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ADVECTION-DRIVEN SUPPORT SHRINKING IN A CHEMOTAXIS MODEL WITH DEGENERATE MOBILITY*

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Abstract. We derive sufficient conditions for advection-driven backward motion of the free boundary in a chemotaxis model with degenerate mobility. In this model, a porous-medium-type diffusive term and an advection term are in competition. The former induces forward motion, the latter may induce backward motion of the free boundary depending on the direction of advection. We deduce conditions on the growth of the initial data at the free boundary which ensure that at least initially the advection term is dominant. This implies local backward motion of the free boundary provided the advection is (locally) directed appropriately. Our result is based on a new class of moving test functions and Stampacchia's lemma. As a by-product of our estimates, we obtain quantitative bounds on the spreading of the support of solutions for the chemotaxis model and provide a proof for the finite speed of the support propagation property of solutions.

Key words. chemotaxis, degenerate parabolic equation, support shrinking, finite speed of propagation, waiting time

AMS subject classifications. 35K55, 35K57, 35K65, 35B05

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1. Introduction and main results.

1.1. Overview. In the previous decades, starting with [18] a large amount of effort has been devoted to research on Keller–Segel-type models. These models are supposed to describe movement of bacteria which release some chemical substance ("chemoattractant") which in turn influences the motion of bacteria of the same species. Thus, the bacteria effectively coordinate their movement. The classical fully parabolic Keller–Segel model consists of two coupled parabolic equations determining evolution of bacteria and chemoattractant density. Denoting bacteria density by u and chemoattractant density by γ, the equations read

$$\begin{aligned} u_t &= \epsilon\Delta u - \operatorname{div}(u\nabla\gamma) \ , \\ \alpha\gamma_t &= \Delta\gamma + u - \gamma \end{aligned}$$

with $\alpha > 0$, $\epsilon > 0$. This model features conservation of bacteria mass if appropriate no-flux boundary conditions are enforced. For $d \geq 2$ and in the case of large initial data, it may suffer from finite time blow-up. For an account on results on Keller–Segel-type models, see the overview articles by Horstmann [16], [17] and the references therein.

In the limit of fast chemoattractant movement (in comparison with bacteria mobility), i.e., small α, one can assume that the chemoattractant density will be close to steady state and that therefore the parabolic equation governing the evolution of chemoattractant density can be replaced by its corresponding elliptic equation, resulting in the parabolic-elliptic system

$$\begin{aligned} u_t &= \epsilon\Delta u - \operatorname{div}(u\nabla\gamma) \ , \\ -\Delta\gamma + \gamma &= u \ . \end{aligned}$$

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This model inherits many properties from the fully parabolic one; again, bacteria mass is conserved in the case of no-flux boundary conditions and again for $d \geq 2$ and large initial data, blow-up may occur in finite time. Given the Bessel potential $\mathcal{B}$ (i.e., the Green's function of the operator $-\Delta + 1$), this system is equivalent to the nonlocal parabolic equation

$$u_t = \epsilon \Delta u - \operatorname{div}(u \nabla (\mathcal{B} * u)) \ .$$

Several models supposed to overcome the drawback of possible finite time blow-up have been proposed; most of them rely on modifying bacteria sensitivity, i.e., bacteria drift velocity is not taken just to be the gradient of γ, but $\nabla \gamma$ is multiplied by some factor leading to a cutoff in case the gradient becomes too steep or in case local bacteria density becomes too large.

One of these models is the model presented by Burger, Di Francesco, and Dolak [6]. This model features degeneration of bacteria diffusivity for bacteria densities approaching either 0 or 1 and degeneration of bacteria mobility for densities approaching 1. It reads

$$u_t = \nabla \cdot (u(1-u)(\epsilon \nabla u - \nabla \gamma)) \, , \tag{1}$$

$$-\Delta \gamma + \gamma = u \ . \tag{2}$$

Burger, Di Francesco, and Dolak have shown existence of global weak solutions in [6]; additionally, they discussed large-time behavior of weak solutions. The corresponding fully parabolic chemotaxis model has been treated by Di Francesco and Rosado [11].

The appearance of waiting times and the finite speed of the support propagation property are typical phenomena arising in solutions of degenerate parabolic equations. In [6], Burger, Di Francesco, and Dolak establish finite speed of propagation for their modified Keller–Segel model in the case of one spatial dimension using a technique from [7]; however, only the qualitative result is established and no quantitative bounds are given; moreover, the method is restricted to the case of a single spatial dimension.

In general, there are different techniques available for deriving estimates on support propagation and sufficient conditions for support shrinking. The earliest approach for proving finite speed of support propagation (applicable, e.g., in the case of the porous medium equation) proceeds by comparing the solution to a self-similar (sub)solution. Another technique which can be used for establishing quantitative bounds for support propagation and waiting times as well as conditions for support shrinking makes use of differential inequalities; see, for example, the articles by Antontsev, Diaz, and Shmarev [3], Bernis [5], [4], Diaz, Galiano, and Jüngel [10], and Galiano and Peletier [13]. Subsequently techniques based on integral estimates have been developed; the first technique in this direction combines integral estimates with functional inequalities and is due to Shishkov and Shchelkov [23]. It has been applied to different situations; see, e.g., the papers by Sapronov and Shishkov [22] and Giacomelli and Shishkov [15]. Another technique based on integral estimates makes use of Stampacchia's lemma: in [8] Dal Passo, Giacomelli, and Grün have proposed a modification of Stampacchia's lemma and used it to derive sufficient conditions for the appearance of a waiting time in the case of the thin-film equation. See the papers by Ansini and Giacomelli [2] and Giacomelli and Grün [14] for other uses of the modified Stampacchia lemma. More recently Carrillo, Gualdani, and Toscani developed a technique to prove finite speed of support propagation which is based on mass transportation [7]. Quite recently finite speed of support propagation for a subcritical

Patlak–Keller–Segel model with porous-medium-type degeneracy has been shown by Kim and Yao [19] using comparison arguments. For a comparison of our finite speed of propagation results to the results by Kim and Yao, see section 3.

In the present work, we derive integral estimates and apply Stampacchia's lemma, proving finite speed of support propagation for the system from [6] and giving sufficient criteria for support shrinking. The main tools used for these results are new test functions whose support moves according to the local advection velocity and whose support shrinks with time to account for the local variation of the advection velocity; moreover, they decay with time to compensate the nonvanishing divergence of the velocity field. This adaption of our test functions to the local advection velocity is crucial for "tracking" the free boundary in our results on support shrinking. See section 2.4 for a detailed discussion. For a result on finite speed of propagation for an advection-diffusion equation, see the paper by Galiano and Peletier [13]. Their paper contains an idea of compensating the advection term using a technique similar in spirit to our approach of using test functions with shrinking support, but their weights neither decay with time nor are adapted to the local advection velocity. Thus they fail to give results in the case of nonvanishing divergence of the velocity field and to derive results on support shrinkage. The same approach of compensating the advection term as in the paper by Galiano and Peletier has been used by Diaz, Galiano, and Jüngel in [10] in order to prove finite speed of support propagation for some charge-transport model with degenerate mobility by means of differential inequalities. Note that Diaz, Galiano, and Jüngel also treat the case of advection velocities with nonvanishing divergence and they also derive sufficient conditions for support shrinking. However, our result on support shrinking covers an entirely different situation: Diaz, Galiano, and Jüngel rely on electron-hole recombination as a mechanism for inducing support shrinking; on the other hand, we rely on advection terms which push the free boundary back. We therefore require the presence of an advection term pushing the front back, but do not need any recombination term; Diaz, Galiano, and Jüngel require the presence of a large recombination term, but no advection term is necessary for their result. Of course, there is no analogue of electron-hole recombination in Keller–Segel-type models, so the sufficient conditions for support shrinking by Diaz, Galiano, and Jüngel do not carry over to the degenerate chemotaxis model.

1.2. Main results. Define $A(u) := \int_0^u (\frac{1}{2}v^2 - \frac{1}{3}v^3)\, dv$. We obtain the following results on support shrinking.

THEOREM 2.23. *Let u, γ be a weak solution of* (1), (2) *with initial data $u_0 \in L^1$, where $0 \le u_0 \le 1$.*

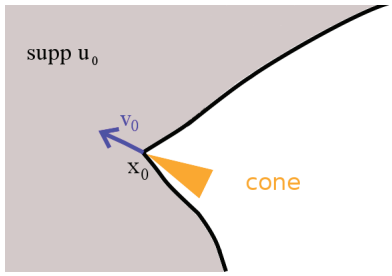
Given a point $x_0 \in \partial \operatorname{supp} u_0$ with $\nabla\gamma(x_0, 0) \neq 0$, assume that there exists a neighborhood U of x_0 and a cone B with vertex x_0 and axis $-\nabla\gamma(x_0, 0)$ such that $(U \cap B) \cap \operatorname{supp} u_0 = \emptyset$. Suppose in addition

$$\lim_{r\to 0} r^{-3} \fint_{B_r(x_0)} A(u_0(x))dx = 0 \ . \tag{3}$$

Then there exist $T^* > 0$ and $c > 0$ such that for any $0 < t \le T^*$ we have $\operatorname{supp} u(.,t) \cap B_{ct}(x_0) = \emptyset$.

