}{R}\right)^4+\left(\varepsilon_R^{2/(d+1)^2}+R^{-2}\max_{E\in\mathcal{E},|E|=1}\fint_{B_R}|\nabla\tilde\psi_E|^2\,dx\right)\left(\frac{r}{R}\right)^{-d}\right]\times\widetilde{\mathrm{Exc}}_2(R).$$

Before proving Lemma 10, we would like to show how it implies Lemma 9.

Proof of Lemma 9. First choose $0<\theta\leq 1/4$ so small that the strict inequality $C(d,\lambda)\theta^4<\theta^{2+2\alpha}$ is satisfied (with $C(d,\lambda)$ being the constant from Lemma 10). Then, choose the threshold ε_{min} for $\varepsilon_{r_0}+\varepsilon_{\tilde\psi,r_0,R}$ so small that the estimate

$$C(d,\lambda)\left[\theta^4+\left(\varepsilon_{r_0}^{2/(d+1)^2}+\varepsilon_{\tilde\psi,r_0,R}^2\right)\theta^{-d}\right]\leq\theta^{2+2\alpha}$$

holds.

Let M be the largest integer for which $\theta^M R\geq r$ holds. Applying Lemma 10 inductively with $R_m:=\theta^{m-1}R$, $r_m:=\theta^m R$ for $1\leq m\leq M$, we infer

$$\widetilde{\mathrm{Exc}}_2(\theta^M R)\leq(\theta^{2+2\alpha})^M\widetilde{\mathrm{Exc}}_2(R).$$

Since we have trivially

$$\widetilde{\mathrm{Exc}}_2(r)\leq\left(\frac{r}{r_M}\right)^{-d}\widetilde{\mathrm{Exc}}_2(r_M)$$

and since by definition of M we have $r>\theta r_M$ and thus $\theta^M<\theta^{-1}\frac{r}{R}$ (where we recall $\theta=\theta(d,\lambda,\alpha)$), we infer

$$\widetilde{\mathrm{Exc}}_2(r)\leq C(d,\lambda,\alpha)\left(\frac{r}{R}\right)^{2+2\alpha}\widetilde{\mathrm{Exc}}_2(R).$$

It remains to show the estimates for $|b^{r,min}-b^{R,min}|$ and $|E^{r,min}-E^{R,min}|$ as well as the bounds for $|b^{r,min}|$ and $|E^{r,min}|$. To do so, let us first estimate the differences $|b^{R_m,min}-b^{r_m,min}|$ and $|E^{R_m,min}-E^{r_m,min}|$. We have the estimate

$$\begin{aligned}
&\fint_{B_{r_m}} \big|\nabla(b_i^{R_m,min}-b_i^{r_m,min})(x_i+\phi_i) \\
&\qquad\qquad + \nabla(E_{ij}^{R_m,min}-E_{ij}^{r_m,min})(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})\big|^2\,dx \\
&\leq 2\fint_{B_{r_m}} \Big|\nabla u-\nabla b_i^{r_m,min}(x_i+\phi_i)-\nabla E_{ij}^{r_m,min}(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})\Big|^2\,dx \\
&\quad + 2\fint_{B_{r_m}} \Big|\nabla u-\nabla b_i^{R_m,min}(x_i+\phi_i)-\nabla E_{ij}^{R_m,min}(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})\Big|^2\,dx \\
&\leq 2\widetilde{\mathrm{Exc}}_2(r_m)+2\left(\frac{R_m}{r_m}\right)^d\widetilde{\mathrm{Exc}}_2(R_m) \\
&\leq C(d,\lambda,\alpha)\left(\frac{r_m}{R}\right)^{2+2\alpha}\widetilde{\mathrm{Exc}}_2(R)+C(d,\lambda,\alpha)\theta^{-d}\left(\frac{R_m}{R}\right)^{2+2\alpha}\widetilde{\mathrm{Exc}}_2(R) \\
&\leq C(d,\lambda,\alpha)\left(\frac{r_m}{R}\right)^2(\theta^{2\alpha})^m\widetilde{\mathrm{Exc}}_2(R).
\end{aligned}$$

From Lemma 11 below, we thus obtain

$$|b^{R_m,min}-b^{r_m,min}|+R|E^{R_m,min}-E^{r_m,min}|\leq C(d,\lambda,\alpha)(\theta^\alpha)^m\sqrt{\widetilde{\mathrm{Exc}}_2(R)}.$$

Note that a similar estimate for the last increment $|b^{r_M,min}-b^{r,min}|+R|E^{r_M,min}-E^{r,min}|$ can be derived analogously. Taking the sum with respect to m and recalling that $R_1=R$ and $r_m=R_{m+1}$, we finally deduce

$$\begin{aligned}
|b^{R,min}-b^{r,min}|+R|E^{R,min}-E^{r,min}| &\leq C(d,\lambda,\alpha)\sum_{m=0}^M(\theta^\alpha)^m\sqrt{\widetilde{\mathrm{Exc}}_2(R)} \\
&\leq C(d,\lambda,\alpha)\sqrt{\widetilde{\mathrm{Exc}}_2(R)}.
\end{aligned}$$

It only remains to establish the last estimate for $|b^{r,min}|$ and $|E^{r,min}|$. By the previous estimate, it is sufficient to prove the corresponding bound for $b^{R,min}$ and $E^{R,min}$. This in turn is a consequence of the inequality

$$\begin{aligned}
&\fint_{B_R}\Big|\nabla b_i^{R,min}(x_i+\phi_i)+\nabla E_{ij}^{R,min}(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})\Big|^2\,dx \\
&\leq 2\widetilde{\mathrm{Exc}}_2(R)+2\fint_{B_R}|\nabla u|^2\,dx\leq 4\fint_{B_R}|\nabla u|^2\,dx
\end{aligned}$$

together with Lemma 11 below. □

The following lemma quantifies the linear independence of the corrected polynomials $x_i+\phi_i$, $E_{ij}(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})$; it is needed in the previous proof.

Lemma 11. *Suppose that for every $E\in\mathcal{E}\setminus\{0\}$, the functions ϕ and $\tilde\psi_E$ satisfy*

$$\rho^{-2}\fint_{B_\rho}|\phi|^2\,dx+\rho^{-2}|E|^{-2}\fint_{B_\rho}|\nabla\tilde\psi_E|^2\,dx\leq\varepsilon_0^2,$$

where $\varepsilon_0 = \varepsilon_0(d)$ is to be defined in the proof below. Then for any $b \in \mathbb{R}^d$ and any $E \in \mathcal{E}$, we have the estimate

$$\begin{aligned} &|b|^2 + \rho^2|E|^2 \\ &\leq C(d) \fint_{B_\rho} |\nabla b_i(x_i+\phi_i) + \nabla E_{ij}(x_ix_j + x_i\phi_j + \phi_ix_j + \tilde{\psi}_{ij})|^2 \, dx. \end{aligned} \tag{22}$$

Proof. Poincaré's inequality (with zero mean) and the triangle inequality imply

$$\begin{aligned} &\left(\fint_{B_\rho} |\nabla b_i(x_i+\phi_i) + \nabla E_{ij}(x_ix_j + x_i\phi_j + \phi_ix_j + \tilde{\psi}_{ij})|^2 \, dx\right)^{1/2} \\ &\geq \frac{1}{C(d)} \frac{1}{\rho} \inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} |b_i(x_i+\phi_i) + E_{ij}(x_ix_j + x_i\phi_j + \phi_ix_j + \tilde{\psi}_{ij}) - a|^2 \, dx\right)^{1/2} \\ &\geq \frac{1}{C(d)} \frac{1}{\rho} \Bigg[\inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} |b_ix_i + E_{ij}x_ix_j - a|^2 \, dx\right)^{1/2} \\ &\qquad - \inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} |b_i\phi_i + E_{ij}(x_i\phi_j + \phi_ix_j + \tilde{\psi}_{ij}) - a|^2 \, dx\right)^{1/2} \Bigg]. \end{aligned}$$

On the one hand, by transversality of constant, linear, and quadratic functions we have

$$\frac{1}{\rho} \inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} |b_ix_i + E_{ij}x_ix_j - a|^2 \, dx\right)^{1/2} \geq \frac{1}{C(d)}(|b| + \rho|E|).$$

On the other hand, we have by the triangle inequality and Poincaré's inequality

$$\begin{aligned} &\frac{1}{\rho} \inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} |b_i\phi_i + E_{ij}(x_i\phi_j + \phi_ix_j + \tilde{\psi}_{ij}) - a|^2 \, dx\right)^{1/2} \\ &\leq C(d) \left[(|b| + \rho|E|) \frac{1}{\rho} \left(\fint_{B_\rho} |\phi|^2 \, dx\right)^{1/2} + \rho|E| \frac{1}{\rho} \max_{\tilde{E}\in\mathcal{E}, |\tilde{E}|=1} \left(\fint_{B_\rho} |\nabla\psi_{\tilde{E}}|^2 \, dx\right)^{1/2} \right]. \end{aligned}$$

Putting these estimates together, by boundedness of the integrals in the previous line by $\varepsilon_0^2\rho^2$ our assertion is established. □

Proof of Lemma 10. In the proof of the lemma, we may assume that

$$\widetilde{\mathrm{Exc}}_2(R) = \fint_{B_R} |\nabla u|^2 \, dx. \tag{23}$$

To see this, recall that the infimum in the definition of $\widetilde{\mathrm{Exc}}_2(R)$ is actually attained. Denote the corresponding choices of b and E by b^{min} and E^{min}. Replacing u by $u - b_i^{min}(x_i + \phi_i) - E_{ij}^{min}(x_ix_j + x_i\phi_j + \phi_ix_j + \tilde{\psi}_{ij})$, we see that we may indeed assume (23): The new function is also a-harmonic due to (6) and Proposition 6.

We then apply Lemma 20 below to our function u. This yields an a_{hom}-harmonic function u_{hom} close to u which in particular satisfies

$$\fint_{B_{R/2}} |\nabla u_{hom}|^2 \, dx \leq C(d,\lambda) \fint_{B_R} |\nabla u|^2 \, dx.$$

By inner regularity theory for elliptic equations with constant coefficients, the a_{hom}-harmonic function u_{hom} satisfies

$$\begin{aligned}
&|\nabla u_{hom}(0)| + R \sup_{B_{R/4}} |\nabla^2 u_{hom}| + R^2 \sup_{B_{R/4}} |\nabla^3 u_{hom}| \\
&\leq C(d,\lambda) \left(\fint_{B_{R/2}} |\nabla u_{hom}|^2 \, dx \right)^{1/2} \leq C(d,\lambda) \left(\fint_{B_R} |\nabla u|^2 \, dx \right)^{1/2}.
\end{aligned}$$

Let us define

$$\begin{aligned}
b^{R,Taylor} &:= \nabla u_{hom}(0), \\
E^{R,Taylor} &:= \nabla^2 u_{hom}(0).
\end{aligned}$$

Since $-\nabla \cdot a_{hom} \nabla u_{hom} = 0$ holds, we infer $E^{R,Taylor}_{ij} (a_{hom})_{ij} = 0$ and therefore $E^{R,Taylor} \in \mathcal{E}$ (note that $E^{R,Taylor}_{ij} = E^{R,Taylor}_{ji}$). By Taylor's expansion of ∇u_{hom} around $x = 0$ we deduce for any $x \in B_{R/4}$ the bound

$$\left| \nabla u_{hom}(x) - b^{R,Taylor} - \frac{1}{2} E^{R,Taylor}_{ij} (x_j e_i + x_i e_j) \right| \leq |x|^2 \sup_{B_{R/4}} |\nabla^3 u_{hom}|.$$

Making use of the identity

$$\begin{aligned}
&(\mathrm{Id} + (\nabla \phi)^t) \nabla u_{hom} - \nabla \left(b^{R,Taylor}_i (x_i + \phi_i) + \frac{1}{2} E^{R,Taylor}_{ij} (x_i x_j + x_i \phi_j + \phi_i x_j) \right) \\
&\quad + \frac{1}{2} E^{R,Taylor}_{ij} (\phi_j e_i + \phi_i e_j) \\
&= (\mathrm{Id} + (\nabla \phi)^t) \left(\nabla u_{hom}(x) - b^{R,Taylor} - \frac{1}{2} E^{R,Taylor}_{ij} (x_j e_i + x_i e_j) \right),
\end{aligned}$$

the previous estimate yields in connection with the bound for $|\nabla^3 u_{hom}|$ and $r < R/4$

$$\begin{aligned}
&\fint_{B_r} \left| (\mathrm{Id} + (\nabla \phi)^t) \nabla u_{hom} - \nabla \left(b^{R,Taylor}_i (x_i + \phi_i) + \frac{1}{2} E^{R,Taylor}_{ij} (x_i x_j + x_i \phi_j + \phi_i x_j) \right) \right. \\
&\qquad \left. + \frac{1}{2} E^{R,Taylor}_{ij} (\phi_j e_i + \phi_i e_j) \right|^2 dx \\
&\leq C(d,\lambda) \left(\frac{r}{R} \right)^4 \fint_{B_R} |\nabla u|^2 \, dx \times \fint_{B_r} |\mathrm{Id} + (\nabla \phi)^t|^2 \, dx.
\end{aligned}$$

By the Caccioppoli inequality for the a-harmonic function $x_i + \phi_i$ (see (6)), we have

$$\fint_{B_r} |\mathrm{Id} + (\nabla \phi)^t|^2 \, dx \leq \frac{C(d,\lambda)}{r^2} \fint_{B_{2r}} |x + \phi|^2 \, dx \leq C(d,\lambda)(1 + \varepsilon_{2r}^2). \tag{24}$$

The approximation property of $u_{hom} + \phi_i \partial_i u_{hom}$ in $B_{R/2}$ from Lemma 20 below implies

$$\fint_{B_r} |\nabla u - \nabla (u_{hom} + \phi_i \partial_i u_{hom})|^2 \, dx \leq C(d,\lambda) \varepsilon_R^{2/(d+1)^2} \left(\frac{r}{R} \right)^{-d} \fint_{B_R} |\nabla u|^2 \, dx.$$