The situation of Theorem 2.23 is depicted in the sketch below. We consider a point $x_0 \in \partial \operatorname{supp} u_0$ at the boundary of the support of the initial data. Suppose that $v_0 = \nabla\gamma(x_0, 0)$, the initial advection velocity at x_0, is directed so as to push the free boundary back (the reader may convince himself that this is implied by the cone condition mentioned in the theorem). If the initial data are "flat enough" at x_0, the

advective term dominates over the degenerate diffusion term and the support shrinks immediately at x_0.



As the interaction between different bacteria is only attractive, we can show the following.

COROLLARY 2.28. *Let u, γ be a weak solution of* (1), (2) *with initial data $u_0 \in L^1$, where $0 \le u_0 \le 1$.*

Suppose $x_0 \in \partial \operatorname{Conv}(\operatorname{supp} u_0)$ (where Conv *denotes the convex hull); then in the case*

$$\lim_{r\to 0} r^{-3} \fint_{B_r(x_0)} A(u_0(x))dx = 0$$

the support of u shrinks immediately near x_0; more precisely, there exist $T^* > 0$, $c > 0$ such that for every $0 < t \le T^*$ we have $\operatorname{supp} u(.,t) \cap B_{ct}(x_0) = \emptyset$.

The condition

$$\lim_{\delta\to 0} r^{-3} \fint_{B_\delta(x_0)} A(u_0)\ dx = 0$$

is trivially satisfied if $x_0 \notin \operatorname{supp} u_0$ and amounts to a local "flatness" condition on the initial data near x_0 if $x_0 \in \partial \operatorname{supp} u_0$. Recall that the condition for the appearance of a waiting time near x_0 for the equation without advection $u_t = \operatorname{div}(u\nabla u)$ is $\limsup_{\delta\to 0} r^{-6} \fint_{B_\delta(x_0)} u_0^3\ dx < \infty$ (see, e.g., [14]). Our condition is significantly weaker, but comes at the cost of imposing a constraint on the initial advection velocity $\nabla\gamma(x_0, 0)$; on the other hand, our assertion is stronger: we do not only show that the support does not spread immediately, but we show that the support even shrinks initially.

The result on finite speed of propagation reads as follows.

THEOREM 2.20. *Let u, γ be a weak solution of* (1), (2) *with initial data $u_0 \in L^1$, where $0 \le u_0 \le 1$.*

Suppose

$$\operatorname{supp} u_0 \cap B_{R_1}(x_0) = \emptyset\ . \tag{4}$$

Then there exists a constant $c > 0$ depending only on d such that

$$\operatorname{supp} u(.,t_1) \cap B_{\frac{R_1}{2}}(x_0) = \emptyset$$

for any

$$t_1 \le \min\left(\frac{cR_1^{\frac{d+6}{3}}}{\epsilon ||u_0||_{L^1}^{\frac{1}{3}}}, \frac{cR_1}{\epsilon^{\frac{3}{d+6}} |\max(\sup(-\Delta\gamma), 0)|^{\frac{d+3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}}}, \frac{cR_1}{\sup|\nabla\gamma|} \right)\ .$$

This will be seen to imply the following corollary.

COROLLARY 2.21. *Let u, γ be a weak solution of* (1), (2) *with initial data $u_0 \in L^1$, where $0 \le u_0 \le 1$.*

Suppose $\operatorname{supp} u_0 \subset B_{R_0}(x_1)$. *Then there exists a constant $C > 0$ depending only on d such that* $\operatorname{supp} u(.,t) \subset B_{R(t)}(x_1)$ *for*

$$R(t) := R_0 + C\epsilon^{\frac{3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}} t^{\frac{3}{d+6}}$$
$$+ C\left(\sup|\nabla\gamma| + \epsilon^{\frac{3}{d+6}} |\max(\sup(-\Delta\gamma), 0)|^{\frac{d+3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}}\right) t \ .$$

Note that the estimates on support propagation are not optimal for large times. For large times, it may be possible to obtain better bounds by taking into account the aggregative nature of the chemotaxis model. However, deriving such sharper bounds is beyond the scope of this paper.

Section 2.1 contains a short description of the model from Burger, Di Francesco, and Dolak [6]. In section 2.2 we will seek to obtain energy estimates for the system. We proceed in section 2.3 by proving some regularity properties for solutions of the system which we will need in what follows. Section 2.4 contains the definition of our class of test functions as well as the application of Stampacchia's lemma, which is based on the previous energy estimates. We therefore obtain a basic theorem on finite propagation. Some considerations regarding the equation for chemoattractant density in section 2.5 will then allow for deriving the above sufficient criterion for the occurrence of support shrinking.

1.3. Notation. By I we denote some time interval $(0, T)$.

The notation $a \wedge b$ (respectively, $a \vee b$) will be used to to denote the minimum (respectively, the maximum) of a and b.

We let $z := (x, t)$ be a point in spacetime.

By abuse of notation, we denote the space $[L^p(I; L^q)]^k$ of vector-valued functions simply as $L^p(I; L^q)$.

We refer to the set of all uniformly continuous functions on $(0, T)$ with values in X, i.e., the set of all functions which can be extended to a continuous function on $[0, T]$ with values in X, by $C^0(I; X)$; by $C^0_{loc}(I; X)$ we denote the set of all continuous functions on $(0, T)$ with values in X. The notation C^∞_{cpt} is used to denote the set of all smooth compactly supported functions.

The symbols c and C denote constants which may change from line to line.

If $v \in L^1$, we denote the essential support of v by $\operatorname{supp} v$.

2. A degenerate model of Keller–Segel-type.

2.1. Description of the model. Let u denote the bacteria density and γ denote the concentration of the chemoattractant. The modified Keller–Segel equations studied by Burger, Di Francesco, and Dolak [6] then read

$$u_t = \nabla \cdot (u(1-u)(\epsilon\nabla u - \nabla\gamma)) \,,$$
$$-\Delta\gamma + \gamma = u \ .$$

The following notion of a weak solution to (1), (2) was introduced in [6].

DEFINITION 2.1. *Let $I := (0,T)$ and $u \in L^2(I; L^2(\mathbb{R}^d))$ as well as $\gamma \in L^2(I; H^1(\mathbb{R}^d))$. We say the pair u, γ is a weak solution to* (1), (2), *if the following conditions are satisfied:*

- *It holds that $0 \le u \le 1$ a.e.;*
- $\nabla\left(\frac{1}{2}u^2 - \frac{1}{3}u^3\right) \in L^2(I; L^2)$;
- $\frac{1}{6}u^3 - \frac{1}{24}u^4 \in L^\infty(I; L^1)$;
- $\int_{\mathbb{R}^d} \nabla\gamma(x,t)\cdot\nabla\phi(x) + \gamma(x,t)\phi(x) - u(x,t)\phi(x)\ dx = 0$ *for any $\phi \in H^1(\mathbb{R}^d)$ and almost every $t \in I$;*
- *For any $\phi \in C^\infty_{cpt}([0,T)\times\mathbb{R}^d)$, we have*

$$\int_0^T \int_{\mathbb{R}^d} u\phi_t\ dxdt - \epsilon \int_0^T \int_{\mathbb{R}^d} \nabla\left(\frac{1}{2}u^2 - \frac{1}{3}u^3\right)\cdot\nabla\phi\ dx\ dt + \int_0^T \int_{\mathbb{R}^d} u(1-u)\nabla\gamma\cdot\nabla\phi\ dx\ dt + \int_{\mathbb{R}^d} u(.,0)\phi(.,0)dx = 0\ . \tag{5}$$

As mentioned in the introduction, Burger, Di Francesco, and Dolak [6] have shown the existence of a weak solution for any $u_0 \in L^1$ with $0 \le u_0 \le 1$.

Remark 2.2. It follows from the above definition that $u \in H^1(I; H^{-1})$. We can therefore rewrite the above equation as

$$\int_0^T \langle u_t, \phi\rangle_{H^{-1}\times H^1} dt = -\epsilon \int_0^T \int_{\mathbb{R}^d} \nabla\left(\frac{1}{2}u^2 - \frac{1}{3}u^3\right)\cdot\nabla\phi\ dxdt + \int_0^T \int_{\mathbb{R}^d} u(1-u)\nabla\gamma\cdot\nabla\phi\ dxdt\ .$$

2.2. Energy estimates. We now seek to obtain energy estimates for the degenerate advection-diffusion equation (1) by plugging in $(\frac{1}{2}u^2 - \frac{1}{3}u^3)\psi$ as a test function, where $\psi \in C^\infty_{cpt}([0,T)\times\mathbb{R}^d)$. Since u does not have sufficient regularity, we need some approximation arguments to perform calculations with the resulting equation.

DEFINITION 2.3. *We use the following abbreviations:*

$$a(u) := \left(\frac{1}{2}u^2 - \frac{1}{3}u^3\right),$$

$$A(u) := \int_0^u \left(\frac{1}{2}v^2 - \frac{1}{3}v^3\right)\ dv = \int_0^u a(v)\ dv,$$

$$H(u) := \int_0^u v(1-v)\frac{d}{dv}\left(\frac{1}{2}v^2 - \frac{1}{3}v^3\right)\ dv = \int_0^u v^2(1-v)^2\ dv = \int_0^u v(1-v)\frac{d}{dv}a(v)\ dv\ .$$

By straightforward calculations one can show the following.

LEMMA 2.4. *For $0 \le v \le 1$ it holds that $cv^2 \le a(v) \le Cv^2$, $cv^3 \le A(v) \le Cv^3$, and $cv^3 \le H(v) \le Cv^3$.*

LEMMA 2.5. *Let $v \in H^1(I; H^{-1})$, $v \in L^2(I; L^2)$, $0 \le v \le 1$, $a(v) \in L^2(I; H^1)$,*

and $\psi \in C^\infty_{cpt}([0,T]\times\mathbb{R}^d)$. *We then have for almost every* $(t_1,t_2)\in I\times I$

$$\begin{aligned}&\int_{t_1}^{t_2}\langle v_t, a(v)\psi^2\rangle_{H^{-1}\times H^1}\ dt\\ &\quad=\int_{\mathbb{R}^d}A(v(x,t_2))\psi^2(x,t_2)dx-\int_{\mathbb{R}^d}A(v(x,t_1))\psi^2(x,t_1)\ dx\\ &\qquad-\int_{t_1}^{t_2}\int_{\mathbb{R}^d}A(v)2\psi\psi_t\ dx\ dt\ .\end{aligned}$$

Additionally, we have

$$v(1-v)\nabla a(v)=\nabla H(v)$$

in $L^2(I;L^2)$.

Proof. Writing $a(.)$ as the difference of an increasing and a decreasing function, the first part of the lemma is a consequence of a technique by Alt and Luckhaus [1].

The second part of the lemma is verified writing $H(v)=H(a^{-1}(a(v)))$ and using the fact that $H(a^{-1}(.))$ is Lipschitz, a fact that is readily verified as

$$\begin{aligned}\frac{d}{dw}H(a^{-1}(w))&=H'(a^{-1}(w))\frac{1}{a'(a^{-1}(w))}\\ &=v^2(1-v)^2\frac{1}{v(1-v)}=v(1-v)\end{aligned}$$

with $v:=a^{-1}(w)$; note that we have used Definition 2.3. Note that no differentiability of v with respect to space is assumed; therefore we need to rewrite $H(v)$ in terms of $a(v)$ (whose weak spatial derivative is known to exist by our assumptions). ∎

PROPOSITION 2.6. *For a weak solution of* (1), (2) *we have for a.e.* $(t_1,t_2)\in I\times I$

$$\begin{aligned}&\int_{\mathbb{R}^d}A(u(x,t_2))\psi^2(x,t_2)dx+\epsilon\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left|\nabla\left(a(u)\psi\right)\right|^2\ dx\ dt\\ &\quad=\int_{\mathbb{R}^d}A(u(x,t_1))\psi^2(x,t_1)\ dx+\int_{t_1}^{t_2}\int_{\mathbb{R}^d}A(u)2\psi\psi_t\ dx\\ &\qquad+\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\nabla H(u)\cdot\nabla\gamma\psi^2\ dx\ dt+\int_{t_1}^{t_2}\int_{\mathbb{R}^d}2\psi u(1-u)a(u)\nabla\gamma\cdot\nabla\psi\ dx\ dt\\ &\qquad+\epsilon\int_{t_1}^{t_2}\int_{\mathbb{R}^d}a(u)^2|\nabla\psi|^2\ dx\ dt\end{aligned}\tag{6}$$

or equivalently

$$\begin{aligned}&\int_{\mathbb{R}^d}A(u(x,t_2))\psi^2(x,t_2)\ dx+\epsilon\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left|\nabla\left(a(u)\psi\right)\right|^2\ dx\ dt\\ &\quad=\int_{\mathbb{R}^d}A(u(x,t_1))\psi^2(x,t_1)\ dx+\int_{t_1}^{t_2}\int_{\mathbb{R}^d}A(u)2\psi\psi_t\ dx\ dt\\ &\qquad-\int_{t_1}^{t_2}\int_{\mathbb{R}^d}2\psi H(u)\nabla\gamma\cdot\nabla\psi\ dx\ dt-\int_{t_1}^{t_2}\int_{\mathbb{R}^d}H(u)\Delta\gamma\ \psi^2\ dx\ dt\\ &\qquad+\int_{t_1}^{t_2}\int_{\mathbb{R}^d}2\psi u(1-u)a(u)\nabla\gamma\cdot\nabla\psi\ dx\ dt+\epsilon\int_{t_1}^{t_2}\int_{\mathbb{R}^d}a(u)^2|\nabla\psi|^2\ dx\ dt\ .\end{aligned}\tag{7}$$

Proof. By a standard approximation argument, we may insert $a(u)\psi^2 \in L^2(I;H^1)$ with $\psi \in C^\infty_{cpt}([0,T)\times\mathbb{R}^d)$ as a test function into (5). Application of the previous lemma and a straightforward calculation yield (6). Note that $|\nabla(a(u)\psi)|^2 = \psi^2|\nabla a(u)|^2 + 2a(u)\psi\nabla a(u)\cdot\nabla\psi + a(u)^2|\nabla\psi|^2$.

By integration by parts applied to the third term on the right-hand side of (6), we obtain (7). ∎

2.3. Additional regularity. We want to apply Stampacchia's lemma using the previous energy estimates. To do so, we need the estimate to hold not only for a.e. $(t_1,t_2)\in I\times I$, but for $t_1=0$. This statement will be a consequence of the regularity $u\in C^0(I;L^2(K))$ for all $K\subset\subset\mathbb{R}^d$, which we show to hold now.

In order to provide a proof for $u\in C^0(I;L^2)$ in the case $u_0\in L^1$, we start off proving that $\int_0^u v^{\frac12}(1-v)^{\frac12}dv \in L^2(I;H^1(K))$ for all $K\subset\subset\mathbb{R}^d$ and utilize this result to obtain $u\in L^2(I;H^1(K))$ for all $K\subset\subset\mathbb{R}^d$.

LEMMA 2.7. *For any solution to our model, we have* $\int_0^u v^{\frac12}(1-v)^{\frac12}dv\in L^2(I;H^1(K))$ *for any* $K\subset\subset\mathbb{R}^d$.

Proof. Set $u_\delta := \min(1-\delta,\max(\delta,u))$, take a smooth cutoff function ϕ with $\{\phi=1\}\supset K$, and let $\xi:=\phi u_\delta$. We know $\xi\in L^2(I;H^1(\mathbb{R}^d))$: it holds that $\frac12u^2-\frac13u^3\in L^2(I;H^1)$; the inverse of $q:[0,1]\to[0,\frac16]$,

$$q(x) := \frac12x^2-\frac13x^3 \tag{8}$$

exists, is continuous, and outside of any neighborhood of $\{q(0)=0,q(1)=\frac16\}$ it is of class C^1. With $\xi=\phi q^{-1}[q(\delta)\vee(q(1-\delta)\wedge(\frac12u^2-\frac13u^3))]$, we see $\xi\in L^2(I;H^1(\mathbb{R}^d))$ and therefore $u_\delta\in L^2(I;H^1(K))$ for any $K\subset\subset\mathbb{R}^d$.

Thus it is possible to plug ξ into (5). Performing some standard rearrangements and using the technique by Alt and Luckhaus [1] to rearrange the term involving the time derivative, we see that it is possible to pass to the limit $\delta\to0$. The desired result then follows. As the proof is mainly standard, we omit the details. ∎

PROPOSITION 2.8. *Any solution to the above equation satisfies* $u\in L^2(I;H^1(K))$ *for all* $K\subset\subset\mathbb{R}^d$. *Moreover, we have* $u\in C^0(I;L^2(K))$ *for all* $K\subset\subset\mathbb{R}^d$.

Proof. Setting $u_\delta:=\min(1-\delta,\max(\delta,u))$, taking a smooth cutoff function ϕ, and using $\xi:=\phi\,(\log u_\delta-\log(1-u_\delta))$ as a test function, the proof of the first assertion is again standard. We again omit it.

It is a well-known result that $u\in L^2(I;H^1)\cap H^1(I;(H^1)')$ implies $u\in C^0(I;L^2)$; thus, the second assertion is immediate. ∎

LEMMA 2.9. *Any weak solution* u *with* $u_0\in L^1$ *satisfies* $u\in C^0(I;L^p)$ *for all* $1\le p<\infty$.

Proof. We only need to show $u\in C^0(I;L^1)$; by boundedness of u the general statement then follows immediately. By the previous corollary, we know $u\in C^0(I;L^2(K))$ and thus $u\in C^0(I;L^1(K))$ for any $K\subset\subset\mathbb{R}^d$. Fix $t\in I$ and choose K so large that $||u(.,t)||_{L^1(\mathbb{R}^d\setminus K)}<\frac\epsilon4$. Now let $\delta>0$ be small enough such that $||u(.,t)-u(.,t')||_{L^1(K)}\le\frac\epsilon4$ for $|t-t'|<\delta$. By conservation of mass, we have $||u(.,t')||_{L^1(\mathbb{R}^d\setminus K)}<\frac\epsilon2$ and therefore $||u(.,t)-u(.,t')||_{L^1(\mathbb{R}^d)}\le\epsilon$. ∎

2.4. Application of Stampacchia's lemma. The aim of this section is to prove the results on finite speed of propagation and support shrinking. We first cast the energy estimate (7) into a simpler form by plugging in appropriate test functions, then use the Gagliardo–Nirenberg–Sobolev inequality and apply Stampacchia's lemma to deduce basic results on finite speed of propagation and support shrinking.