Combining the last three estimates and the equality

$$\begin{aligned}
&\nabla u-\nabla\left(b_i^{R,Taylor}(x_i+\phi_i)+\frac{1}{2}E_{ij}^{R,Taylor}(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})\right)\\
=&\Bigg[(\mathrm{Id}+(\nabla\phi)^t)\nabla u_{hom}-\nabla\left(b_i^{R,Taylor}(x_i+\phi_i)+\frac{1}{2}E_{ij}^{R,Taylor}(x_ix_j+x_i\phi_j+\phi_ix_j)\right)\\
&\quad+\frac{1}{2}E_{ij}^{R,Taylor}(\phi_je_i+\phi_ie_j)\Bigg]-\frac{1}{2}E_{ij}^{R,Taylor}(\phi_je_i+\phi_ie_j+\nabla\tilde\psi_{ij})\\
&+\Big[\nabla u-\nabla(u_{hom}+\phi_i\partial_iu_{hom})\Big]+\phi_i\nabla\partial_iu_{hom},
\end{aligned}$$

we infer

$$\begin{aligned}
&\fint_{B_r}\left|\nabla u-\nabla\left(b_i^{R,Taylor}(x_i+\phi_i)+\frac{1}{2}E_{ij}^{R,Taylor}(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})\right)\right|^2dx\\
&\leq 4\fint_{B_r}\Bigg|(\mathrm{Id}+(\nabla\phi)^t)\nabla u_{hom}-\nabla\left(b_i^{R,Taylor}(x_i+\phi_i)+\frac{1}{2}E_{ij}^{R,Taylor}(x_ix_j+x_i\phi_j+\phi_ix_j)\right)\\
&\qquad\qquad+\frac{1}{2}E_{ij}^{R,Taylor}(\phi_je_i+\phi_ie_j)\Bigg|^2dx\\
&\quad+4\fint_{B_r}\left|\frac{1}{2}E_{ij}^{R,Taylor}(\phi_je_i+\phi_ie_j+\nabla\tilde\psi_{ij})\right|^2dx\\
&\quad+4\fint_{B_r}|\nabla u-\nabla(u_{hom}+\phi_i\partial_iu_{hom})|^2dx\\
&\quad+4\fint_{B_r}|\phi_i\nabla\partial_iu_{hom}|^2dx\\
&\leq C(d,\lambda)\left(\frac{r}{R}\right)^4\left(1+\varepsilon_r^2\right)\fint_{B_R}|\nabla u|^2dx\\
&\quad+C(d)|E^{R,Taylor}|^2\left(r^2\varepsilon_r^2+\max_{E\in\mathcal{E},|E|=1}\fint_{B_r}|\nabla\tilde\psi_E|^2dx\right)\\
&\quad+C(d,\lambda)\varepsilon_R^{2/(d+1)^2}\left(\frac{r}{R}\right)^{-d}\fint_{B_R}|\nabla u|^2dx\\
&\quad+C(d)r^2\varepsilon_r^2\sup_{B_{R/4}}|\nabla^2u_{hom}|^2.
\end{aligned}$$

This finally yields in connection with the above bounds on ∇^2u_{hom} in $B_{R/4}$ (recall that $E^{R,Taylor}=\nabla^2u_{hom}(0)$)

$$\begin{aligned}
&\fint_{B_r}\left|\nabla u-\nabla\left(b_i^{R,Taylor}(x_i+\phi_i)+\frac{1}{2}E_{ij}^{R,Taylor}(x_ix_j+x_i\phi_j+\phi_ix_j+\tilde\psi_{ij})\right)\right|^2dx\\
&\leq C(d,\lambda)\left(\frac{r}{R}\right)^4\left(1+\varepsilon_r^2\right)\fint_{B_R}|\nabla u|^2dx\\
&\quad+C(d,\lambda)R^{-2}\fint_{B_R}|\nabla u|^2dx\left(r^2\varepsilon_r^2+\max_{E\in\mathcal{E},|E|=1}\fint_{B_r}|\nabla\tilde\psi_E|^2dx\right)\\
&\quad+C(d,\lambda)\varepsilon_R^{2/(d+1)^2}\left(\frac{r}{R}\right)^{-d}\fint_{B_R}|\nabla u|^2dx
\end{aligned}$$

$$
\begin{aligned}
&+ C(d,\lambda) r^2 \varepsilon_r^2 R^{-2} \fint_{B_R} |\nabla u|^2 \, dx \\
&\le C(d,\lambda) \left[\left(\frac{r}{R}\right)^4 + \left(\varepsilon_R^{2/(d+1)^2} + R^{-2} \max_{E\in\mathcal{E},|E|=1} \fint_{B_R} |\nabla \tilde\psi_E|^2 \, dx \right) \left(\frac{r}{R}\right)^{-d} \right] \\
&\quad \times \fint_{B_R} |\nabla u|^2 \, dx,
\end{aligned}
$$

where in the last step we have used the inequality $\varepsilon_r^2 \le \left(\frac{R}{r}\right)^d \varepsilon_R^2 \le \left(\frac{R}{r}\right)^d \varepsilon_R^{2/(d+1)^2}$. The new bound directly implies the desired estimate. □

3.2. **The $C^{1,1}$ excess-decay estimate.** We now show how our $C^{2,\alpha}$ excess-decay estimate for the second-order excess $\widetilde{\mathrm{Exc}}_2$ from Lemma 9 entails a $C^{1,1}$ excess-decay estimate for the first-order excess Exc.

Lemma 12. *Let $d \ge 2$ and $R > 0$. For any $E \in \mathcal{E}$, denote by $\tilde\psi_E$ a solution to the equation of the second-order corrector* (14) *on the ball B_R (without boundary conditions); assume that $\tilde\psi_E$ depends linearly on E. There exists a constant $\varepsilon_{min} > 0$ depending only on d and λ such that the following assertion holds:*

Suppose $r_0 \in (0,R]$ is so large that $\varepsilon_{r_0} \le \varepsilon_{min}$ and

$$
\sup_{r_0 \le \rho \le R} \rho^{-1} \left(\max_{E\in\mathcal{E},|E|=1} \fint_{B_\rho} |\nabla \tilde\psi_E|^2 \, dx \right)^{1/2} \le \varepsilon_{min}
$$

hold. Let u be an a-harmonic function on B_R. Then there exists $b^R \in \mathbb{R}^d$ for which the estimate

$$
\fint_{B_r} |\nabla u - \nabla b_i^R (x_i + \phi_i)|^2 \, dx \le C(d,\lambda) \left(\frac{r}{R}\right)^2 \fint_{B_R} |\nabla u|^2 \, dx
$$

holds for any $r \in [r_0, R]$. Furthermore, b^R depends linearly on u and satisfies

$$
|b^R|^2 \le C(d,\lambda) \fint_{B_R} |\nabla u|^2 \, dx.
$$

Proof. In Lemma 9, fix $\alpha := 1/2$. We then easily verify that Lemma 9 is applicable in our situation. Set $b^R := b^{r_0,min}$ and $E^R := E^{r_0,min}$; this implies that b^R depends linearly on u. The estimate (21) takes the form

$$
R^2 |E^R|^2 + |b^R|^2 \le C(d,\lambda) \fint_{B_R} |\nabla u|^2 \, dx.
$$

Furthermore, applying Lemma 9 with r_0 playing the role of r and r playing the role of R, we deduce from (20)

$$
\begin{aligned}
&r^2 |E^R - E^{r,min}|^2 + |b^R - b^{r,min}|^2 \le C(d,\lambda) \widetilde{\mathrm{Exc}}_2(r) \\
&\overset{(19)}{\le} C(d,\lambda) \left(\frac{r}{R}\right)^{2+2\alpha} \widetilde{\mathrm{Exc}}_2(R) \le C(d,\lambda) \left(\frac{r}{R}\right)^{2+2\alpha} \fint_{B_R} |\nabla u|^2 \, dx.
\end{aligned}
$$

We now estimate

$$
\begin{aligned}
&\fint_{B_r} |\nabla u - \nabla b_i^R (x_i+\phi_i)|^2 \, dx \\
&\leq 3\fint_{B_r} \left|\nabla u - \nabla b_i^{r,min}(x_i+\phi_i) - \nabla E_{ij}^{r,min}(x_i x_j + x_i\phi_j + \phi_i x_j + \tilde\psi_{ij})\right|^2 \, dx \\
&\quad + 3\fint_{B_r} \left|\nabla E_{ij}^{r,min}(x_i x_j + x_i\phi_j + \phi_i x_j + \tilde\psi_{ij})\right|^2 \, dx \\
&\quad + 3\fint_{B_r} |(b_i^{r,min} - b_i^R)\nabla(x_i+\phi_i)|^2 \, dx \\
&\leq 3\widetilde{\mathrm{Exc}}_2(r) \\
&\quad + C(d)|E^{r,min}|^2 \left(\fint_{B_r} |\phi|^2 + r^2|\operatorname{Id} + (\nabla\phi)^t|^2 \, dx + \max_{E\in\mathcal{E},|E|=1} \fint_{B_r} |\nabla\tilde\psi_E|^2 \, dx\right) \\
&\quad + 3|b^{r,min} - b^R|^2 \fint_{B_r} |\operatorname{Id} + (\nabla\phi)^t|^2 \, dx \\
&\overset{(19,24)}{\leq} C(d,\lambda)\left(\frac{r}{R}\right)^{2+2\alpha} \widetilde{\mathrm{Exc}}_2(R) + C(d,\lambda)|E^{r,min}|^2 r^2 (\varepsilon_r^2 + (1+\varepsilon_{2r}^2) + \varepsilon_{\tilde\psi,r_0,R}^2) \\
&\quad + C(d,\lambda)|b^{r,min} - b^R|^2 (1+\varepsilon_{2r}^2) \\
&\leq C(d,\lambda)\left(\frac{r}{R}\right)^{2+2\alpha} \widetilde{\mathrm{Exc}}_2(R) + C(d,\lambda)|E^{r,min}|^2 r^2 + C(d,\lambda)|b^{r,min} - b^R|^2.
\end{aligned}
$$

In conjunction with the two previous estimates, we infer

$$
\begin{aligned}
&\fint_{B_r} |\nabla u - \nabla b_i^R(x_i+\phi_i)|^2 \, dx \\
&\leq C(d,\lambda)\left[\left(\frac{r}{R}\right)^{2+2\alpha} + \left(\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^{2+2\alpha}\right) + \left(\frac{r}{R}\right)^{2+2\alpha}\right] \fint_{B_R} |\nabla u|^2 \, dx.
\end{aligned}
$$

Our lemma is therefore established. □

3.3. **Construction of second-order correctors.** Using the $C^{1,1}$ theory established in the previous subsection, we now proceed to the construction of our second-order corrector. The following lemma provides the inductive step; starting from a function which acts as a corrector on a ball B_R, we construct a function acting as a corrector on the ball B_{2R}.

Lemma 13. *Let $d \geq 2$ and let $r_0 > 0$ satisfy the estimate $\varepsilon_{2,r_0} \leq \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(d,\lambda)$ is to be chosen in the proof below. Then the following implication holds:*

Let $R = 2^M r_0$ for some $M \in \mathbb{N}_0$. Suppose that for every $E \in \mathbb{R}^{d\times d}$ we have a solution ψ_E^R to the equation

$$
-\nabla\cdot a\nabla\psi_E^R = E_{ij}\,\nabla\cdot\chi_{B_R}[\sigma_{ij} + \sigma_{ji} + a(\phi_i e_j + \phi_j e_i)]
$$

subject to the growth condition

$$
r^{-1}\left(\fint_{B_r} |\nabla\psi_E^R|^2 \, dx\right)^{1/2} \leq C_1(d,\lambda)|E| \sum_{m=0}^{M} \min\{1, 2^m r_0/r\}\varepsilon_{2^m r_0}
$$

for all $r \geq r_0$, where $C_1(d,\lambda)$ is a sufficiently large constant to be chosen in the proof below. Assume furthermore that ψ_E^R depends linearly on E.

Then for every $E \in \mathbb{R}^{d\times d}$ there exists a solution ψ_E^{2R} to the equation

$$-\nabla\cdot a\nabla\psi_E^{2R} = E_{ij}\,\nabla\cdot[\chi_{B_{2R}}(\sigma_{ij}+\sigma_{ji}+a(\phi_i e_j+\phi_j e_i))]$$

subject to the growth condition

$$r^{-1}\left(\fint_{B_r}|\nabla\psi_E^{2R}|^2\,dx\right)^{1/2} \leq C_1(d,\lambda)|E|\sum_{m=0}^{M+1}\min\{1,2^m r_0/r\}\varepsilon_{2^m r_0}$$

for all $r \geq r_0$. Furthermore, ψ_E^{2R} depends linearly on E and we have

$$r^{-1}\left(\fint_{B_r}|\nabla\psi_E^{2R}-\nabla\psi_E^{R}|^2\,dx\right)^{1/2} \leq C_1(d,\lambda)|E|\varepsilon_{2^{M+1}r_0}.$$

Proof. To establish the lemma, we first note that the assumptions of the lemma ensure that the $C^{1,1}$ excess-decay lemma (Lemma 12) is applicable on B_R with $\tilde\psi_E := \psi_E^R$. To see this, we estimate for any $r \in [r_0, R]$

$$r^{-1}\left(\fint_{B_r}|\nabla\psi_E^{R}|^2\,dx\right)^{1/2} \leq C_1(d,\lambda)|E|\varepsilon_{2,r_0} \leq C_1(d,\lambda)|E|\varepsilon_0.$$

By choosing $\varepsilon_0 > 0$ small enough depending only on d and λ and C_1 (which is to be chosen at the end of this proof), we can ensure that the assumption of Lemma 12 regarding smallness of $\varepsilon_{\tilde\psi,r_0,R}$ is satisfied.