Define

$$G_T(\xi) := \int_0^T \int_{B_{R-\xi-C^* M_{adv} t}(x_0+tv)} a(u)^2 dx dt \ , \tag{9}$$

where C^* is a constant which is to be selected depending only on d while deriving (19) and (20) and where $M_{adv} \geq 0$, $v \in \mathbb{R}^d$ are parameters. The function $G_T(\xi)$ will play the role of the so-called *scanning function*. When deriving estimates on support propagation via Stampacchia's lemma, the *scanning function* is the function to which Stampacchia's iteration lemma is actually applied. Having found a zero ξ of the scanning function, we may conclude that no mass is present within the corresponding domain of integration (i.e., in our case that no mass is present in $B_{R-\xi-C^* M_{adv} t}(x_0 + tv)$ for any $t \in [0, T]$). The scanning function bears its name since it "scans" the corresponding domain of integration for the presence of mass.

Let

$$S(R) := \sup_{r \in (R', R]} (r - R')^{-6-d} \int_{B_r(x_0)} A(u(.,0)) dx \ . \tag{10}$$

We suppress the dependence of $S(R)$ on R' in our notation in the sequel. It is immediate that G is nonincreasing in ξ and S is nondecreasing in R. Note that $S(R) < \infty$ if $\operatorname{dist}(x_0, \operatorname{supp} u_0) \geq R'$ and if in addition u_0 is "flat enough" near $\partial B_{R'}(x_0)$.

This section is devoted to the proof of the following three results.

LEMMA 2.10. *Suppose that there exist positive constants M_{adv}, M_{div}, R', R, s, and a constant vector $v \in \mathbb{R}^d$ such that the following conditions are satisfied:*

(H1) $0 < R' < R \leq \frac{\epsilon}{M_{adv}+|v|}$;

(H2) $|\nabla\gamma - v| \leq M_{adv}$ *for all* (x,t) *such that* $|x - x_0| \leq R + |v|s$ *and* $t \leq s$;

(H3) $\Delta\gamma \geq -M_{div}$ *for all* (x,t) *such that* $|x - x_0| \leq R + |v|s$ *and* $t \leq s$;

(H4) $S(R) < \infty$ *(note that this implies* $u_0(.) \equiv 0$ *on* $B_{R'}(x_0)$*).*

Then there exist positive universal constants c_S, C^* (i.e., constants depending only on d) such that the following holds:

$$u(.,t) \equiv 0 \ \textit{on} \ B_{R'-C^* M_{adv} t}(x_0 + tv)$$
$$\textit{for any}$$
$$t \in \left[0, \min\left(s, \frac{c_S}{M_{div}}, \frac{c_S R'}{M_{adv}}, c_S \epsilon^{-\frac{4}{3}} \left(G_s(0)(R-R')^{-d-8} + \epsilon^{-1} S(R)\right)^{-\frac{1}{3}}\right)\right].$$

In case of vanishing v, we obtain a sharper result on finite propagation.

LEMMA 2.11. *Suppose that there exist positive constants M_{adv}, M_{div}, R', R, s such that the following conditions are satisfied:*

(H1′) $0 < R' < R$;

(H2′) $|\nabla\gamma| \leq M_{adv}$ *for all* (x,t) *such that* $|x - x_0| \leq R$ *and* $t \leq s$;

(H3′) $\Delta\gamma \geq -M_{div}$ *for all* (x,t) *such that* $|x - x_0| \leq R$ *and* $t \leq s$;

(H4′) $S(R) < \infty$ *(note that this implies* $u_0(.) \equiv 0$ *on* $B_{R'}(x_0)$*).*

Then there exist positive universal constants c, C^* (i.e., constants depending only on d) such that the following holds:

$$u(.,t) \equiv 0 \ \textit{on} \ B_{R'-C^* M_{adv} t}(x_0)$$
$$\textit{for any}$$
$$t \in \left[0, \min\left(s, \frac{c}{M_{div}}, \frac{c R'}{M_{adv}}, c \epsilon^{-\frac{4}{3}} \left(G_s(0)(R-R')^{-d-8} + \epsilon^{-1} S(R)\right)^{-\frac{1}{3}}\right)\right].$$

We now replace $S(R)$ by $\sup_R S(R)$ and then let $R \to \infty$ to see the following corollary.

COROLLARY 2.12. *Suppose that globally* $|\nabla\gamma| \le M_{adv}$, $\Delta\gamma \ge -M_{div}$. *Assume* $\sup_R S(R) < \infty$ *which in particular implies* $u(.,0) \equiv 0$ *on* $B_{R'}(x_0)$. *Then there exist universal constants* $c > 0$, $C^* < \infty$ *(i.e., constants depending only on* d*) such that* $u(.,t) \equiv 0$ *in* $B_{R'-tC^*M_{adv}}(x_0)$ *if* $t \le \frac{c}{M_{div}}$ *and* $t \le \frac{cR'}{M_{adv}}$ *and*

$$t \le c\epsilon^{-1} \inf_{r\in(R',\infty)} (r-R')^{\frac{d+6}{3}} \left(\int_{B_r(x_0)} A(u_0(.)) \right)^{-\frac{1}{3}}$$

are satisfied.

Proof of Lemma 2.10. The following extension of Stampacchia's lemma is given in [14].

LEMMA 2.13. *Let* $G : [0,\hat{R}] \to [0,\infty)$ *be nonincreasing and suppose that*

$$G(\xi) \le \frac{K}{(\xi-\eta)^\alpha} \left(G(\eta) + \tilde{S} \cdot (\hat{R}-\eta)^{\frac{\alpha}{\beta-1}} \right)^\beta$$

for all $0 \le \eta < \xi \le \hat{R}$ *with* $K > 0$, $\tilde{S} \ge 0$, $\alpha > 0$, *and* $\beta > 1$. *Suppose furthermore that* $\hat{R}^{\frac{\alpha}{\beta-1}} \ge (2^{\frac{\beta(\alpha+\beta-1)}{\beta-1}} K)^{\frac{1}{\beta-1}} (G(0) + \tilde{S} \cdot \hat{R}^{\frac{\alpha}{\beta-1}})$. *Then* $G(\hat{R}) = 0$ *holds.*

Set $\theta = \frac{d}{6+d}$. We will use the energy estimates to deduce the inequality

$$G_T(\xi) \le \frac{C_A T^{1-\theta} \epsilon^{\frac{4-4\theta}{3}}}{(\xi-\eta)^{\frac{8-2\theta}{3}}} \left(G_T(\eta) + \epsilon^{-1} (R-R'-\eta)^{\frac{8-2\theta}{1-\theta}} S(R) \right)^{\frac{4-\theta}{3}} \tag{11}$$

for $R - R' \ge \xi > \eta \ge 0$ if $|\nabla\gamma - v| \le M_{adv}$, $\Delta\gamma \ge -M_{div}$ holds for all

$$(x,t) \in \cup_{t\in[0,T]} B_R(x_0 + tv) \times \{t\} \ ,$$

and if additionally $T \le \frac{c_A(R-\xi)}{M_{adv}}$, $T \le \frac{c_A}{M_{div}}$, and $R \le \frac{\epsilon}{M_{adv}+|v|}$ are satisfied. Here C_A, c_A are positive constants depending only on d which will be determined in the course of the proof of (11) below. The constant $C^* > 0$ which $G_T(\xi)$ implicitly depends on shall also be determined in the proof of (11) below.

Note that the conditions (H1), (H2), (H3) then ensure the validity of (11) for $0 \le \xi \le R - R'$ if the condition

$$T \le \min \left[s, \frac{c_A}{M_{div}}, \frac{c_A R'}{M_{adv}} \right]$$

is satisfied. In this case we can apply the extension of Stampacchia's lemma. Set $\hat{R} = R - R'$, $K = C_A T^{1-\theta} \epsilon^{\frac{4-4\theta}{3}}$, $\alpha = \frac{8-2\theta}{3}$, $\beta = \frac{4-\theta}{3}$, $\tilde{S} := \epsilon^{-1} S(R)$. We see that $\frac{\alpha}{\beta-1} = \frac{8-2\theta}{1-\theta}$. Consequently Lemma 2.13 gives $G_T(R-R') = 0$ in the case

$$(R-R')^\alpha \ge C T^{1-\theta} \epsilon^{\frac{4-4\theta}{3}} \left(G_T(0) + \epsilon^{-1} (R-R')^{\frac{\alpha}{\beta-1}} S(R) \right)^{\beta-1}$$

or equivalently

$$T \leq c\epsilon^{-\frac{4}{3}} \left((R-R')^{\frac{2\theta-8}{1-\theta}} G_T(0) + \epsilon^{-1} S(R) \right)^{-\frac{1}{3}} .$$

We set $c_S := \min(c_A, c)$, where c is the constant in the inequality directly above. Since $\frac{2\theta-8}{1-\theta} = -d-8$ we have established Lemma 2.10, as $T \leq s$ and $G_t(0)$ is a nondecreasing function with respect to t (this property is immediate by (9)). ∎

It remains to derive inequality (11).

We turn our attention to the cutoff function ψ which is to be used in the energy estimate (7). Assume that both $\nabla\gamma$ and its divergence $\Delta\gamma$ are bounded; we will later show that this is indeed the case. More precisely, let $\vec{v} \in \mathbb{R}^d$, let U be a domain with

$$\sup_U |\nabla\gamma - \vec{v}| \leq M_{adv} < \infty \tag{12}$$

and

$$\max\left(\sup_U -\Delta\gamma, 0 \right) \leq M_{div} < \infty . \tag{13}$$

We shall need to select the parameters in the definition of ψ (see (15)) in such a way that

$$\operatorname{supp}\psi \subset U \tag{14}$$

is satisfied.

Choose a smooth monotonous cutoff $\nu : \mathbb{R} \to [0,1]$ with $\nu \equiv 1$ for $x < 0$ and $\nu \equiv 0$ for $x > 1$. Fix $r > 0$, $R > 0$, $T_f > 0$, $\vec{v} \in \mathbb{R}^d$. Let $\mu > 0$ be some constant. We define our test function as follows (with C^*, C^- some sufficiently large universal constants):

$$\psi := \nu\left(\frac{|x - x_0 - vt| - r + C^* M_{adv} t}{R-r} \right) \cdot \exp(-C^- M_{div} t) \cdot \nu\left(\frac{t - T_f}{\mu} \right) . \tag{15}$$

For

$$C^* M_{adv} \cdot (T_f + \mu) < r \tag{16}$$

and

$$T_f + \mu \leq \frac{\ln 2}{C^- M_{div}} \tag{17}$$

this test function has the following features:

T1. $\psi \in C^\infty_{cpt}$ and $0 \leq \psi \leq 1$;
T2. $\operatorname{supp}\psi$ moves with velocity v and shrinks with time with speed $C^* M_{adv}$;
T3. $\psi(.,t) \equiv 0$ outside of $B_{R-C^* M_{adv} t}(x_0 + vt)$ for any t and $\psi(.,t) \equiv 0$ if $t > T_f + \mu$;
T4. $\psi \geq \frac{1}{2}$ in $B_{r-C^* M_{adv} t}(x_0 + vt)$ for $t \leq T_f$;
T5. $\psi_t \leq -\nabla\psi \cdot v - C^* M_{adv} |\nabla\psi| - C^- M_{div} \psi$;
T6. $|\nabla\psi| \leq \frac{C}{R-r}$.

The first three properties are obvious; the fourth property holds by (17). The fifth property follows by the usual calculus using the monotonicity of ν: We have

$$\begin{aligned}
\psi_t &= \mu^{-1}\nu\left(\frac{|x-x_0-vt|-r+C^*M_{adv}t}{R-r}\right)\cdot\exp(-C^-M_{div}t)\cdot\nu'\left(\frac{t-T_f}{\mu}\right)\\
&\quad+\left(\frac{C^*M_{adv}}{R-r}-\frac{(x-x_0-vt)\cdot v}{|x-x_0-vt|(R-r)}\right)\cdot\nu'\left(\frac{|x-x_0-vt|-r+C^*M_{adv}t}{R-r}\right)\\
&\quad\cdot\exp(-C^-M_{div}t)\cdot\nu\left(\frac{t-T_f}{\mu}\right)\\
&\quad-C^-M_{div}\nu\left(\frac{|x-x_0-vt|-r+C^*M_{adv}t}{R-r}\right)\cdot\exp(-C^-M_{div}t)\cdot\nu\left(\frac{t-T_f}{\mu}\right)\\
&\le\left(\frac{C^*M_{adv}}{R-r}-\frac{(x-x_0-vt)\cdot v}{|x-x_0-vt|(R-r)}\right)\cdot\nu'\left(\frac{|x-x_0-vt|-r+C^*M_{adv}t}{R-r}\right)\\
&\quad\cdot\exp(-C^-M_{div}t)\cdot\nu\left(\frac{t-T_f}{\mu}\right)\\
&\quad-C^-M_{div}\nu\left(\frac{|x-x_0-vt|-r+C^*M_{adv}t}{R-r}\right)\cdot\exp(-C^-M_{div}t)\cdot\nu\left(\frac{t-T_f}{\mu}\right)
\end{aligned}$$

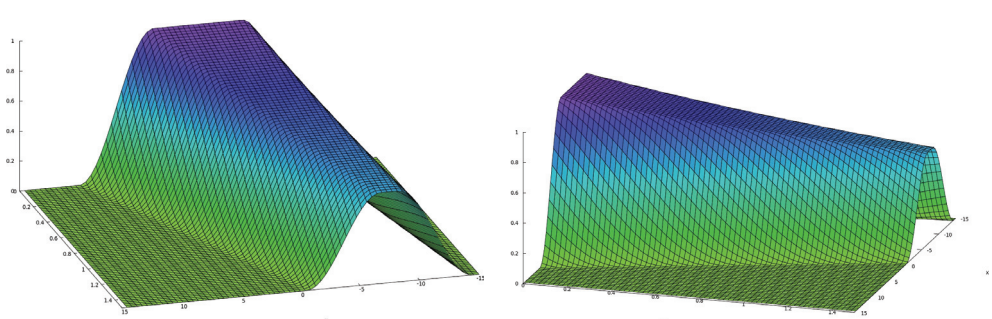
and

$$\begin{aligned}
\nabla\psi &= \frac{x-x_0-vt}{|x-x_0-vt|(R-r)}\nu'\left(\frac{|x-x_0-vt|-r+C^*M_{adv}t}{R-r}\right)\\
&\quad\cdot\exp(-C^-M_{div}t)\cdot\nu\left(\frac{t-T_f}{\mu}\right)
\end{aligned}$$

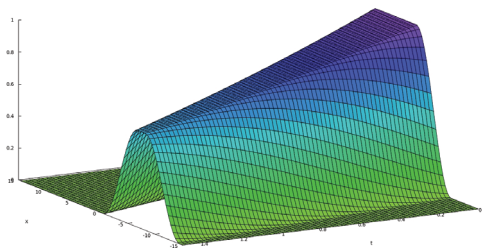
from which the sixth property is immediate. Thus

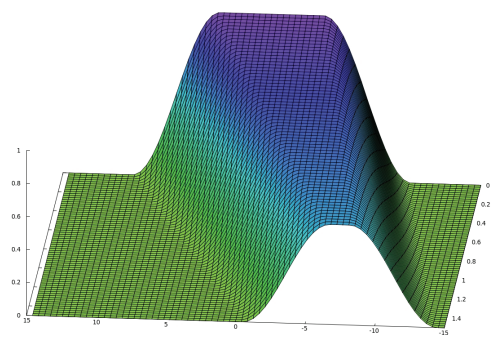
$$\psi_t \le -\nabla\psi\cdot v - C^*M_{adv}\,|\nabla\psi| - C^-M_{div}\psi\ .$$

Our test function moves with speed v and therefore can be adapted to the advection speed. Since the advection velocity is usually not constant, we cannot choose v to match the advection speed exactly; therefore the support shrinking property of the test function is introduced in order to compensate for the difference between v and the actual advection speed. The exponential decay is used to treat the terms which occur as a consequence of the nonvanishing divergence of the advection velocity field. The plots which are to follow show the evolution of a one-dimensional cross-section of the test function from different perspectives; it is easily seen that the support moves and shrinks with time and that the test function decays exponentially. The cutoff at time T_f is not visible in the pictures.



Galiano and Peletier [13] deal with a divergence-free advection term using an idea which is close in spirit to our approach of test functions with shrinking support, proving finite speed of support propagation for their equation; however, our test functions additionally decay exponentially with time and move over time, thereby enabling us to treat advection velocity fields with nonvanishing divergence and derive results on support shrinkage.





Plugging in this test function in the energy estimate (7), for C^*, C^- chosen large enough but constant, property T5 of the test function ensures that

$$\int_0^T \int_{\mathbb{R}^d} A(u) 2\psi\psi_t dx$$

is negative enough to dominate the possibly positive terms

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} 2\psi \left(u(1-u)a(u) - H(u)\right) \nabla\gamma \cdot \nabla\psi dxdt \\ &\quad - \int_0^T \int_{\mathbb{R}^d} H(u)\Delta\gamma\ \psi^2 dxdt\end{aligned}$$

since we know that $v(1-v)a(v) \leq Cv^3$, $H(v) \leq Cv^3$, and $v^3 \leq CA(u)$ (see Lemma 2.4 and Definition 2.3).

More precisely, using (7) and T5 we get the estimate

$$\begin{aligned}&\int_{\mathbb{R}^d} A(u(x,T))\psi^2(x,T)dx + \epsilon \int_0^T \int_{\mathbb{R}^d} \left|\nabla\left(a(u)\psi\right)\right|^2 dxdt \\ &\quad \leq \int_{\mathbb{R}^d} A(u(x,0))\psi^2(x,0)dx - \int_0^T \int_{\mathbb{R}^d} A(u) 2\psi v \cdot \nabla\psi dx \\ &\quad - C^- \int_0^T \int_{\mathbb{R}^d} A(u) 2\psi^2 M_{div} dx - C^* \int_0^T \int_{\mathbb{R}^d} A(u) 2\psi M_{adv} |\nabla\psi| dx \\ &\quad - \int_0^T \int_{\mathbb{R}^d} 2\psi H(u) \nabla\gamma \cdot \nabla\psi dxdt - \int_0^T \int_{\mathbb{R}^d} H(u)\Delta\gamma\ \psi^2 dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} 2\psi u(1-u)a(u)\nabla\gamma \cdot \nabla\psi dxdt + \epsilon \int_0^T \int_{\mathbb{R}^d} a(u)^2 |\nabla\psi|^2 dxdt\end{aligned} \tag{18}$$

for any $T > 0$ and having chosen C^- large enough we obtain, by the condition on