Let now ξ_E^R be the weak solution on $\mathbb{R}^d$ with square integrable gradient, which is unique up to additive constants and whose existence follows from the Lax-Milgram theorem, to the problem

$$-\nabla\cdot a\nabla\xi_E^R = E_{ij}\,\nabla\cdot\chi_{B_{2R}-B_R}(\sigma_{ij}+\sigma_{ji}) + E_{ij}\,\nabla\cdot\chi_{B_{2R}-B_R}a(\phi_i e_j+\phi_j e_i).$$

Obviously, $\nabla\xi_E^R$ depends linearly on E; after fixing the additive constant e.g. by requiring $\int_{B_1}\xi_E^R\,dx = 0$, ξ_E^R itself depends linearly on E. Furthermore, we have the bound

$$\int_{\mathbb{R}^d}|\nabla\xi_E^R|^2\,dx \leq C(\lambda)|E|^2\int_{\mathbb{R}^d}\chi_{B_{2R}-B_R}|\sigma|^2+\chi_{B_{2R}-B_R}|\phi|^2\,dx$$

and therefore

$$\int_{\mathbb{R}^d}|\nabla\xi_E^R|^2\,dx \leq C(\lambda)|E|^2R^{2+d}\varepsilon_{2R}^2. \tag{25}$$

As ξ_E^R is a-harmonic in B_R, Lemma 12 now implies the existence of some $b_E^R \in \mathbb{R}^d$ for which the estimates

$$|b_E^R|^2 \leq C(d,\lambda)\fint_{B_R}|\nabla\xi_E^R|^2\,dx \leq C(d,\lambda)|E|^2R^2\varepsilon_{2R}^2 \tag{26}$$

and

$$\begin{aligned}\fint_{B_r}|\nabla\xi_E^R-\nabla(b_E^R)_i(x_i+\phi_i)|^2\,dx &\leq C(d,\lambda)\left(\frac{r}{R}\right)^2\fint_{B_R}|\nabla\xi_E^R|^2\,dx\\ &\leq C(d,\lambda)|E|^2r^2\varepsilon_{2R}^2\end{aligned}$$

hold for all $r \in [r_0, R]$ and which linearly depends on E.

Furthermore, we have for $r > R$

$$\begin{aligned}
&\fint_{B_r} |\nabla \xi_E^R - \nabla (b_E^R)_i (x_i + \phi_i)|^2 \, dx \\
&\overset{(24)}{\leq} C(d,\lambda) \left(r^{-d} \int_{B_r} |\nabla \xi_E^R|^2 \, dx + |b_E^R|^2 (1 + \varepsilon_{2r}^2) \right) \\
&\overset{(25,26)}{\leq} C(d,\lambda) |E|^2 R^2 \left(\left(\frac{R}{r} \right)^d + 1 + \varepsilon_{2r}^2 \right) \varepsilon_{2R}^2 \\
&\leq C(d,\lambda) |E|^2 R^2 \varepsilon_{2R}^2.
\end{aligned}$$

The combination of both r-ranges yields

$$\frac{1}{r} \left(\fint_{B_r} |\nabla \xi_E^R - \nabla (b_E^R)_i (x_i + \phi_i)|^2 \, dx \right)^{1/2} \leq C(d,\lambda) |E| \min\{1, 2R/r\} \varepsilon_{2R}. \tag{27}$$

In total, we see that

$$\psi_E^{2R} := \psi_E^R + \xi_E^R - (b_E^R)_i (x_i + \phi_i)$$

is the desired function (note in particular that the last term is a-harmonic), provided we choose C_1 to be the constant appearing in (27). □

We now establish existence of second-order correctors by means of the previous lemma.

Proof of Theorem 5. We just need to construct an "initial" second-order corrector $\psi_E^{r_0}$ subject to the properties of Lemma 13; then Lemma 13 yields a sequence $(\psi_E^{2^m r_0})_m$ which is a Cauchy sequence in $H^1(B_R)$ for every $R > 0$ due to the last estimate in the lemma and our assumption (4) which implies summability of $\varepsilon_{2^m r_0}$. Thus, the limit ψ_E satisfies the equation (14) in the whole space, depends linearly on E, and satisfies the estimate

$$r^{-1} \left(\fint_{B_r} |\nabla \psi_E|^2 \, dx \right)^{1/2} \leq C_1(d,\lambda) |E| \sum_{m=0}^{\infty} \min\{1, 2^m r_0 / r\} \varepsilon_{2^m r_0}$$

for any $r \geq r_0$.

To construct $\psi_E^{r_0}$, just use Lax-Milgram to find the solution $\psi_E^{r_0}$ on $\mathbb{R}^d$ with square-integrable gradient (unique up to an additive constant) to the equation

$$-\nabla \cdot a \nabla \psi_E^{r_0} = E_{ij} \, \nabla \cdot [\chi_{B_{r_0}} (\sigma_{ij} + \sigma_{ji} + a(\phi_i e_j + \phi_j e_i))].$$

Obviously, after fixing the additive constant appropriately $\psi_E^{r_0}$ depends linearly on E. Furthermore, we have the energy estimate

$$\int_{\mathbb{R}^d} |\nabla \psi_E^{r_0}|^2 \, dx \leq C(\lambda) |E|^2 \int_{\mathbb{R}^d} |\chi_{B_{r_0}} \sigma|^2 + |\chi_{B_{r_0}} a \phi|^2 \, dx,$$

i.e. for any $r \geq r_0$

$$\int_{B_r} |\nabla \psi_E^{r_0}|^2 \, dx \leq C(d,\lambda) |E|^2 \int_{B_{r_0}} |\phi|^2 + |\sigma|^2 \, dx$$

and therefore

$$\begin{aligned}\fint_{B_r} |\nabla \psi_E^{r_0}|^2 \, dx &\leq C(d,\lambda)|E|^2 r^{-d} \varepsilon_{r_0}^2 r_0^{2+d} \\ &\leq C(d,\lambda)|E|^2 r^2 \min\{1,(r_0/r)^2\}\varepsilon_{r_0}^2.\end{aligned}$$

We note that this provides the starting point for Lemma 13, possibly after enlarging the constant C_1 in the statement thereof. □

3.4. **Proof of the $C^{2,\alpha}$ Liouville principle.** The $C^{2,\alpha}$ Liouville principle (Corollary 8) is an easy consequence of our large-scale excess-decay estimate (Theorem 7).

Proof of Corollary 8. Let $\alpha \in (0,1)$ be such that

$$\lim_{R\to\infty} \frac{1}{R^{2+\alpha}} \left(\fint_{B_R} |u|^2 \, dx\right)^{1/2} = 0$$

holds. By the Caccioppoli estimate, we deduce

$$\lim_{R\to\infty} \frac{1}{R^{1+\alpha}} \left(\fint_{B_R} |\nabla u|^2 \, dx\right)^{1/2} = 0.$$

Fix $r \geq r_0$. The excess-decay estimate from Theorem 7 yields together with the trivial bound $\mathrm{Exc}_2(R) \leq \fint_{B_R} |\nabla u|^2 \, dx$ that

$$\begin{aligned}\mathrm{Exc}_2(r) &\leq C(d,\lambda,\alpha)\left(\frac{r}{R}\right)^{2+2\alpha} \mathrm{Exc}_2(R) \\ &\leq C(d,\lambda,\alpha) r^{2+2\alpha}\left(\frac{1}{R^{1+\alpha}}\left(\fint_{B_R} |\nabla u|^2 \, dx\right)^{1/2}\right)^2.\end{aligned}$$

Passing to the limit $R \to \infty$, we deduce that

$$\mathrm{Exc}_2(r) = 0$$

holds for every $r \geq r_0$. Therefore, on every B_r with $r \geq r_0$, ∇u can be represented *exactly* as the derivative of a corrected polynomial of second order (since the infimum in the definition of Exc_2 is actually attained, as noted at the beginning of the proof of Lemma 10), i.e. we have

$$\nabla u = \nabla b_i^r (x_i + \phi_i) + \nabla E_{ij}^r (x_i x_j + x_i \phi_j + \phi_i x_j + \psi_{ij})$$

in B_r for some $b^r \in \mathbb{R}^d$ and some $E^r \in \mathcal{E}$. It is not difficult to show that for r large enough, the b^r and E^r are actually independent of r and define some common $b \in \mathbb{R}^d$ and $E \in \mathcal{E}$: For example, one may use Lemma 9 to compare the b^r, E^r for two different radii $r_1, r_2 \geq r_0$; the estimate for $|b^{r_1} - b^{r_2}|$ and $|E^{r_1} - E^{r_2}|$ then contains the factor $\mathrm{Exc}_2(\max(r_1,r_2))$ and is therefore zero. Moreover, the gradient ∇u determines the function u itself up to a constant, i.e. we have

$$u = a + b_i(x_i + \phi_i) + E_{ij}(x_i x_j + x_i \phi_j + \phi_i x_j + \psi_{ij})$$

for some $a \in \mathbb{R}$, some $b \in \mathbb{R}^d$, and some $E \in \mathcal{E} \subset \mathbb{R}^{d\times d}$. □

4. A $C^{k,\alpha}$ Large-Scale Regularity Theory for Elliptic Equations with Random Coefficients

We now generalize our proofs from the $C^{2,\alpha}$ case in order to correct polynomials of order k and obtain our $C^{k,\alpha}$ large-scale regularity theory. We proceed by induction in k.

In order to establish our $C^{k,\alpha}$ regularity theory, let us first show Proposition 2, which – like the proof of Proposition 6 in the $C^{2,\alpha}$ case – only requires a simple computation.

Proof of Proposition 2. Making use of the fact that we have $(a_{hom})_{ij}\partial_i\partial_j P = 0$ (in the third step below), we obtain

$$\begin{aligned}
&-\nabla\cdot(\sigma_i\nabla\partial_i P)\\
&=(\nabla\cdot\sigma_i)\cdot\nabla\partial_i P\\
&\overset{(8)}{=} q_i\cdot\nabla\partial_i P\\
&\overset{(7)}{=} a(e_i+\nabla\phi_i)\cdot\nabla\partial_i P\\
&\overset{(6)}{=} \nabla\cdot(\partial_i P\, a(e_i+\nabla\phi_i)).
\end{aligned}$$

This yields

$$\begin{aligned}
&\nabla\cdot((\phi_i a-\sigma_i)\nabla\partial_i P)\\
&=\nabla\cdot a(\phi_i\nabla\partial_i P+\partial_i P e_i+\partial_i P\nabla\phi_i)\\
&=\nabla\cdot a\nabla(P+\phi_i\partial_i P),
\end{aligned}$$

which together with (9) implies our proposition. □

4.1. The $C^{k,\alpha}$ excess-decay estimate.

To establish our $C^{k,\alpha}$ excess-decay estimate, we make use of the following lemma, which essentially generalizes Theorem 3 to correctors that are only available on balls B_R.

Lemma 14. *Let $d\geq 2$ and $k\geq 2$. Suppose that Theorem 1 holds for orders $2,\dots,k-1$, and set $\psi_P\equiv 0$ for first-order polynomials P to simplify notation. For any $P\in\mathcal{P}^k_{a_{hom}}$, denote by $\tilde\psi_P$ a solution to the equation (9) on the ball B_R (without boundary conditions); assume that the $\tilde\psi_P$ depend linearly on P. Set*

$$\varepsilon_{\tilde\psi,r,R} := \sup_{r\leq\rho\leq R}\rho^{-(k-1)}\left(\max_{P\in\mathcal{P}^k_{a_{hom}},||P||=1}\fint_{B_\rho}|\nabla\tilde\psi_P|^2\,dx\right)^{1/2}. \tag{28}$$

For an a-harmonic function u in B_R, consider the kth-order excess

(29)

$$\widetilde{\mathrm{Exc}}_k(r) := \inf_{P_\kappa\in\mathcal{P}^\kappa_{a_{hom}}}\fint_{B_r}\left|\nabla u-\nabla\Big(\sum_{\kappa=1}^{k-1}(P_\kappa+\phi_i\partial_i P_\kappa+\psi_{P_\kappa})+(P_k+\phi_i\partial_i P_k+\tilde\psi_{P_k})\Big)\right|^2 dx.$$

For any $0<\alpha<1$ there exists a constant $\varepsilon_{min}>0$ depending only on d, k, λ, and α such that the following assertion holds:

Suppose that $r_0 > 0$ satisfies $\varepsilon_{2,r_0} + \varepsilon_{\tilde\psi,r_0,R} \leq \varepsilon_{min}$. Then for all $r \in [r_0, R]$ the $C^{k,\alpha}$ excess-decay estimate

$$\widetilde{\mathrm{Exc}}_k(r) \leq C(d,k,\lambda,\alpha)\left(\frac{r}{R}\right)^{2(k-1)+2\alpha}\widetilde{\mathrm{Exc}}_k(R) \tag{30}$$

is satisfied.

Note that the infimum in (29) is actually attained, as the average integral in the definition of $\widetilde{\mathrm{Exc}}_2(\rho)$ is a quadratic functional of P_κ. Denote by $P_\kappa^{\rho,min}$ a corresponding optimal choice of P_κ in (29). We then have the estimates

$$\sum_{\kappa=1}^{k} R^{2(\kappa-1)}||P_\kappa^{r,min} - P_\kappa^{R,min}||^2 \leq C(d,k,\lambda,\alpha)\widetilde{\mathrm{Exc}}_k(R) \tag{31}$$

and

$$\sum_{\kappa=1}^{k} R^{2(\kappa-1)}||P_\kappa^{r,min}||^2 \leq C(d,k,\lambda,\alpha)\fint_{B_R} |\nabla u|^2 \,dx. \tag{32}$$

Proof of Theorem 3. Once we have shown Theorem 1, Theorem 3 obviously follows from Lemma 14 by setting $\tilde\psi_{P_k} := \psi_{P_k}$, with ψ_{P_k} being the kth-order corrector whose existence is established in Theorem 1. □

The following lemma is essentially a special case of our $C^{k,\alpha}$ large-scale excess-decay estimate Lemma 14; it entails the general case of Lemma 14 (see below).