$-\Delta\gamma$ (see (13)),

$$\begin{aligned}
&\int_{\mathbb{R}^d} A(u(x,T))\psi^2(x,T)dx + \epsilon\int_0^T\int_{\mathbb{R}^d} |\nabla\left(a(u)\psi\right)|^2\,dxdt \\
&\quad\le \int_{\mathbb{R}^d} A(u(x,0))\psi^2(x,0)dx - \int_0^T\int_{\mathbb{R}^d} A(u)2\psi v\cdot\nabla\psi dx \\
&\qquad - C^*\int_0^T\int_{\mathbb{R}^d} A(u)2\psi M_{adv}|\nabla\psi|dx - \int_0^T\int_{\mathbb{R}^d} 2\psi H(u)\nabla\gamma\cdot\nabla\psi dxdt \\
&\qquad + \int_0^T\int_{\mathbb{R}^d} 2\psi u(1-u)a(u)\nabla\gamma\cdot\nabla\psi dxdt + \epsilon\int_0^T\int_{\mathbb{R}^d} a(u)^2|\nabla\psi|^2 dxdt\ .
\end{aligned}$$

Since $|H(u)-\frac{1}{3}u^3|\le Cu^4$ and thus $|H(u)-2A(u)|\le Cu^4$ and since $|u(1-u)a(u)-3A(u)|\le Cu^4$ (see Definition 2.3) we have for any $T>0$

$$\begin{aligned}
&\int_{\mathbb{R}^d} A(u(x,T))\psi^2(x,T)dx + \epsilon\int_0^T\int_{\mathbb{R}^d} |\nabla\left(a(u)\psi\right)|^2\,dxdt \\
&\quad\le \int_{\mathbb{R}^d} A(u(x,0))\psi^2(x,0)dx + \epsilon\int_0^T\int_{\mathbb{R}^d} a(u)^2|\nabla\psi|^2 dxdt \\
&\qquad + \int_0^T\int_{\mathbb{R}^d} \psi A(u)\left(2\nabla\gamma - 2v\right)\cdot\nabla\psi dxdt \\
&\qquad - C^*\int_0^T\int_{\mathbb{R}^d} A(u)2\psi M_{adv}|\nabla\psi|dx + C\int_0^T\int_{\mathbb{R}^d} \psi u^4|\nabla\gamma||\nabla\psi|dxdt\ .
\end{aligned}$$

Since $u^4\le C(a(u))^2$ we obtain, for C^* chosen large enough but constant, by the condition on $|\nabla\gamma - v|$ (see (12))

$$\begin{aligned}
&\int_{\mathbb{R}^d} A(u(x,T))\psi^2(x,T)dx + \epsilon\int_0^T\int_{\mathbb{R}^d} |\nabla\left(a(u)\psi\right)|^2\,dxdt \\
&\quad\le \int_{\mathbb{R}^d} A(u(x,0))\psi^2(x,0)dx + \epsilon\int_0^T\int_{\mathbb{R}^d} a(u)^2|\nabla\psi|^2 dxdt \\
&\qquad + C\int_0^T\int_{\mathbb{R}^d} \psi a(u)^2|\nabla\gamma||\nabla\psi|dxdt\ .
\end{aligned} \tag{19}$$

A sharper estimate holds in case of vanishing v:

$$\begin{aligned}
&\int_{\mathbb{R}^d} A(u(x,T))\psi^2(x,T)dx + \epsilon\int_0^T\int_{\mathbb{R}^d} |\nabla\left(a(u)\psi\right)|^2\,dxdt \\
&\quad\le \int_{\mathbb{R}^d} A(u(x,0))\psi^2(x,0)dx + \epsilon\int_0^T\int_{\mathbb{R}^d} a(u)^2|\nabla\psi|^2 dxdt\ .
\end{aligned} \tag{20}$$

This estimate can be derived directly from (18) by choosing C^-, C^* sufficiently large and utilizing the fact that $|H(u)|+|u(1-u)a(u)|\le CA(u)$.

The estimates (19) and (20) in conjunction with the Gagliardo–Nirenberg interpolation inequality yield the desired estimates. Let us recall the Gagliardo–Nirenberg interpolation inequality (for proofs see [12], [20], [21], [9]).

THEOREM 2.14. *Suppose $0<q<p$, $1\le r\le\infty$. Let $\Omega=\mathbb{R}^d$. For $u\in L^q$ with $Du\in L^r$ we have the estimate*

$$||u||_{L^p}\le C||Du||_{L^r}^{\theta}\cdot||u||_{L^q}^{1-\theta}$$

with

$$\frac{1}{p} = \theta\left(\frac{1}{r} - \frac{1}{d}\right) + (1-\theta)\frac{1}{q} \ .$$

Using the Gagliardo–Nirenberg inequality on $a(u)\psi$, $p = 2$, $r = 2$, and $q = \frac{3}{2}$, we get $\theta = \frac{d}{6+d}$ and

$$\int_{\mathbb{R}^d} a(u)^2\psi^2 dx \le C\left(\int_{\mathbb{R}^d} |\nabla(a(u)\psi)|^2\, dx\right)^{\theta} \cdot \left(\int_{\mathbb{R}^d} a(u)^{\frac{3}{2}}\psi^{\frac{3}{2}} dx\right)^{\frac{4(1-\theta)}{3}} \ .$$

Integrating with respect to t and using Jensen's inequality yields

$$\begin{aligned}(21)\quad & \int_0^T \int_{\mathbb{R}^d} a(u)^2\psi^2 dxdt \\ & \le CT^{1-\theta}\left(\int_0^T \int_{\mathbb{R}^d} |\nabla(a(u)\psi)|^2\, dxdt\right)^{\theta} \cdot \sup_{t\in[0,T]}\left(\int_{\mathbb{R}^d} a(u)^{\frac{3}{2}}\psi^{\frac{3}{2}} dx\right)^{\frac{4(1-\theta)}{3}} \ .\end{aligned}$$

We now are in position to prove inequality (11).

Let $r_a := R - \xi$ and $R_a := R - \eta$. Note that by the conditions on ξ and η, we have (see the conditions following (11)) $R' \le r_a < R_a \le R$.

We make use of two test functions $\tilde{\psi}$, $\hat{\psi}$ from our class of test functions defined in (15). The parameters are chosen as follows: we use r_a and $\frac{r_a+R_a}{2}$ (respectively, $\frac{r_a+R_a}{2}$ and R_a) as the small and large radius—i.e., in place of r and R—as parameters in the definition of $\tilde{\psi}$ (respectively, $\hat{\psi}$). The other parameters of the test functions T_f, μ, $\vec{v}$, M_{adv}, M_{div} are chosen to be the same for both $\tilde{\psi}$ and $\hat{\psi}$. The constants M_{adv} and M_{div} as well as $\vec{v}$ are taken over from the conditions following (11); we furthermore require $T_f + \mu < T$, where T is taken over from the conditions following (11).

Set

$$U := \bigcup_{t\in[0,T]} B_R(x_0 + tv) \times \{t\} \ .$$

As a consequence of the conditions following (11), the conditions (12), (13) are satisfied for this choice of U. Using $r_a \le R_a \le R$ we have by property T3

$$\operatorname{supp}\hat{\psi}(.,t) \subset B_{R_a - C^* M_{adv} t}(x_0 + t\vec{v}) \subset B_R(x_0 + t\vec{v})$$

and

$$(22)\qquad \operatorname{supp}\tilde{\psi}(.,t) \subset B_{\frac{r_a+R_a}{2} - C^* M_{adv} t}(x_0 + t\vec{v}) \subset B_R(x_0 + t\vec{v})$$

which together with $T_f + \mu < T$ imply $\operatorname{supp}\tilde{\psi} \subset U$, $\operatorname{supp}\hat{\psi} \subset U$. This proves condition (14) for both $\tilde{\psi}$ and $\hat{\psi}$.

It remains to check conditions (16) and (17). Recall that c_A in the conditions following (11) is still to be determined. We set

$$(23)\qquad c_A := \min\left(\frac{1}{2C^*}, \frac{\ln 2}{C^-}\right) \ .$$

As C^* and C^- depend only on d, c_A also only depends on d.

Knowing that $T_f + \mu < T$, the conditions (16) and (17) are now a consequence of the conditions following (11): We have $T_f + \mu \le T \le \frac{c_A(R-\xi)}{M_{adv}} = \frac{c_A r_a}{M_{adv}} < \frac{r_a}{C^* M_{adv}}$ by

the conditions following (11) and by (23). Thus (16) is satisfied as both

$$C^* M_{adv} \cdot (T_f + \mu) < r_a$$

and

$$C^* M_{adv} \cdot (T_f + \mu) < \frac{r_a + R_a}{2}$$

hold. Condition (17) is checked similarly: we have $T_f + \mu \leq T \leq \frac{c_A}{M_{div}}$ and therefore $T_f + \mu \leq \frac{\ln 2}{C^- M_{div}}$ by (23).