Lemma 15. *Let $d \geq 2$, $k \geq 2$, and let $R, r > 0$ satisfy $r < R/4$ and $\varepsilon_{2,R} \leq \varepsilon_0(d,k-1,\lambda)$, with $\varepsilon_0(d,k-1,\lambda)$ being the constant from Theorem 1 for the orders $2, \ldots, k-1$. Assume that Theorem 1 holds for orders $2, \ldots, k-1$, and let $\psi_P \equiv 0$ for linear polynomials P in order to simplify notation. For any $P \in \mathcal{P}^\kappa_{a_{hom}}$, denote by $\tilde\psi_P$ a solution to the equation (9) on the ball B_R (without boundary conditions); assume that $\tilde\psi_P$ depends linearly on P. For an a-harmonic function u on B_R, consider again again the kth-order excess (29). Then the excess on the smaller ball B_r is estimated in terms of the excess on the larger ball B_R and our quantities $\varepsilon_{2,R}$ and $\nabla\tilde\psi_P$: We have*

$$\begin{aligned}\widetilde{\mathrm{Exc}}_k(r) &\leq C(d,k,\lambda)\widetilde{\mathrm{Exc}}_k(R)\\ &\times\left[\left(\frac{r}{R}\right)^{2k} + \left(\varepsilon_{2,R}^{2/(d+1)^2} + R^{-2(k-1)}\max_{P\in\mathcal{P}^k_{a_{hom}},||P||=1}\fint_{B_R}|\nabla\tilde\psi_P|^2\,dx\right)\left(\frac{r}{R}\right)^{-d}\right].\end{aligned}$$

Before proving Lemma 15, we would like to show how it implies Lemma 14.

Proof of Lemma 14. First choose $0 < \theta \leq 1/4$ so small that the strict inequality $C(d,k,\lambda)\theta^{2k} < \theta^{2(k-1)+2\alpha}$ is satisfied (with $C(d,k,\lambda)$ being the constant from Lemma 15). Then, choose the threshold ε_{min} for $\varepsilon_{2,r_0} + \varepsilon_{\tilde\psi,r_0,R}$ so small that the estimate

$$C(d,k,\lambda)\left[\theta^{2k} + \left(\varepsilon_{2,r_0}^{2/(d+1)^2} + \varepsilon^2_{\tilde\psi,r_0,R}\right)\theta^{-d}\right] \leq \theta^{2(k-1)+2\alpha}$$

holds.

Let M be the largest integer for which $\theta^M R \geq r$ holds. Applying Lemma 15 inductively with $R_m := \theta^{m-1}R$, $r_m := \theta^m R$ for $1 \leq m \leq M$, we infer

$$\widetilde{\mathrm{Exc}}_k(\theta^M R) \leq (\theta^{2(k-1)+2\alpha})^M\widetilde{\mathrm{Exc}}_k(R).$$

Since we have trivially

$$\widetilde{\mathrm{Exc}}_k(r) \leq \left(\frac{r}{r_M}\right)^{-d} \widetilde{\mathrm{Exc}}_k(r_M)$$

and since by definition of M we have $r > \theta r_M$ and thus $\theta^M < \theta^{-1}\frac{r}{R}$ (where we recall $\theta = \theta(d,k,\lambda,\alpha)$), we infer

$$\widetilde{\mathrm{Exc}}_k(r) \leq C(d,k,\lambda,\alpha)\left(\frac{r}{R}\right)^{2(k-1)+2\alpha} \widetilde{\mathrm{Exc}}_k(R).$$

It remains to show the estimates for $||P_\kappa^{r,min} - P_\kappa^{R,min}||$ as well as the bounds for $||P_\kappa^{r,min}||$. To do so, let us first estimate the differences $||P_\kappa^{R_m,min} - P_\kappa^{r_m,min}||$ of two successive polynomials. We have the estimate

$$\begin{aligned}
&\fint_{B_{r_m}} \Bigg|\nabla \sum_{\kappa=1}^{k-1} \Big(P_\kappa^{R_m,min} - P_\kappa^{r_m,min} + \phi_i\partial_i(P_\kappa^{R_m,min} - P_\kappa^{r_m,min}) + \psi_{P_\kappa^{R_m,min} - P_\kappa^{r_m,min}}\Big) \\
&\qquad + \nabla\Big(P_k^{R_m,min} - P_k^{r_m,min} + \phi_i\partial_i(P_k^{R_m,min} - P_k^{r_m,min}) + \tilde\psi_{P_k^{R_m,min} - P_k^{r_m,min}}\Big)\Bigg|^2 \,dx \\
&\leq 2\fint_{B_{r_m}} \Bigg|\nabla u - \nabla \sum_{\kappa=1}^{k-1} \Big(P_\kappa^{r_m,min} + \phi_i\partial_i P_\kappa^{r_m,min} + \psi_{P_\kappa^{r_m,min}}\Big) \\
&\qquad\qquad - \nabla\Big(P_k^{r_m,min} + \phi_i\partial_i P_k^{r_m,min} + \tilde\psi_{P_k^{r_m,min}}\Big)\Bigg|^2 \,dx \\
&\quad + 2\fint_{B_{r_m}} \Bigg|\nabla u - \nabla \sum_{\kappa=1}^{k-1} \Big(P_\kappa^{R_m,min} + \phi_i\partial_i P_\kappa^{R_m,min} + \psi_{P_\kappa^{R_m,min}}\Big) \\
&\qquad\qquad - \nabla\Big(P_k^{R_m,min} + \phi_i\partial_i P_k^{R_m,min} + \tilde\psi_{P_k^{R_m,min}}\Big)\Bigg|^2 \,dx \\
&\leq 2\widetilde{\mathrm{Exc}}_k(r_m) + 2\left(\frac{R_m}{r_m}\right)^d \widetilde{\mathrm{Exc}}_k(R_m) \\
&\leq C(d,k,\lambda,\alpha)\left(\frac{r_m}{R}\right)^{2(k-1)+2\alpha} \widetilde{\mathrm{Exc}}_k(R) + C(d,k,\lambda,\alpha)\theta^{-d}\left(\frac{R_m}{R}\right)^{2(k-1)+2\alpha} \widetilde{\mathrm{Exc}}_k(R) \\
&\leq C(d,k,\lambda,\alpha)\left(\frac{r_m}{R}\right)^{2(k-1)} (\theta^{2\alpha})^m \widetilde{\mathrm{Exc}}_k(R).
\end{aligned}$$

From Lemma 16 below, we thus obtain

$$\sum_{\kappa=1}^{k} R^{\kappa-1}||P_\kappa^{R_m,min} - P_\kappa^{r_m,min}|| \leq C(d,k,\lambda,\alpha)(\theta^\alpha)^m \sqrt{\widetilde{\mathrm{Exc}}_k(R)}.$$

A similar estimate for the last increment $\sum_{\kappa=1}^k R^{\kappa-1}||P_\kappa^{r_M,min} - P_\kappa^{r,min}||$ can be derived analogously. Taking the sum with respect to m and recalling that $R_1 = R$

and $r_m = R_{m+1}$, we finally deduce

$$\begin{aligned}
&\sum_{\kappa=1}^{k} R^{\kappa-1}||P_\kappa^{R,min} - P_\kappa^{r,min}|| \\
&\leq C(d,k,\lambda,\alpha) \sum_{m=1}^{M} (\theta^\alpha)^m \sqrt{\widetilde{\mathrm{Exc}}_k(R)} \\
&\leq C(d,k,\lambda,\alpha)\sqrt{\widetilde{\mathrm{Exc}}_k(R)}.
\end{aligned}$$

It only remains to establish the last estimate for $||P_\kappa^{r,min}||$. By the previous estimate, it is sufficient to prove the corresponding bound for $||P_\kappa^{R,min}||$. This in turn is a consequence of the obvious inequality

$$\begin{aligned}
&\fint_{B_R} \bigg|\nabla \sum_{\kappa=1}^{k-1} \Big(P_\kappa^{R,min} + \phi_i\partial_i P_\kappa^{R,min} + \psi_{P_\kappa^{R,min}}\Big) \\
&\qquad + \nabla\Big(P_k^{R,min} + \phi_i\partial_i P_k^{R,min} + \tilde\psi_{P_k^{R,min}}\Big)\bigg|^2 \,dx \\
&\leq 2\widetilde{\mathrm{Exc}}_k(R) + 2\fint_{B_R} |\nabla u|^2 \,dx \leq 4\fint_{B_R} |\nabla u|^2 \,dx
\end{aligned}$$

in conjunction with Lemma 16 below. □

The following lemma quantifies the linear independence of the corrected polynomials $P_\kappa + \phi_i\partial_i P_\kappa + \psi_{P_\kappa}$ (with $1 \leq \kappa \leq k$); it is needed for the previous proof.

Lemma 16. *Suppose that the functions ϕ and $\tilde\psi_{P_\kappa}$ ($2 \leq \kappa \leq k$) satisfy*

$$\rho^{-2}\fint_{B_\rho} |\phi|^2 \,dx + \sum_{\kappa=2}^{k} \rho^{-2(\kappa-1)} \max_{P\in\mathcal{P}^\kappa_{a_{hom}}, ||P||=1} ||P||^{-2}\fint_{B_\rho} |\nabla\tilde\psi_P|^2 \,dx \leq \varepsilon_0^2,$$

where $\varepsilon_0 = \varepsilon_0(d,k)$ is to be defined in the proof below. Set $\tilde\psi_P \equiv 0$ for linear polynomials P in order to simplify notation. Then for any $P_\kappa \in \mathcal{P}^\kappa_{a_{hom}}$ ($1 \leq \kappa \leq k$) we have the estimate

$$\sum_{\kappa=1}^{k} \rho^{2(\kappa-1)}||P_\kappa||^2 \leq C(d,k)\fint_{B_\rho} \bigg|\nabla \sum_{\kappa=1}^{k} (P_\kappa + \phi_i\partial_i P_\kappa + \tilde\psi_{P_\kappa})\bigg|^2 \,dx. \tag{33}$$

Proof. Poincaré's inequality (with zero mean) and the triangle inequality imply

$$\begin{aligned}
&\left(\fint_{B_\rho} \bigg|\nabla \sum_{\kappa=1}^{k} (P_\kappa + \phi_i\partial_i P_\kappa + \tilde\psi_{P_\kappa})\bigg|^2 \,dx\right)^{1/2} \\
&\geq \frac{1}{C(d)}\frac{1}{\rho}\inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} \bigg|\sum_{\kappa=1}^{k} (P_\kappa + \phi_i\partial_i P_\kappa + \tilde\psi_{P_\kappa}) - a\bigg|^2 \,dx\right)^{1/2} \\
&\geq \frac{1}{C(d)}\frac{1}{\rho}\Bigg[\inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} \bigg|\sum_{\kappa=1}^{k} P_\kappa - a\bigg|^2 \,dx\right)^{1/2} \\
&\qquad - \inf_{a\in\mathbb{R}} \left(\fint_{B_\rho} \bigg|\sum_{\kappa=1}^{k} (\phi_i\partial_i P_\kappa + \tilde\psi_{P_\kappa}) - a\bigg|^2 \,dx\right)^{1/2}\Bigg].
\end{aligned}$$

On the one hand, by transversality of constant, linear, homogeneous second-order, ..., and homogeneous kth-order polynomials we have

$$\frac{1}{\rho}\inf_{a\in\mathbb{R}}\left(\fint_{B_\rho}\left|\sum_{\kappa=1}^{k}P_\kappa - a\right|^2\,dx\right)^{1/2} \geq \frac{1}{C(d,k)}\sum_{\kappa=1}^{k}\rho^{\kappa-1}||P_\kappa||.$$

On the other hand, we have by the triangle inequality and Poincaré's inequality

$$\begin{aligned}
&\frac{1}{\rho}\inf_{a\in\mathbb{R}}\left(\fint_{B_\rho}\left|\sum_{\kappa=1}^{k}(\phi_i\partial_i P_\kappa + \tilde{\psi}_{P_\kappa}) - a\right|^2\,dx\right)^{1/2} \\
&\leq C(d,k)\Bigg[\left(\sum_{\kappa=1}^{k}\rho^{\kappa-1}||P_\kappa||\right)\frac{1}{\rho}\left(\fint_{B_\rho}|\phi|^2\,dx\right)^{1/2} \\
&\qquad + \sum_{\kappa=2}^{k}\rho^{\kappa-1}||P_\kappa||\frac{1}{\rho^{\kappa-1}}\max_{P\in\mathcal{P}^\kappa,||P||=1}\left(\fint_{B_\rho}|\nabla\tilde{\psi}_P|^2\,dx\right)^{1/2}\Bigg].
\end{aligned}$$

Putting these estimates together, by boundedness of the integrals in the previous line by $\varepsilon_0^2\rho^{2(\kappa-1)}$ our assertion is established. □

Proof of Lemma 15. In the proof of the lemma, we may assume that

$$\widetilde{\mathrm{Exc}}_k(R) = \fint_{B_R}|\nabla u|^2\,dx. \tag{34}$$

To see this, recall that the infimum in the definition of $\widetilde{\mathrm{Exc}}_k(R)$ is actually attained. Denote the corresponding choices of P_κ by P_κ^{min}. Replacing u by $u-\sum_{\kappa=1}^{k-1}(P_\kappa^{min}+\phi_i\partial_i P_\kappa^{min}+\psi_{P_\kappa^{min}})-(P_k^{min}+\phi_i\partial_i P_k^{min}+\tilde{\psi}_{P_k^{min}})$, we see that we may indeed assume (34): The new function is also a-harmonic due to (6) and Proposition 2.

We then apply Lemma 20 below to our function u. This yields an a_{hom}-harmonic function u_{hom} close to u which in particular satisfies

$$\fint_{B_{R/2}}|\nabla u_{hom}|^2\,dx \leq C(d,\lambda)\fint_{B_R}|\nabla u|^2\,dx.$$

By inner regularity theory for elliptic equations with constant coefficients, the a_{hom}-harmonic function u_{hom} satisfies

$$\begin{aligned}
&|\nabla u_{hom}(0)| + R\sup_{B_{R/4}}|\nabla^2 u_{hom}| + \sum_{\kappa=2}^{k}R^\kappa\sup_{B_{R/4}}|\nabla^{\kappa+1}u_{hom}| \\
&\leq C(d,k,\lambda)\left(\fint_{B_{R/2}}|\nabla u_{hom}|^2\,dx\right)^{1/2} \leq C(d,k,\lambda)\left(\fint_{B_R}|\nabla u|^2\,dx\right)^{1/2}.
\end{aligned}\tag{35}$$

Let $P_\kappa^{R,Taylor}$ (for $1\leq\kappa\leq k$) be the term of order κ in the Taylor expansion of u_{hom} at $x_0=0$. We now show (for $\kappa\geq 2$, as for $\kappa=1$ this assertion is trivial) that $P_\kappa^{R,Taylor}\in\mathcal{P}_{a_{hom}}^\kappa$. The term-wise Hessian of the Taylor series of u_{hom} yields the Taylor series of $\nabla^2 u_{hom}$. We now know that $a_{hom}:\nabla^2 u_{hom}=0$; thus, the Taylor series of $a_{hom}:\nabla^2 u_{hom}$ is identically zero and by equating the coefficients we deduce $a_{hom}:\nabla^2 P_\kappa^{R,Taylor}=0$ for $2\leq\kappa\leq k$.

As the term-wise derivative of the Taylor series of u_{hom} yields the Taylor series of ∇u_{hom}, we obtain by the standard error estimate for the Taylor expansion of ∇u_{hom} at $x_0 = 0$ for any $x \in B_{R/4}$ the estimate

$$\left|\nabla u_{hom}(x) - \sum_{\kappa=1}^{k} \nabla P_\kappa^{R,Taylor}(x)\right| \leq |x|^k \sup_{B_{R/4}} |\nabla^{k+1} u_{hom}|.$$

Making use of the identity

$$\begin{aligned}
&(\mathrm{Id} + (\nabla\phi)^t)\nabla u_{hom} - \nabla \sum_{\kappa=1}^{k} (P_\kappa^{R,Taylor} + \phi_i \partial_i P_\kappa^{R,Taylor}) \\
&+ \sum_{\kappa=2}^{k} \phi_i \nabla \partial_i P_\kappa^{R,Taylor} \\
&= (\mathrm{Id} + (\nabla\phi)^t)\left(\nabla u_{hom}(x) - \sum_{\kappa=1}^{k} \nabla P_\kappa^{R,Taylor}(x)\right),
\end{aligned}$$

the previous estimate yields in connection with the bound for $|\nabla^{k+1} u_{hom}|$ and $r < R/4$

$$\begin{aligned}
&\fint_{B_r} \bigg|(\mathrm{Id} + (\nabla\phi)^t)\nabla u_{hom} - \nabla \sum_{\kappa=1}^{k} (P_\kappa^{R,Taylor} + \phi_i \partial_i P_\kappa^{R,Taylor}) \\
&\qquad + \sum_{\kappa=2}^{k} \phi_i \nabla \partial_i P_\kappa^{R,Taylor}\bigg|^2 \,dx \\
&\leq C(d,k,\lambda) \left(\frac{r}{R}\right)^{2k} \fint_{B_R} |\nabla u|^2 \,dx \times \fint_{B_r} |\mathrm{Id} + (\nabla\phi)^t|^2 \,dx.
\end{aligned}$$

By the Caccioppoli inequality for the a-harmonic function $x_i + \phi_i$ (see (6)), we have

$$\fint_{B_r} |\mathrm{Id} + (\nabla\phi)^t|^2 \,dx \leq \frac{C(d,\lambda)}{r^2} \fint_{B_{2r}} |x + \phi|^2 \,dx \leq C(d,\lambda)(1 + \varepsilon_{2r}^2). \tag{36}$$

The approximation property of $u_{hom} + \phi_i \partial_i u_{hom}$ in $B_{R/2}$ from Lemma 20 below implies

$$\fint_{B_r} |\nabla u - \nabla(u_{hom} + \phi_i \partial_i u_{hom})|^2 \,dx \leq C(d,\lambda) \varepsilon_R^{2/(d+1)^2} \left(\frac{r}{R}\right)^{-d} \fint_{B_R} |\nabla u|^2 \,dx.$$

Combining the last three estimates and the equality

$$\begin{aligned}
&\nabla u - \nabla \sum_{\kappa=1}^{k-1} \left(P_\kappa^{R,Taylor} + \phi_i \partial_i P_\kappa^{R,Taylor} + \psi_{P_\kappa^{R,Taylor}}\right) \\
&- \nabla\Big(P_k^{R,Taylor} + \phi_i \partial_i P_k^{R,Taylor} + \tilde\psi_{P_k^{R,Taylor}}\Big) \\
=&\bigg[(\mathrm{Id} + (\nabla\phi)^t)\nabla u_{hom} - \nabla \sum_{\kappa=1}^{k} \left(P_\kappa^{R,Taylor} + \phi_i \partial_i P_\kappa^{R,Taylor}\right) \\
&\quad + \sum_{\kappa=2}^{k} \phi_i \nabla \partial_i P_\kappa^{R,Taylor}\bigg] - \sum_{\kappa=2}^{k} \phi_i \nabla \partial_i P_\kappa^{R,Taylor} - \sum_{\kappa=2}^{k-1} \nabla \psi_{P_\kappa^{R,Taylor}} - \nabla \tilde\psi_{P_k^{R,Taylor}} \\
&+ \Big[\nabla u - \nabla(u_{hom} + \phi_i \partial_i u_{hom})\Big] + \phi_i \nabla \partial_i u_{hom},
\end{aligned}$$

we infer

$$
\begin{aligned}
&\fint_{B_r} \bigg|\nabla u-\nabla \sum_{\kappa=1}^{k-1}\Big(P_\kappa^{R,Taylor}+\phi_i\partial_i P_\kappa^{R,Taylor}+\psi_{P_\kappa^{R,Taylor}}\Big)\\
&\qquad -\nabla\Big(P_k^{R,Taylor}+\phi_i\partial_i P_k^{R,Taylor}+\tilde\psi_{P_k^{R,Taylor}}\Big)\bigg|^2\,dx\\
&\leq 6\fint_{B_r}\bigg|(\mathrm{Id}+(\nabla\phi)^t)\nabla u_{hom}-\nabla\sum_{\kappa=1}^{k}\Big(P_\kappa^{R,Taylor}+\phi_i\partial_i P_\kappa^{R,Taylor}\Big)\\
&\qquad\qquad +\sum_{\kappa=2}^{k}\phi_i\nabla\partial_i P_\kappa^{R,Taylor}\bigg|^2\,dx\\
&\quad + C(k)\fint_{B_r}\sum_{\kappa=2}^{k}|\nabla^2 P_\kappa^{R,Taylor}|^2|\phi|^2\,dx\\
&\quad + C(k)\fint_{B_r}\sum_{\kappa=2}^{k-1}\big|\nabla\psi_{P_\kappa}^{R,Taylor}\big|^2\,dx\\
&\quad + 6\fint_{B_r}\big|\nabla\tilde\psi_{P_k^{R,Taylor}}\big|^2\,dx\\
&\quad + 6\fint_{B_r}|\nabla u-\nabla(u_{hom}+\phi_i\partial_i u_{hom})|^2\,dx\\
&\quad + 6\fint_{B_r}|\phi_i\nabla\partial_i u_{hom}|^2\,dx\\
&\leq C(d,k,\lambda)\left(\frac{r}{R}\right)^{2k}\left(1+\varepsilon_r^2\right)\fint_{B_R}|\nabla u|^2\,dx\\
&\quad + C(d,k)\sum_{\kappa=2}^{k} r^{2(\kappa-1)}||P_\kappa^{R,Taylor}||^2\varepsilon_r^2\\
&\quad + C(d,k)\sum_{\kappa=2}^{k-1}||P_\kappa^{R,Taylor}||^2\max_{P\in\mathcal{P}^\kappa}||P||^{-2}\fint_{B_r}|\nabla\psi_P|^2\,dx\\
&\quad + C(d,k)||P_k^{R,Taylor}||^2\max_{P\in\mathcal{P}^k_{a_{hom}}}||P||^{-2}\fint_{B_r}|\nabla\tilde\psi_P|^2\,dx\\
&\quad + C(d,\lambda)\varepsilon_R^{2/(d+1)^2}\left(\frac{r}{R}\right)^{-d}\fint_{B_R}|\nabla u|^2\,dx\\
&\quad + C(d)r^2\varepsilon_r^2\sup_{B_{R/4}}|\nabla^2 u_{hom}|^2.
\end{aligned}
$$

This finally yields in connection with the bounds on $\nabla^\kappa u_{hom}$ in $B_{R/4}$ (see (35)) which in particular imply $||P_\kappa^{R,Taylor}||\leq C(d,k,\lambda)R^{1-\kappa}\left(\fint_{B_R}|\nabla u|^2\,dx\right)^{1/2}$

$$
\begin{aligned}
&\fint_{B_r}\bigg|\nabla u-\nabla\sum_{\kappa=1}^{k-1}\Big(P_\kappa^{R,Taylor}+\phi_i\partial_i P_\kappa^{R,Taylor}+\psi_{P_\kappa^{R,Taylor}}\Big)\\
&\qquad -\nabla\Big(P_k^{R,Taylor}+\phi_i\partial_i P_k^{R,Taylor}+\tilde\psi_{P_k^{R,Taylor}}\Big)\bigg|^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(d,k,\lambda)\left(\frac{r}{R}\right)^{2k}\left(1+\varepsilon_r^2\right)\fint_{B_R}|\nabla u|^2\,dx\\
&\quad+C(d,k,\lambda)\varepsilon_r^2\sum_{\kappa=2}^{k}\left(\frac{r}{R}\right)^{2(\kappa-1)}\fint_{B_R}|\nabla u|^2\,dx\\
&\quad+C(d,k,\lambda)\fint_{B_R}|\nabla u|^2\,dx\sum_{\kappa=2}^{k-1}R^{-2(\kappa-1)}\max_{P\in\mathcal{P}^\kappa}||P||^{-2}\fint_{B_r}|\nabla\psi_P|^2\,dx\\
&\quad+C(d,k,\lambda)\fint_{B_R}|\nabla u|^2\,dx\times R^{-2(k-1)}\max_{P\in\mathcal{P}^k_{a_{hom}}}||P||^{-2}\fint_{B_r}|\nabla\tilde{\psi}_P|^2\,dx\\
&\quad+C(d,\lambda)\varepsilon_R^{2/(d+1)^2}\left(\frac{r}{R}\right)^{-d}\fint_{B_R}|\nabla u|^2\,dx\\
&\quad+C(d,\lambda)r^2\varepsilon_r^2R^{-2}\fint_{B_R}|\nabla u|^2\,dx\\
&\leq C(d,k,\lambda)\fint_{B_R}|\nabla u|^2\,dx\bigg[\left(\frac{r}{R}\right)^{2k}\\
&\qquad+\left(\varepsilon_{2,R}^{2/(d+1)^2}+R^{-2(k-1)}\max_{P\in\mathcal{P}^k_{a_{hom}},||P||=1}\fint_{B_R}|\nabla\tilde{\psi}_P|^2\,dx\right)\left(\frac{r}{R}\right)^{-d}\bigg],
\end{aligned}
$$

where in the last step we have used the inequality $\varepsilon_r^2\leq\left(\frac{R}{r}\right)^d\varepsilon_R^2\leq\left(\frac{R}{r}\right)^d\varepsilon_R^{2/(d+1)^2}$ and $\varepsilon_R\leq\varepsilon_{2,R}$ as well as (10) for $2\leq\kappa\leq k-1$. Our new estimate now implies the desired bound. □

4.2. **The $C^{k-1,1}$ excess-decay estimate.** Like in the $C^{2,\alpha}$ case, we now show how the $C^{k,\alpha}$ excess-decay estimate for the kth-order excess $\widetilde{\mathrm{Exc}}_k$ (in Lemma 14) entails a $C^{k-1,1}$ excess-decay estimate for the $(k-1)$th-order excess Exc_{k-1}.

Lemma 17. *Let $d\geq 2$, $k\geq 2$, and $R>0$. Assume that Theorem 1 holds for the orders $2,\ \ldots,\ k-1$, and let $\psi_P\equiv 0$ for linear polynomials P in order to simplify notation. For any $P\in\mathcal{P}^\kappa_{a_{hom}}$, denote by $\tilde{\psi}_P$ a solution to the equation (9) on the ball B_R (without boundary conditions); assume that the $\tilde{\psi}_P$ depend linearly on P. Then there exists a constant $\varepsilon_{min}>0$ depending only on d, k, and λ such that the following assertion holds:*

Suppose $r_0\in(0,R]$ is so large that $\varepsilon_{2,r_0}\leq\varepsilon_{min}$ and

$$
\sup_{r_0\leq\rho\leq R}\rho^{-(k-1)}\left(\max_{P\in\mathcal{P}^k_{a_{hom}},||P||=1}\fint_{B_\rho}|\nabla\tilde{\psi}_P|^2\,dx\right)^{1/2}\leq\varepsilon_{min}
$$

hold. Let u be an a-harmonic function on B_R. Then there exist $P_\kappa^R\in\mathcal{P}^\kappa_{a_{hom}}$ $(1\leq\kappa\leq k-1)$ for which the estimate

$$
\begin{aligned}
&\fint_{B_r}\left|\nabla u-\nabla\sum_{\kappa=1}^{k-1}\left(P_\kappa^R+\phi_i\partial_iP_\kappa^R+\psi_{P_\kappa^R}\right)\right|^2\,dx\\
&\leq C(d,k,\lambda)\left(\frac{r}{R}\right)^{2(k-1)}\fint_{B_R}|\nabla u|^2\,dx
\end{aligned}
$$

holds for any $r \in [r_0, R]$. *Furthermore, the* P^R_κ *depend linearly on* u *and satisfy*

$$\sum_{\kappa=1}^{k-1} R^{2(\kappa-1)} ||P^R_\kappa||^2 \leq C(d,k,\lambda) \fint_{B_R} |\nabla u|^2 \, dx.$$

Proof. In Lemma 14, fix $\alpha := 1/2$. We then easily verify that Lemma 14 is applicable in our situation. Set $P^R_\kappa := P^{r_0,min}_\kappa$; this implies that the P^R_κ depend linearly on u. The estimate (32) takes the form

$$\sum_{\kappa=1}^{k} R^{2(\kappa-1)} ||P^R_\kappa||^2 \leq C(d,k,\lambda) \fint_{B_R} |\nabla u|^2 \, dx.$$