By property T4 and (22) we obtain that for $t < T_f$

$$\hat{\psi}(.,t) \geq \frac{1}{2} \text{ on } B_{\frac{r_a+R_a}{2} - C^* M_{adv} t}(x_0 + t\vec{v}) \supset \operatorname{supp} \tilde{\psi}(.,t) \ .$$

Using this assertion as well as $a^{\frac{3}{2}}(v) \leq Cv^3 \leq CA(v)$ (see Definition 2.3 and Lemma 2.4) and (19), replacing ψ by $\tilde{\psi}$ we obtain from (21) for any $\tilde{T} < T_f$

$$\begin{aligned}
&\int_0^{\tilde{T}} \int_{\mathbb{R}^d} a(u)^2 \tilde{\psi}^2 dxdt \\
&\leq C\tilde{T}^{1-\theta} \left(\int_0^{\tilde{T}} \int_{\mathbb{R}^d} \left| \nabla (a(u)\tilde{\psi}) \right|^2 dxdt \right)^{\theta} \cdot \sup_{t \in [0,\tilde{T}]} \left(\int_{\mathbb{R}^d} a(u)^{\frac{3}{2}} \hat{\psi}^2 dx \right)^{\frac{4(1-\theta)}{3}} \\
&\leq C\tilde{T}^{1-\theta} \left(\int_0^{\tilde{T}} \int_{\mathbb{R}^d} \left| \nabla (a(u)\tilde{\psi}) \right|^2 dxdt \right)^{\theta} \cdot \sup_{t \in [0,\tilde{T}]} \left(\int_{\mathbb{R}^d} A(u) \hat{\psi}^2 dx \right)^{\frac{4(1-\theta)}{3}} \\
&\leq C\tilde{T}^{1-\theta} \left(\epsilon^{-1} \int_{\mathbb{R}^d} A(u(x,0)) \tilde{\psi}^2(x,0) dx + C \int_0^{\tilde{T}} \int_{\mathbb{R}^d} a(u)^2 |\nabla \tilde{\psi}|^2 dx \right. \\
&\qquad \left. + C \int_0^{\tilde{T}} \int_{\mathbb{R}^d} |\nabla \gamma| a(u)^2 |\nabla \tilde{\psi}| \tilde{\psi} dxdt \right)^{\theta} \\
&\quad \cdot \left(\int_{\mathbb{R}^d} A(u(x,0)) \hat{\psi}^2(x,0) dx + C\epsilon \int_0^{\tilde{T}} \int_{\mathbb{R}^d} a(u)^2 |\nabla \hat{\psi}|^2 dxdt \right. \\
&\qquad \left. + C \int_0^{\tilde{T}} \int_{\mathbb{R}^d} |\nabla \gamma| a(u)^2 |\nabla \hat{\psi}| \hat{\psi} dx \right)^{\frac{4(1-\theta)}{3}} .
\end{aligned}$$

We deduce by the properties $T3$, $T4$, and $T6$ of our test functions

$$\begin{aligned}
&\int_0^{\tilde{T}} \int_{B_{r_a - C^* M_{adv} t}(x_0 + tv)} a(u)^2 dxdt \\
&\leq \frac{C\tilde{T}^{1-\theta} \epsilon^{\frac{4-4\theta}{3}}}{(R_a - r_a)^{\frac{8-2\theta}{3}}} \left(\frac{(R_a - r_a)^2}{\epsilon} \int_{B_{R_a}(x_0)} A(u(x,0)) dx \right. \\
&\qquad + \int_0^{\tilde{T}} \int_{B_{R_a - C^* M_{adv} t}(x_0 + tv)} a(u)^2 dxdt \\
&\qquad \left. + \frac{M_{adv} + |v|}{\epsilon} (R_a - r_a) \int_0^{\tilde{T}} \int_{B_{R_a - C^* M_{adv} t}(x_0 + tv)} a(u)^2 dx \right)^{\frac{4-\theta}{3}}
\end{aligned} \tag{24}$$

since $|\nabla\gamma| \le |v| + M_{adv}$ (see (12)). We have $R_a \le R \le \frac{\epsilon}{M_{adv}+|v|}$ (see the conditions following (11)); thus

$$\begin{aligned}\int_0^{\tilde T}\int_{B_{r_a - C^* M_{adv} t}(x_0+tv)} a(u)^2 dxdt \\ \le \frac{C\tilde T^{1-\theta}\epsilon^{\frac{4-4\theta}{3}}}{(R_a-r_a)^{\frac{8-2\theta}{3}}}\Bigg(\frac{(R_a-r_a)^2}{\epsilon}\int_{B_{R_a}(x_0)} A(u(x,0))dx \\ + 2\int_0^{\tilde T}\int_{B_{R_a - C^* M_{adv} t}(x_0+tv)} a(u)^2 dxdt\Bigg)^{\frac{4-\theta}{3}}.\end{aligned}$$

Letting $T_f \uparrow T$, this implies by (9) and (10) as well as $R' \le r_a$

$$\begin{aligned}G_T(R-r_a) \le \frac{CT^{1-\theta}\epsilon^{\frac{4-4\theta}{3}}}{(R_a-r_a)^{\frac{8-2\theta}{3}}} \\ \cdot\Bigg(\epsilon^{-1}(R_a-R')^{8+d}\cdot(R_a-R')^{-6-d}\int_{B_{R_a}(x_0)} A(u(.,0))\ dx \\ + G_T(R-R_a)\Bigg)^{\frac{4-\theta}{3}} \\ \le \frac{CT^{1-\theta}\epsilon^{\frac{4-4\theta}{3}}}{(R_a-r_a)^{\frac{8-2\theta}{3}}}\cdot\left(\epsilon^{-1}(R_a-R')^{8+d}S(R) + G_T(R-R_a)\right)^{\frac{4-\theta}{3}}\end{aligned}$$

from which inequality (11) follows by noting that $\frac{8-2\theta}{1-\theta} = 8+d$, recalling that $r_a = R-\xi$, $R_a = R-\eta$, and setting $C_A := C$. This finishes the proof of Lemma 2.10.

Lemma 2.11 is proven in a similar way; using inequality (20) in place of (19) and performing the same steps, we obtain

$$\begin{aligned}\int_0^{T}\int_{B_{r - C^* M_{adv} t}(x_0)} a(u)^2 dxdt \\ \le \frac{CT^{1-\theta}\epsilon^{\frac{4-4\theta}{3}}}{(R-r)^{\frac{8-2\theta}{3}}}\Bigg(\frac{(R-r)^2}{\epsilon}\int_{B_{R}(x_0)} A(u(x,0))dx \\ + \int_0^{T}\int_{B_{R - C^* M_{adv} t}(x_0)} a(u)^2 dxdt\Bigg)^{\frac{4-\theta}{3}}\end{aligned} \tag{25}$$

without any additional restriction on R of the form $R \le \frac{\epsilon}{M_{adv}+|v|}$. We then follow the lines of the proof of Lemma 2.10.

2.5. Proof of the main results. In order to cast the results from the previous sections into a more usable form, we need to take a closer look at the advection velocity $\nabla\gamma$. γ is uniquely determined by the equation $-\Delta\gamma + \gamma = u$. This is a linear elliptic equation with constant coefficients and thus can be solved by taking the convolution of u with the corresponding fundamental solution. Recall that the fundamental solution is defined to be the distribution solving $-\Delta e + e = \delta$ with δ

denoting Dirac's distribution. In this case, e is given by the Bessel potential

$$\mathcal{B}(x) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^\infty \frac{e^{-t-\frac{|x|^2}{4t}}}{t^{\frac{d}{2}}} dt$$

(see [6]). The following two estimates on the Bessel potential are well known and can be derived using the above representation formula for the Bessel potential by differentiating under the integral sign and using standard estimates.

LEMMA 2.15. *We have* $|\nabla \mathcal{B}(x)| \leq C|x|^{-d+1} e^{-\frac{|x|}{8}}$.

LEMMA 2.16. *We have* $|\mathcal{B}(x)| \leq C|x|^{2-d} e^{-\frac{|x|}{8}}$ *in the case* $d > 2$*; in the case* $d = 2$*, we obtain* $|\mathcal{B}(x)| \leq C|1+|\log|x||| e^{-\frac{|x|}{8}}$*. In the case* $d = 1$*, it holds that* $|\mathcal{B}(x)| \leq Ce^{-\frac{|x|}{8}}$.

Notice that for $1 \leq p < \frac{d}{d-2}$ we have $\mathcal{B} \in L^p$; for $1 \leq p < \frac{d}{d-1}$, it holds that $\nabla \mathcal{B} \in L^p$.

We now show continuity of the gradient of the chemoattractant concentration. Note that this continuity property can also be derived using standard techniques from regularity theory.

LEMMA 2.17. *We have* $\nabla \gamma \in C^0(I; C^0)$.

Proof. We know $u \in C^0(I; L^2(K))$ for all bounded K. Since $u \in L^\infty(I; L^\infty)$, we have $u \in C^0(I; L^p(B_R))$ for any $p > 2$ since convergence in L^q and boundedness in L^∞ imply convergence in any L^p with $q \leq p < \infty$.

Using $\gamma(x,t) = (\mathcal{B}(.) * u(.,t))(x)$, choose $1 < \beta$ small enough such that $\mathcal{B}, \nabla \mathcal{B} \in L^\beta$, and $q \geq 2$, where $\frac{1}{\beta} + \frac{1}{q} = 1 + \frac{1}{\infty}$. Splitting u into two parts, one supported in B_R and one supported in $\mathbb{R}^d \setminus B_R$, and applying Young's inequality for convolutions, we have

$$\begin{aligned} |\gamma(x,t) - \gamma(x,t')| &= \int \mathcal{B}(y)(u(x-y,t) - u(x-y,t'))\ dy \\ &\leq ||\mathcal{B}(.)||_{L^\beta(B_R)} ||u(.,t) - u(.,t')||_{L^q(B_R(x))} \\ &\quad + ||\mathcal{B}(.)||_{L^1(\mathbb{R}^d \setminus B_R)} \left(||u(.,t)||_{L^\infty} + ||u(.,t')||_{L^\infty}\right) \end{aligned}$$

and since $\mathcal{B} \in L^1$, first choosing R large and then $|t' - t|$ small, we see $|\gamma(x,t) - \gamma(x,t')| \to 0$ as $t' \to t$ uniformly in $x \in K$ and $t \in \bar{I}$, the latter since $\bar{I}$ is compact.