Furthermore, applying Lemma 14 with r_0 playing the role of r and r playing the role of R, we deduce from (31)

$$\begin{aligned}
&\sum_{\kappa=1}^{k} r^{2(\kappa-1)} ||P^R_\kappa - P^{r,min}_\kappa||^2 \\
&\leq C(d,k,\lambda) \widetilde{\mathrm{Exc}}_k(r) \\
&\overset{(30)}{\leq} C(d,k,\lambda) \left(\frac{r}{R}\right)^{2(k-1)+2\alpha} \widetilde{\mathrm{Exc}}_k(R) \\
&\leq C(d,k,\lambda) \left(\frac{r}{R}\right)^{2(k-1)+2\alpha} \fint_{B_R} |\nabla u|^2 \, dx.
\end{aligned}$$

We now estimate

$$\begin{aligned}
&\fint_{B_r} \left|\nabla u - \nabla \sum_{\kappa=1}^{k-1} \left(P^R_\kappa + \phi_i \partial_i P^R_\kappa + \psi_{P^R_\kappa}\right)\right|^2 \, dx \\
&\leq 3 \fint_{B_r} \bigg|\nabla u - \nabla \sum_{\kappa=1}^{k-1} \Big(P^{r,min}_\kappa + \phi_i \partial_i P^{r,min}_\kappa + \psi_{P^{r,min}_\kappa}\Big) \\
&\qquad\qquad - \nabla \Big(P^{r,min}_k + \phi_i \partial_i P^{r,min}_k + \tilde\psi_{P^{r,min}_k}\Big)\bigg|^2 \, dx \\
&\quad + 3 \fint_{B_r} \left|\nabla \Big(P^{r,min}_k + \phi_i \partial_i P^{r,min}_k + \tilde\psi_{P^{r,min}_k}\Big)\right|^2 \, dx \\
&\quad + 3 \fint_{B_r} \left|\nabla \sum_{\kappa=1}^{k-1} \Big(P^{r,min}_\kappa - P^R_\kappa + \phi_i \partial_i (P^{r,min}_\kappa - P^R_\kappa) + \psi_{P^{r,min}_\kappa - P^R_\kappa}\Big)\right|^2 \, dx \\
&\leq 3 \widetilde{\mathrm{Exc}}_k(r) \\
&\quad + C(d,k) ||P^{r,min}_k||^2 r^{2(k-2)} \Bigg(\fint_{B_r} |\phi|^2 + r^2 |\mathrm{Id} + (\nabla\phi)^t|^2 \, dx \\
&\qquad\qquad\qquad + r^{-2(k-2)} \max_{P \in \mathcal{P}^k_{a_{hom}}, ||P||=1} \fint_{B_r} |\nabla \tilde\psi_P|^2 \, dx \Bigg) \\
&\quad + C(d,k) \sum_{\kappa=1}^{k-1} r^{2(\kappa-1)} ||P^{r,min}_\kappa - P^R_\kappa||^2 \fint_{B_r} |\mathrm{Id} + (\nabla\phi)^t|^2 \, dx
\end{aligned}$$

$$
\begin{aligned}
&+ C(d,k)\sum_{\kappa=2}^{k-1}||P_\kappa^{r,min}-P_\kappa^R||^2 \max_{P\in\mathcal{P}^\kappa_{a_{hom}},||P||=1} \fint_{B_r} r^{2(\kappa-2)}|\phi|^2+|\nabla\psi_P|^2\,dx\\
&\overset{(10,30,36)}{\leq} C(d,k,\lambda)\left(\frac{r}{R}\right)^{2(k-1)+2\alpha}\widetilde{\mathrm{Exc}}_k(R)\\
&+ C(d,k,\lambda)||P_k^{r,min}||^2 r^{2(k-1)}(\varepsilon_r^2+(1+\varepsilon_{2r}^2)+\varepsilon^2_{\tilde\psi,r_0,R})\\
&+ C(d,k,\lambda)\sum_{\kappa=1}^{k-1} r^{2(\kappa-1)}||P_\kappa^{r,min}-P_\kappa^R||^2(1+\varepsilon_{2r}^2)\\
&+ C(d,k,\lambda)\sum_{\kappa=2}^{k-1} r^{2(\kappa-1)}||P_\kappa^{r,min}-P_\kappa^R||^2(\varepsilon_r^2+\varepsilon_{2,r}^2)\\
&\leq C(d,k,\lambda)\left(\frac{r}{R}\right)^{2(k-1)+2\alpha}\widetilde{\mathrm{Exc}}_k(R)+C(d,k,\lambda)||P_k^{r,min}||^2 r^{2(k-1)}\\
&+ C(d,k,\lambda)\sum_{\kappa=1}^{k-1} r^{2(\kappa-1)}||P_\kappa^{r,min}-P_\kappa^R||^2.
\end{aligned}
$$

In conjunction with the two previous estimates, we infer

$$
\begin{aligned}
&\fint_{B_r}\left|\nabla u-\nabla\sum_{\kappa=1}^{k-1}\left(P_\kappa^R+\phi_i\partial_i P_\kappa^R+\psi_{P_\kappa^R}\right)\right|^2\,dx\\
&\leq C(d,k,\lambda)\left[\left(\frac{r}{R}\right)^{2(k-1)+2\alpha}+\left(\left(\frac{r}{R}\right)^{2(k-1)}+\left(\frac{r}{R}\right)^{2(k-1)+2\alpha}\right)+\left(\frac{r}{R}\right)^{2(k-1)+2\alpha}\right]\\
&\quad\times\fint_{B_R}|\nabla u|^2\,dx.
\end{aligned}
$$

Our lemma is therefore established. □

4.3. **Construction of correctors of order k.** Using the $C^{k-1,1}$ theory established in the previous subsection, we now proceed to the construction of our kth-order corrector. The following lemma provides the inductive step; starting from a function which acts as a kth-order corrector on a ball B_R, we construct a function acting as a kth-order corrector on the ball B_{2R}.

Lemma 18. *Let $d\geq 2$, $k\geq 2$, and assume that Theorem 1 holds for the orders 2, …, $k-1$. Let $r_0>0$ satisfy the estimate $\varepsilon_{2,r_0}\leq\varepsilon_0$, where $\varepsilon_0=\varepsilon_0(d,k,\lambda)$ is to be chosen in the proof below. Then the following implication holds:*

Let $R=2^M r_0$ for some $M\in\mathbb{N}_0$. Suppose that for every $P\in\mathcal{P}^k$ we have a solution ψ_P^R to the equation

$$
-\nabla\cdot a\nabla\psi_P^R=\nabla\cdot(\chi_{B_R}(\phi_i a-\sigma_i)\nabla\partial_i P)
$$

subject to the growth condition

$$
r^{-(k-1)}\left(\fint_{B_r}|\nabla\psi_P^R|^2\,dx\right)^{1/2}\leq C_1(d,k,\lambda)||P||\sum_{m=0}^{M}\min\{1,2^m r_0/r\}\varepsilon_{2^m r_0}
$$

for all $r\geq r_0$, where $C_1(d,k,\lambda)$ is a sufficiently large constant to be chosen in the proof below. Assume furthermore that ψ_P^R depends linearly on P.

Then for every $P\in\mathcal{P}^k$ there exists a solution ψ_P^{2R} to the equation

$$
-\nabla\cdot a\nabla\psi_P^{2R}=\nabla\cdot(\chi_{B_{2R}}(\phi_i a-\sigma_i)\nabla\partial_i P)
$$

subject to the growth condition

$$r^{-(k-1)}\left(\fint_{B_r}|\nabla\psi_P^{2R}|^2\,dx\right)^{1/2}\leq C_1(d,k,\lambda)||P||\sum_{m=0}^{M+1}\min\{1,2^m r_0/r\}\varepsilon_{2^m r_0}$$

for all $r\geq r_0$. Furthermore, ψ_P^{2R} depends linearly on P and we have

$$r^{-(k-1)}\left(\fint_{B_r}|\nabla\psi_P^{2R}-\nabla\psi_P^{R}|^2\,dx\right)^{1/2}\leq C_1(d,k,\lambda)||P||\varepsilon_{2^{M+1}r_0}.$$

Proof. To establish the lemma, we first note that the assumptions of the lemma ensure that the $C^{k-1,1}$ excess-decay lemma (Lemma 17) is applicable on B_R with $\tilde\psi_P:=\psi_P^R$. To see this, we estimate for any $r\in[r_0,R]$

$$r^{-(k-1)}\left(\fint_{B_r}|\nabla\psi_P^{R}|^2\,dx\right)^{1/2}\leq C_1(d,k,\lambda)||P||\varepsilon_{2,r_0}\leq C_1(d,k,\lambda)||P||\varepsilon_0.$$

By choosing $\varepsilon_0>0$ small enough depending only on d, k, λ, and C_1 (which is to be chosen at the end of this proof), we can ensure that the assumption of Lemma 17 regarding smallness of $\varepsilon_{\tilde\psi,r_0,R}$ is satisfied.

We now turn to the construction of $\psi_P^{2R}-\psi_P^R$ and to that purpose denote by ξ_P^R the weak solution on $\mathbb{R}^d$ with zero mean in B_{2R} and square integrable gradient, whose existence and uniqueness follows by the Lax-Milgram theorem, to the problem

$$-\nabla\cdot a\nabla\xi_P^R=\nabla\cdot(\chi_{B_{2R}-B_R}(\phi_i a-\sigma_i)\nabla\partial_i P).$$

Obviously, ξ_P^R depends linearly on P. Furthermore, by ellipticity we have the estimate

$$\begin{aligned}&\int_{\mathbb{R}^d}|\nabla\xi_P^R|^2\,dx\\&\leq C(d,\lambda)\sup_{B_{2R}}|\nabla^2 P|\left(\int_{\mathbb{R}^d}\chi_{B_{2R}-B_R}(|\phi a|^2+|\sigma|^2)\,dx\right)^{1/2}\left(\int_{\mathbb{R}^d}|\nabla\xi_P^R|^2\,dx\right)^{1/2}\end{aligned}$$

which gives

$$\left(\int_{\mathbb{R}^d}|\nabla\xi_P^R|^2\,dx\right)^{1/2}\leq C(d,\lambda)\sup_{B_{2R}}|\nabla^2 P|\left(\int_{B_{2R}}|\phi|^2+|\sigma|^2\,dx\right)^{1/2}.$$

The last estimate in turn implies

$$\int_{\mathbb{R}^d}|\nabla\xi_P^R|^2\,dx\leq C(d,k,\lambda)||P||^2R^{2(k-2)}\varepsilon_{2R}^2R^{2+d}. \tag{37}$$

We now obtain $\psi_P^{2R}-\psi_P^R$ by modifying ξ_P^R by an a-harmonic function of degree $k-1$. As ξ_P^R is a-harmonic in B_R, Lemma 17 now implies the existence of some $P_{\kappa,P}^R\in\mathcal{P}^\kappa$ for $1\leq\kappa\leq k-1$ which depend linearly on P and for which the estimates

$$||P_{\kappa,P}^R||^2\leq C(d,k,\lambda)R^{-2(\kappa-1)}\fint_{B_R}|\nabla\xi_P^R|^2\,dx\overset{(37)}{\leq}C(d,k,\lambda)||P||^2R^{2(k-\kappa)}\varepsilon_{2R}^2 \tag{38}$$

and

$$\begin{aligned}
&\fint_{B_r} \left|\nabla \xi_P^R - \nabla \sum_{\kappa=1}^{k-1} \left(P_{\kappa,P}^R + \phi_i \partial_i P_{\kappa,P}^R + \psi_{P_{\kappa,P}^R}\right)\right|^2 dx \\
&\leq C(d,k,\lambda) \left(\frac{r}{R}\right)^{2(k-1)} \fint_{B_R} |\nabla \xi_P^R|^2 \, dx \\
&\overset{(37)}{\leq} C(d,k,\lambda) ||P||^2 r^{2(k-1)} \varepsilon_{2R}^2
\end{aligned}$$

hold for all $r \in [r_0, R]$.

Furthermore, we have for $r > R$

$$\begin{aligned}
&\fint_{B_r} \left|\nabla \xi_P^R - \nabla \sum_{\kappa=1}^{k-1} \left(P_{\kappa,P}^R + \phi_i \partial_i P_{\kappa,P}^R + \psi_{P_{\kappa,P}^R}\right)\right|^2 dx \\
&\overset{(36,10)}{\leq} C(d,k,\lambda) \Bigg(r^{-d} \int_{B_r} |\nabla \xi_P^R|^2 \, dx + ||P_{1,P}^R||^2 (1 + \varepsilon_{2r}^2) \\
&\qquad + \sum_{\kappa=2}^{k-1} r^{2(\kappa-1)} ||P_{\kappa,P}^R||^2 (1 + \varepsilon_{2r}^2 + \varepsilon_{2,r}^2) \Bigg) \\
&\overset{(37,38)}{\leq} C(d,k,\lambda) ||P||^2 R^{2(k-1)} \left(\left(\frac{R}{r}\right)^d + 1 + \varepsilon_{2r}^2 + (1 + \varepsilon_{2r} + \varepsilon_{2,r}) \left(\frac{r}{R}\right)^{2(k-2)} \right) \varepsilon_{2R}^2 \\
&\leq C(d,k,\lambda) ||P||^2 r^{2(k-2)} R^2 \varepsilon_{2R}^2.
\end{aligned}$$

The combination of both r-ranges yields

$$\begin{aligned}
&\frac{1}{r^{k-1}} \left(\fint_{B_r} \left|\nabla \xi_P^R - \nabla \sum_{\kappa=1}^{k-1} \left(P_{\kappa,P}^R + \phi_i \partial_i P_{\kappa,P}^R + \psi_{P_{\kappa,P}^R}\right)\right|^2 dx \right)^{1/2} \\
&\leq C(d,k,\lambda) ||P|| \min\{1, 2R/r\} \varepsilon_{2R}.
\end{aligned} \tag{39}$$

In total, we see that

$$\psi_P^{2R} := \psi_P^R + \xi_P^R - \sum_{\kappa=1}^{k-1} \left(P_{\kappa,P}^R + \phi_i \partial_i P_{\kappa,P}^R + \psi_{P_{\kappa,P}^R}\right)$$

is the desired function (note in particular that the last term is a-harmonic), provided we choose C_1 to be the constant appearing in (39). □

We now establish existence of kth-order correctors by means of the previous lemma.

Proof of Theorem 1. We just need to construct an "initial" kth-order corrector $\psi_P^{r_0}$ subject to the properties of Lemma 18; then Lemma 18 yields a sequence $(\psi_P^{2^m r_0})_m$ which (after subtracting appropriate constants) is a Cauchy sequence in $H^1(B_R)$ for every $R > 0$ due to the last estimate in the lemma and our assumption (4) which implies summability of $\varepsilon_{2^m r_0}$. Thus, the limit ψ_P satisfies the equation (9) in the whole space, depends linearly on P, and satisfies the estimate

$$r^{-(k-1)}\left(\fint_{B_r}|\nabla\psi_P|^2\,dx\right)^{1/2}\leq C_1(d,k,\lambda)||P||\sum_{m=0}^{\infty}\min\{1,2^m r_0/r\}\varepsilon_{2^m r_0}$$
$$\leq C_1(d,k,\lambda)||P||\varepsilon_{2,r}$$

for any $r\geq r_0$.

To construct $\psi_P^{r_0}$, we use Lax-Milgram to find the (unique) solution $\psi_P^{r_0}$ on $\mathbb{R}^d$ with square-integrable gradient and zero mean on B_{r_0} to the equation

$$-\nabla\cdot a\nabla\psi_P^{r_0}=\nabla\cdot(\chi_{B_{r_0}}(\phi_i a-\sigma_i)\nabla\partial_i P).$$

Obviously, $\psi_P^{r_0}$ depends linearly on P. Furthermore, we have the energy estimate

$$\int_{\mathbb{R}^d}|\nabla\psi_P^{r_0}|^2\,dx$$
$$\leq C(d,\lambda)\sup_{B_{r_0}}|\nabla^2 P|\left(\int_{\mathbb{R}^d}|\chi_{B_{r_0}}a\phi|^2+|\chi_{B_{r_0}}\sigma|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^d}|\nabla\psi_P^{r_0}|^2\,dx\right)^{1/2}.$$

We therefore get

$$\left(\int_{\mathbb{R}^d}|\nabla\psi_P^{r_0}|^2\,dx\right)^{1/2}\leq C(d,\lambda)\sup_{B_{r_0}}|\nabla^2 P|\left(\int_{B_{r_0}}|\phi|^2+|\sigma|^2\,dx\right)^{1/2}.$$

This yields in particular for any $r\geq r_0$

$$\int_{B_r}|\nabla\psi_P^{r_0}|^2\,dx\leq C(d,k,\lambda)||P||^2 r_0^{2(k-2)}\int_{B_{r_0}}|\phi|^2+|\sigma|^2\,dx$$

and therefore

$$\fint_{B_r}|\nabla\psi_P^{r_0}|^2\,dx\leq C(d,k,\lambda)||P||^2 r^{-d}r_0^{2(k-2)}\varepsilon_{r_0}^2 r_0^{2+d}$$
$$\leq C(d,k,\lambda)||P||^2 r^{2(k-1)}\min\{1,(r_0/r)^2\}\varepsilon_{r_0}^2.$$

We note that this provides the starting point for Lemma 18, possibly after enlarging the constant C_1 in the statement thereof. □

4.4. **Proof of the kth-order Liouville principle.** Like in the $C^{2,\alpha}$ case, the $C^{k,\alpha}$ Liouville principle (Lemma 19 below) is an easy consequence of our large-scale excess-decay estimate (Theorem 3). The kth-order Liouville principle (Corollary 4) in turn is an easy consequence of the $C^{k+1,\alpha}$ Liouville principle.

Lemma 19. *Let $d\geq 2$, $k\geq 2$, and suppose that the assumption (4) is satisfied. Then the following property holds: Any a-harmonic function u satisfying the growth condition*

$$\liminf_{r\to\infty}\frac{1}{r^{k+\alpha}}\left(\fint_{B_r}|u|^2\,dx\right)^{1/2}=0 \qquad (40)$$

for some $\alpha\in(0,1)$ is of the form

$$u=a+b_i(x_i+\phi_i)+\sum_{\kappa=2}^{k}(P_\kappa+\phi_i\partial_i P_\kappa+\psi_{P_\kappa})$$

with some $a\in\mathbb{R}$, $b\in\mathbb{R}^d$, and $P_\kappa\in\mathcal{P}^\kappa_{a_{hom}}$ for $2\leq\kappa\leq k$ (i.e. P_κ is a homogeneous a_{hom}-harmonic polynomial of degree κ). Here, the ψ_P denote the higher-order correctors whose existence is guaranteed by Theorem 1.

Proof of Corollary 4. Obviously, (13) entails (40) with $k+1$ in place of k and e.g. $\alpha := \frac{1}{2}$. By Lemma 19, any a-harmonic function u subject to condition (13) must be of the form

$$u = a + b_i(x_i+\phi_i) + \sum_{\kappa=2}^{k+1}(P_\kappa + \phi_i\partial_i P_\kappa + \psi_{P_\kappa}) \tag{41}$$

with some $a\in\mathbb{R}$, $b\in\mathbb{R}^d$, and $P_\kappa\in\mathcal{P}^\kappa_{a_{hom}}$ for $2\leq\kappa\leq k+1$. Our stronger growth condition (13) however shows that we have $P_{k+1}\equiv 0$: Since the ϕ_i grow sublinearly (see (2)) and since $\psi_{P_{k+1}}$ grows slower than a polynomial of degree $k+1$ (see (10)), we see that for large $|x|$ the term P_{k+1} would be the dominating term in (41) if it were nonzero, contradicting our growth condition (13). □

Proof of Lemma 19. Let $\alpha\in(0,1)$ be such that

$$\liminf_{R\to\infty}\frac{1}{R^{k+\alpha}}\left(\fint_{B_R}|u|^2\ dx\right)^{1/2} = 0$$

holds. By the Caccioppoli estimate, we deduce

$$\liminf_{R\to\infty}\frac{1}{R^{k-1+\alpha}}\left(\fint_{B_R}|\nabla u|^2\ dx\right)^{1/2} = 0.$$

Fix $r\geq r_0$. The excess-decay estimate from Theorem 3 together with the trivial bound $\operatorname{Exc}_k(R)\leq\fint_{B_R}|\nabla u|^2\ dx$ yields

$$\begin{aligned}\operatorname{Exc}_k(r) &\leq C(d,k,\lambda,\alpha)\left(\frac{r}{R}\right)^{2(k-1)+2\alpha}\operatorname{Exc}_k(R)\\ &\leq C(d,k,\lambda,\alpha)r^{2(k-1)+2\alpha}\left(\frac{1}{R^{k-1+\alpha}}\left(\fint_{B_R}|\nabla u|^2\ dx\right)^{1/2}\right)^2.\end{aligned}$$

Passing to the lim inf $R\to\infty$, we deduce that

$$\operatorname{Exc}_k(r) = 0$$

holds for every $r\geq r_0$. Therefore, on every B_r with $r\geq r_0$, ∇u can be represented *exactly* as the derivative of a corrected polynomial of kth order (since the infimum in the definition of Exc_k is actually attained, as noted at the beginning of the proof of Lemma 15), i.e. we have

$$\nabla u = \nabla b_i^r(x_i+\phi_i) + \nabla\sum_{\kappa=2}^{k}(P_\kappa^r + \phi_i\partial_i P_\kappa^r + \psi_{P_\kappa^r})$$

in B_r for some $b^r\in\mathbb{R}^d$ and some $P^r_\kappa\in\mathcal{P}^\kappa_{a_{hom}}$ ($2\leq\kappa\leq k$); recall that we have used the convention $\psi_P\equiv 0$ for linear polynomials P. It is not difficult to show that for r large enough, the b^r and P^r_κ are actually independent of r and define some common $b\in\mathbb{R}^d$ and $P_\kappa\in\mathcal{P}^\kappa_{a_{hom}}$: For example, one may use Lemma 14 to compare the b^r, P^r_κ for two different radii $r_1, r_2\geq r_0$; the estimate for $|b^{r_1}-b^{r_2}|$ and $||P^{r_1}_\kappa - P^{r_2}_\kappa||$ then contains the factor $\operatorname{Exc}_k(\max(r_1,r_2))$ and is therefore zero.

Moreover, the gradient ∇u determines the function u itself up to a constant, i.e. we have

$$u = a + b_i(x_i + \phi_i) + \sum_{\kappa=2}^{k} (P_\kappa + \phi_i \partial_i P_\kappa + \psi_{P_\kappa})$$

for some $a \in \mathbb{R}$, $b \in \mathbb{R}^d$, and $P_\kappa \in \mathcal{P}^\kappa_{a_{hom}}$ $(2 \le \kappa \le k)$. □

Appendix A. Approximation of a-Harmonic Functions by Corrected a_{hom}-Harmonic Functions

Our proofs make use of the following lemma, which is implicitly derived in the course of the proof of Lemma 2 in [11]. For the reader's convenience, we recall its proof here.

The lemma essentially states that an a-harmonic function u on a ball B_R may be approximated on the ball $B_{R/2}$ up to a small error (of order $\varepsilon_R^{1/(d+1)^2}$) by an appropriate a_{hom}-harmonic function u_{hom} and correcting this function u_{hom} using the first-order corrector ϕ_i.

The purpose of the lemma is the same as in classical elliptic regularity theory: The function u_{hom} satisfies an elliptic equation with constant coefficients, i.e. it is smooth and good estimates for its higher derivatives are available. In our proof above, we show by means of the present lemma that this high regularity of u_{hom} transfers (in an appropriate sense) to u itself.

Lemma 20. *Let $R > 0$ and let u be a-harmonic on B_R. Suppose that $\varepsilon_R \le 1$ (with ε_R as defined in* (3)*). Then there exists an a_{hom}-harmonic function u_{hom} on $B_{R/2}$ satisfying the following two properties: First, we have the energy estimate*

$$\fint_{B_{R/2}} |\nabla u_{hom}|^2 \, dx \le C(d,\lambda) \fint_{B_R} |\nabla u|^2 \, dx. \tag{42}$$

Second, the "corrected" function $u_{hom} + \phi_i \partial_i u_{hom}$ is a good approximation for u in the sense that

$$\fint_{B_{R/2}} |\nabla u - \nabla(u_{hom} + \phi_i \partial_i u_{hom})|^2 \, dx \le C(d,\lambda) \varepsilon_R^{2/(d+1)^2} \fint_{B_R} |\nabla u|^2 \, dx.$$

Proof. Choose some $R' \in [\frac{3}{4}R, R]$ for which

$$R' \fint_{\partial B_{R'}} |\nabla u|^2 \, dS \le C(d) \fint_{B_R} |\nabla u|^2 \, dx \tag{43}$$

holds. Let u_{hom} be the a_{hom}-harmonic function in $B_{R'}$ which coincides with u on $\partial B_{R'}$. Testing the equation $-\nabla \cdot a_{hom} \nabla u_{hom} = 0$ with $u_{hom} - u$ (note that this test function is admissible since we have $u_{hom} - u = 0$ on $\partial B_{R'}$), we infer by ellipticity of a and (in the second step) Young's inequality

$$\begin{aligned} \fint_{B_{R'}} |\nabla u_{hom}|^2 \, dx &\le C(\lambda) \fint_{B_{R'}} |\nabla u||\nabla u_{hom}| \, dx \\ &\le \frac{1}{2} \fint_{B_{R'}} |\nabla u_{hom}|^2 \, dx + C(\lambda) \fint_{B_{R'}} |\nabla u|^2 \, dx, \end{aligned} \tag{44}$$

which because of $R/2 \le R' \le R$ gives the desired energy estimate. It remains to establish the approximation property of $u_{hom} + \phi_i \partial_i u_{hom}$.

Denote by $\eta_0 : \mathbb{R} \to \mathbb{R}$ a smooth function with $\eta_0(s) = 1$ for $s \geq 1$ and $\eta_0(s) = 0$ for $s \leq 0$. Let $0 < \rho < R/4$ and set $\eta(x) := \eta_0(2(R' - \rho/2 - |x|)/\rho)$. Note that we have $|\nabla\eta| \leq C(d)/\rho$ as well as $\eta \equiv 0$ outside of $B_{R'-\rho/2}$ and $\eta \equiv 1$ in $B_{R'-\rho}$. Due to $\rho \leq R/4$, we also have $R' - \rho \geq R/2$. We will optimize in this "boundary layer thickness" ρ at the end of the proof.

Let us abbreviate

$$v := u - u_{hom} - \eta\phi_i\partial_i u_{hom}.$$

where the purpose of η is to have $v \equiv 0$ on $\partial B_{R'}$. The desired approximation property of $u_{hom} + \phi_i\partial_i u_{hom}$ as stated in the lemma will be a consequence of an appropriate energy estimate for v (recall that we have $\eta \equiv 1$ in $B_{R/2}$ since $\rho < R/4$ and $R' > 3R/4$).

To derive this energy estimate, we would like to show that v is approximately a-harmonic. We first compute using the fact that u and $x_i + \phi_i$ are a-harmonic (see (6))

$$\begin{aligned}
&-\nabla\cdot a\nabla v\\
&= -\nabla\cdot a\nabla u + \nabla\cdot(1-\eta)a\nabla u_{hom} + \nabla\cdot a(e_i + \nabla\phi_i)\eta\partial_i u_{hom} + \nabla\cdot \phi_i a\nabla(\eta\partial_i u_{hom})\\
&\overset{(6)}{=} \nabla\cdot(1-\eta)a\nabla u_{hom} + a(e_i + \nabla\phi_i)\cdot\nabla(\eta\partial_i u_{hom}) + \nabla\cdot\phi_i a\nabla(\eta\partial_i u_{hom})\\
&= \nabla\cdot(1-\eta)(a - a_{hom})\nabla u_{hom} + (a(e_i + \nabla\phi_i) - a_{hom}e_i)\cdot\nabla(\eta\partial_i u_{hom})\\
&\quad + \nabla\cdot\phi_i a\nabla(\eta\partial_i u_{hom}),
\end{aligned}$$

where in the last step we have used the a_{hom}-harmonicity of u_{hom} in form of the equality $-\nabla\cdot(1-\eta)a_{hom}\nabla u_{hom} - a_{hom}e_i\cdot\nabla(\eta\partial_i u_{hom}) = 0$. Taking into account the formula $a(e_i + \nabla\phi_i) - a_{hom}e_i = \nabla\cdot\sigma_i$ (see (7),(8)) and the fact that

$$(\nabla\cdot\sigma_i)\cdot\nabla w = \partial_k\sigma_{ijk}\partial_j w = \partial_k(\sigma_{ijk}\partial_j w) = -\partial_k(\sigma_{ikj}\partial_j w) = -\nabla\cdot(\sigma_i\nabla w)$$

holds for any function w by skew-symmetry of σ_i, we may rewrite the right-hand side in divergence form:

$$-\nabla\cdot a\nabla v = \nabla\cdot(1-\eta)(a - a_{hom})\nabla u_{hom} + \nabla\cdot(\phi_i a - \sigma_i)\nabla(\eta\partial_i u_{hom}).$$

Testing the weak formulation of this equation with v (recall that $v \equiv 0$ on $\partial B_{R'}$) and using the ellipticity of a, we deduce using Young's inequality and the properties of η

$$\begin{aligned}
&\int_{B_{R'}} |\nabla v|^2\,dx\\
&\leq C(\lambda)\int_{B_{R'}} |(1-\eta)(a - a_{hom})\nabla u_{hom}|^2 + |\phi_i a - \sigma_i|^2|\nabla(\eta\partial_i u_{hom})|^2\,dx\\
&\leq C(d,\lambda)\int_{B_{R'}} |1-\eta|^2|\nabla u_{hom}|^2\,dx\\
&\quad + C(d,\lambda)\int_{B_{R'}} (|\phi|^2 + |\sigma|^2)(|\nabla\eta|^2|\nabla u_{hom}|^2 + \eta^2|\nabla^2 u_{hom}|^2)\,dx\\
&\leq C(d,\lambda)\int_{B_{R'}-B_{R'-\rho}} |\nabla u_{hom}|^2\,dx\\
&\quad + C(d,\lambda)\sup_{B_{R'-\rho/2}}\left(\frac{1}{\rho^2}|\nabla u_{hom}|^2 + |\nabla^2 u_{hom}|^2\right)\int_{B_{R'}} |\phi|^2 + |\sigma|^2\,dx.
\end{aligned}$$

Since our function u_{hom} is a_{hom}-harmonic, we have the regularity estimates

$$\sup_{B_{R'-\rho/2}} \left(\frac{1}{\rho^2}|\nabla u_{hom}|^2 + |\nabla^2 u_{hom}|^2\right) \leq \frac{C(d,\lambda)}{\rho^2} \sup_{y\in B_{R'-\rho/2}} \fint_{B_{\rho/2}(y)} |\nabla u_{hom}|^2 \, dx,$$

$$\left(\int_{B_{R'}} |\nabla u_{hom}|^p \, dx\right)^{2/p} \leq C(d,\lambda) \int_{\partial B_{R'}} |\nabla^{tan} u_{hom}|^2 \, dS,$$

where $p := 2d/(d-1)$: The first estimate is a standard constant-coefficient interior regularity estimate (which is a consequence e.g. of an iterative application of Theorem 4.9 in [9] and the Sobolev embedding). The second estimate follows by combining 1) the existence of an extension $\bar{u}$ of u_{hom} subject to the estimate $||\nabla \bar{u}||_{L^p(B_{R'})} \leq C(d)||\nabla^{tan} u_{hom}||_{L^2(\partial B_{R'})}$ and 2) the Calderon-Zygmund estimate on $B_{R'}$, which reads $||\nabla w||_{L^p(B_{R'})} \leq C(d,\lambda)||\nabla \bar{u}||_{L^p(B_{R'})}$ for any solution $w \in H^1(B_{R'})$ with $w - \bar{u} \in H^1_0(B_{R'})$ to the equation $-\nabla\cdot a_{hom}\nabla w = 0$. For the latter estimate, see Theorem 7.1 in [9].

Using these regularity estimates, the equality $\nabla^{tan} u_{hom} = \nabla^{tan} u$ on ∂B_R, as well as the obvious inequality

$$\sup_{y\in B_{R'-\rho/2}} \fint_{B_{\rho/2}(y)} |\nabla u_{hom}|^2 \, dx \leq \left(\frac{2R'}{\rho}\right)^d \fint_{B_{R'}} |\nabla u_{hom}|^2 \, dx,$$

we infer by $\rho \leq R'/4$ and $3R/4 \leq R' \leq R$

$$\begin{aligned}
\int_{B_{R'}} |\nabla v|^2 \, dx &\leq C(d,\lambda)|B_{R'} - B_{R'-\rho}|^{1-2/p} \left(\int_{B_{R'}-B_{R'-\rho}} |\nabla u_{hom}|^p \, dx\right)^{2/p} \\
&\quad + C(d,\lambda)\frac{1}{R'^2}\left(\frac{R'}{\rho}\right)^{d+2} \fint_{B_{R'}} |\nabla u_{hom}|^2 \, dx \cdot (R')^d \fint_{B_{R'}} |\phi|^2 + |\sigma|^2 \, dx \\
&\overset{(44)}{\leq} C(d,\lambda)\rho^{1/d} R'^{(d-1)/d} \int_{\partial B_{R'}} |\nabla^{tan} u|^2 \, dS \\
&\quad + C(d,\lambda)\varepsilon_R^2 \left(\frac{R'}{\rho}\right)^{d+2} \int_{B_{R'}} |\nabla u|^2 \, dx \\
&\overset{(43)}{\leq} C(d,\lambda)\left(\frac{\rho}{R'}\right)^{1/d} \int_{B_R} |\nabla u|^2 \, dx \\
&\quad + C(d,\lambda)\varepsilon_R^2 \left(\frac{R'}{\rho}\right)^{d+2} \int_{B_{R'}} |\nabla u|^2 \, dx.
\end{aligned}$$

We optimize in ρ by choosing $\rho := \frac{1}{4}\varepsilon_R^{2d/(d+1)^2} R'$ (which thanks to the assumption $\varepsilon_R \leq 1$ is admissible in the sense of $\rho \leq \frac{1}{4}R'$). This yields

$$\int_{B_{R'}} |\nabla v|^2 \, dx \leq C(d,\lambda)\varepsilon_R^{2/(d+1)^2} \left(\int_{B_{R'}} |\nabla u|^2 \, dx + \int_{B_R} |\nabla u|^2 \, dx\right)$$

which together with the estimate $3R/4 \leq R' \leq R$ and $\eta \equiv 1$ in $B_{R/2}$ proves the desired approximation result. □

Appendix B. Failure of Liouville principle for smooth uniformly elliptic coefficient fields

We now provide the argument that smoothness of a uniformly elliptic coefficient field does not prevent Liouville's theorem from failing: Even for smooth uniformly elliptic coefficient fields, sublinearly growing harmonic functions are not necessarily constant, implying a failure even of the zero-th order Liouville theorem.

Proposition 21. *For any $\alpha \in (0,1)$ there exists a smooth, bounded, and uniformly elliptic symmetric coefficient field a on $\mathbb{R}^2$ such that the following holds: There exists a smooth function u which is a-harmonic and satisfies*

$$\left(\fint_{B_R} u^2 \, dx\right)^{\frac{1}{2}} \sim R^\alpha \quad \textit{for } R \gg 1. \tag{45}$$

Proof. By a classical example in dimension $d = 2$ (see e. g. [18]), for any exponent $\alpha \in (0,1)$, there exists a uniformly elliptic, symmetric coefficient field a_0 of a scalar equation and a weakly a_0-harmonic function u_0 (in particular, it is locally integrable and of locally integrable gradient) whose modulus on average grows like $|x|^\alpha$, for instance as expressed by

$$\left(\fint_{B_R} u_0^2 \, dx\right)^{\frac{1}{2}} \sim R^\alpha. \tag{46}$$

Moreover, in this example

$$a_0 \text{ and } u_0 \text{ are homogeneous and smooth outside the origin.} \tag{47}$$

We now argue that this example may be post-processed to an example of an *everywhere smooth* uniformly elliptic symmetric coefficient field a and a smooth a-harmonic function u such that still (45) holds.

Indeed, because of (47) we can easily construct a uniformly elliptic coefficient field a that agrees with a_0 outside of B_1 and is smooth. Next we observe that (47) also implies (using $d = 2$ and $\alpha > 0$) that ∇u_0 is locally *square* integrable, so that by Riesz' representation theorem, there exists a weak solution of

$$-\nabla \cdot a\nabla w = \nabla \cdot (a - a_0)\nabla u_0 \tag{48}$$

in the sense that w and its gradient are locally integrable and that

$$\int |\nabla w|^2 \, dx \le C(\lambda). \tag{49}$$

Equation (48) is made such that $u = u_0 + w$ is a weak solution (i. e. locally integrable with locally integrable gradient) of

$$-\nabla \cdot a\nabla u = 0,$$

and thus smooth since a is smooth by classical uniqueness and regularity results. It remains to give the argument in favor of (45), which in view of (46) follows once we show that (49) implies in particular for large R

$$\left(\fint_{B_R} w^2 \, dx\right)^{\frac{1}{2}} = o(R^\alpha). \tag{50}$$

This is a well-known argument related to "bounded mean oscillation": By Poincaré's estimate with mean value zero we have on every dyadic ball around the origin

$$\left(\int_{B_{2^n}} (w - \fint_{B_{2^n}} w)^2 \, dx\right)^{\frac{1}{2}} \leq C(d) \cdot 2^n \left(\int_{B_{2^n}} |\nabla w|^2 \, dx\right)^{\frac{1}{2}},$$

which for $d = 2$ takes on the form

$$\left(\fint_{B_{2^n}} (w - \fint_{B_{2^n}} w)^2 \, dx\right)^{\frac{1}{2}} \leq C \left(\int_{B_{2^n}} |\nabla w|^2 \, dx\right)^{\frac{1}{2}} \overset{(49)}{\leq} C(\lambda). \tag{51}$$

By Jensen's and the triangle inequality, this yields in particular $|\fint_{B_{2^{n-1}}} w \, dx - \fint_{B_{2^n}} w \, dx| \leq C(\lambda)$ and thus, since we may w. l. o. g. assume $\int_{B_1} w \, dx = 0$, $|\fint_{B_{2^n}} w \, dx| \leq nC(\lambda)$. Inserting this back into (51) gives

$$\left(\fint_{B_{2^n}} w^2 \, dx\right)^{\frac{1}{2}} \leq nC(\lambda),$$

that is, (50) in the stronger form of

$$\left(\fint_{B_R} w^2 \, dx\right)^{\frac{1}{2}} \leq C(d) \log R.$$

□

Proposition 22. *There exists a smooth, bounded, and uniformly elliptic symmetric coefficient field a on $\mathbb{R}^3$ such that the following holds: There exists a smooth map $u : \mathbb{R}^3 \to \mathbb{R}^3$ which is a-harmonic and satisfies*

$$\left(\fint_{B_R} u^2 \, dx\right)^{\frac{1}{2}} \sim R^{-\alpha} \quad \text{for } R \gg 1, \tag{52}$$

where $\alpha = \frac{1}{2}(1 - \frac{3}{\sqrt{17}})$.

Proof. By a classical example of De Giorgi in dimension $d = 3$ (see e. g. Chapter 9.1.1 in [9]), there exists a bounded, symmetric, and uniformly elliptic coefficient field a_0 which is radial and smooth away from the origin, for which the map

$$u_0(x) := \frac{x}{|x|^\gamma} \tag{53}$$

with $\gamma := \frac{3}{2}(1 - \frac{1}{\sqrt{17}})$ is a_0-harmonic. Choose a to be a smooth, bounded, and uniformly elliptic coefficient field which agrees with a_0 outside of the unit ball B_1.

We now show that the a_0-harmonic map u_0 may be modified to yield an a-harmonic map u with the same decay properties on large scales. To construct the difference $u - u_0$, let w be the Lax-Milgram solution (which is unique up to a constant) to the problem

$$-\nabla \cdot a \nabla w = \nabla \cdot (a - a_0) \nabla u_0. \tag{54}$$

Since $a - a_0$ is supported in B_1, since a and a_0 are bounded, and since ∇u_0 belongs to $L^2_{loc}(\mathbb{R}^3)$, we deduce by the standard energy estimate

$$\int |\nabla w|^2 \, dx \leq C \int |(a - a_0) \nabla u_0|^2 \, dx \leq C. \tag{55}$$

Poincaré's inequality now implies for any $R>0$

$$\fint_{B_R} \left|w-\fint_{B_R} w\right|^2 dx \leq CR^2 \fint_{B_R} |\nabla w|^2 \, dx \leq CR^{-1} \int_{B_R} |\nabla w|^2 \, dx \overset{(55)}{\leq} CR^{-1}, \tag{56}$$

which entails

$$\left|\fint_{B_R} w \, dx - \fint_{B_{2R}} w \, dx\right| \leq CR^{-1/2}.$$

We therefore deduce that the sequence $\fint_{B_{2^n}} w \, dx$ is Cauchy: We have for any $N>n\geq 0$

$$\begin{aligned}\left|\fint_{B_{2^n}} w \, dx - \fint_{B_{2^N}} w \, dx\right| &\leq \sum_{m=n}^{N-1} \left|\fint_{B_{2^m}} w \, dx - \fint_{B_{2^{m+1}}} w \, dx\right| \\ &\leq \sum_{m=n}^{N-1} C2^{-m/2} \leq C2^{-n/2}.\end{aligned}$$

Possibly adding a constant to w (to ensure that the limit of the above sequence is zero), we therefore may assume that

$$\left|\fint_{B_{2^n}} w \, dx\right| \leq C2^{-n/2}.$$

In conjunction with (56), we infer for any $R\geq 1$

$$\left(\fint_{B_R} |w|^2 \, dx\right)^{1/2} \leq CR^{-1/2}. \tag{57}$$

By (54), the map $u := u_0 + w$ is a-harmonic. As u solves a linear elliptic system with smooth coefficients and belongs to $H^1_{loc}(\mathbb{R}^3)$, u itself is smooth. Since we have $\alpha = \gamma - 1 < \frac{1}{2}$, the estimate (57) in conjunction with (53) entails (52). □

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Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany, E-Mail: julian.fischer@mis.mpg.de

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany, E-Mail: otto@mis.mpg.de