It remains to show $|\gamma(x,t) - \gamma(x',t)| \to 0$ as $x' \to x$ uniformly in $x \in K$ and $t \in \bar{I}$. We compute for $|x - x'| \leq R$ by writing $\mathcal{B}(x) = \mathcal{B}(x)\chi_{|x| \leq 2R} + \mathcal{B}(x)\chi_{|x|>2R}$

$$\begin{aligned} &|\gamma(x,t) - \gamma(x',t)| \\ &\quad \leq ||\mathcal{B}(.)||_{L^\beta(B_{2R}(0))} ||u(.+x,t) - u(.+x',t)||_{L^q(B_{2R}(0))} \\ &\qquad + ||\mathcal{B}(.)||_{L^1(\mathbb{R}^d \setminus B_{2R}(x))} \cdot 2||u||_{L^\infty}\ . \end{aligned}$$

The second term can be made arbitrarily small by choosing R large. The result follows since we have (recall $q \geq 2$)

$$\begin{aligned} &||u(.+x,t) - u(.+x',t)||_{L^q(B_{2R}(x))} \\ &\quad = \left(\int_{B_{2R}(x)} |u(.+x,t) - u(.+x',t)|^q\ dx \right)^{\frac{1}{q}} \\ &\quad \leq C||u||_{L^\infty}^{\frac{q-2}{q}} ||u(.+x,t) - u(.+x',t)||_{L^2(B_{2R}(x))}^{\frac{2}{q}} \end{aligned}$$

and since the set $B := \{v \in L^2(B_{3R}(x)) : v(.) = u(.,t) \text{ for some } t \in I\}$ is compact in L^2 because it is the image of the continuous mapping $[0,1] \to L^2(B_{3R}(x))$, $t \mapsto u(.,t)$. It thus follows that $\lim_{y\to 0} \sup_{v\in B} ||v(y+.) - v(.)||_{L^2(B_{2R}(x))} = 0$ uniformly in $v \in B$ (see, e.g., [24]).

A similar calculation for $\nabla\gamma$ yields the corresponding result for $\nabla\gamma$; here the representation $\nabla\gamma = \nabla\mathcal{B} * u$ is used and the estimates for $\nabla\mathcal{B}$ (namely, $\nabla\mathcal{B} \in L^\beta$ and $\nabla\mathcal{B} \in L^1$) are employed. ∎

LEMMA 2.18. *We have $\gamma \in L^\infty(I; L^\infty)$ and $\nabla\gamma \in L^\infty(I; L^\infty)$ as well as $\Delta\gamma \in L^\infty(I; L^\infty)$, with some uniform bound depending only on the spatial dimension d.*

Proof. The first two estimates are an immediate consequence of $\gamma = \mathcal{B} * u$ and Young's inequality; the last estimate follows since $\gamma \in L^2(I; H^2(K))$ for any $K \subset\subset \mathbb{R}^d$ by elliptic regularity and since $\Delta\gamma = \gamma - u \in L^\infty$. ∎

We are now ready to show a result on finite propagation.

LEMMA 2.19. *Set $S := \int_{\mathbb{R}^d} A(u_0(x))\, dx$. Fix $x_0 \in \mathbb{R}^d$, $x_0 \notin \operatorname{supp} u_0$. Assume $R \le \operatorname{dist}(x_0, \operatorname{supp} u_0)$. We then have $u(x,t) = 0$ for all $x \in B_{\frac{R}{2}}(x_0)$ and all*

$$t \le \min\left\{ \frac{cR^{\frac{d+6}{3}}}{\epsilon S^{\frac{1}{3}}}, \frac{c}{\max(\sup(-\Delta\gamma), 0)}, \frac{cR}{\sup|\nabla\gamma|} \right\}$$

for some constant $c > 0$ depending only on d. The second term is to be read as ∞ in the case $\sup -\Delta\gamma \le 0$, which corresponds to a noncompressive advection velocity field.

Proof. The assertion follows directly from Corollary 2.12: we set $R' := \frac{2R}{3}$ and use the upper bounds for $|\nabla\gamma|$ and $-\Delta\gamma$ from the previous lemma as M_{adv} and M_{div}. The estimate

$$\begin{aligned} S(R) &= \inf_{r\in(\frac{2}{3}R,\infty)} \left(r - \frac{2}{3}R\right)^{\frac{d+6}{3}} \left(\int_{B_r(x_0)} A(u_0(.))\right)^{-\frac{1}{3}} \\ &= \inf_{r\in(R,\infty)} \left(r - \frac{2}{3}R\right)^{\frac{d+6}{3}} \left(\int_{B_r(x_0)} A(u_0(.))\right)^{-\frac{1}{3}} \ge cR^{\frac{d+6}{3}} \left(\int_{\mathbb{R}^d} A(u_0(.))\right)^{-\frac{1}{3}} \end{aligned}$$

holds because $\int_{B_r(x_0)} A(u_0(.)) = 0$ for $r < R$. We therefore obtain $\operatorname{supp} u(.,t) \cap B_{\frac{2}{3}R - C^* M_{adv} t}(x_0) = \emptyset$ for all t subject to the assumptions of the lemma. If additionally $t \le \frac{R}{6C^* M_{adv}}$ holds, i.e., $t \le \frac{cR}{M_{adv}}$ for c a constant which is chosen small enough (depending only on d), we therefore obtain $\operatorname{supp} u(.,t) \cap B_{\frac{1}{2}R}(x_0) = \emptyset$. ∎

THEOREM 2.20. *Let u, γ be a weak solution of* (1), (2) *with initial data $u_0 \in L^1$, where $0 \le u_0 \le 1$.*

Suppose

$$\operatorname{supp} u_0 \cap B_{R_1}(x_0) = \emptyset\ . \tag{26}$$

Then there exists a constant $c > 0$ depending only on d such that

$$\operatorname{supp} u(.,t_1) \cap B_{\frac{R_1}{2}}(x_0) = \emptyset$$

for any

$$t_1 \le \min\left(\frac{cR_1^{\frac{d+6}{3}}}{\epsilon ||u_0||_{L^1}^{\frac{1}{3}}}, \frac{cR_1}{\epsilon^{\frac{3}{d+6}} |\max(\sup(-\Delta\gamma), 0)|^{\frac{d+3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}}}, \frac{cR_1}{\sup|\nabla\gamma|} \right)\ .$$

Proof. For

$$\min\left(\frac{cR_1^{\frac{d+6}{3}}}{\epsilon||u_0||_{L^1}^{\frac{1}{3}}}, \frac{cR_1}{\epsilon^{\frac{3}{d+6}}|\max(\sup(-\Delta\gamma),0)|^{\frac{d+3}{d+6}}||u_0||_{L^1}^{\frac{1}{d+6}}}, \frac{cR_1}{\sup|\nabla\gamma|}\right) \leq \frac{c}{\max(\sup(-\Delta\gamma),0)}$$

the assertion is immediate from the previous lemma. Note that for all $t \geq 0$, we have $S = \int A(u(.,t))\ dx \leq \int u(.,t)\ dx = ||u_0||_{L^1}$.

If $\frac{c}{\max(\sup(-\Delta\gamma),0)}$ is smaller, we iterate the estimate from the previous lemma, as the previous lemma only allows advancing in time by $\frac{c}{\max(\sup(-\Delta\gamma),0)}$. To this aim, choose $\tilde{R}$ so as to satisfy

$$\tilde{R}^{\frac{d+6}{3}} = \frac{c\epsilon||u_0||_{L^1}^{\frac{1}{3}}}{\max(\sup(-\Delta\gamma),0)} \tag{27}$$

and set

$$\tilde{t} = \min\left(\frac{c}{\max(\sup(-\Delta\gamma),0)}, \frac{c\tilde{R}}{\sup|\nabla\gamma|}\right) . \tag{28}$$

Note that $\tilde{R}$ and $\tilde{t}$ have been chosen in such a way that

$$\frac{c\tilde{R}^{\frac{d+6}{3}}}{\epsilon||u_0||_{L^1}^{\frac{1}{3}}} = \frac{c}{\max(\sup(-\Delta\gamma),0)}$$

and therefore

$$\tilde{t} = \min\left(\frac{c\tilde{R}^{\frac{d+6}{3}}}{\epsilon||u_0||_{L^1}^{\frac{1}{3}}}, \frac{c}{\max(\sup(-\Delta\gamma),0)}, \frac{c\tilde{R}}{\sup|\nabla\gamma|}\right) .$$

Note also that $\tilde{R} \leq R_1$. We now apply the previous lemma to *all* balls of the form $B_{\tilde{R}}(x)$ with $x \in \mathbb{R}^d$ subject to the condition $B_{\tilde{R}}(x) \cap \operatorname{supp} u_0 = \emptyset$. The parameters in the previous lemma are chosen as follows: we set $R := \tilde{R}$, $t := \tilde{t}$, and use x in place of x_0. The lemma now yields $\operatorname{supp} u(.,\tilde{t}) \cap B_{\frac{\tilde{R}}{2}}(x) = \emptyset$, due to our choice of $\tilde{t}$ and due to the estimate $S = \int A(u_0(.)) \leq ||u_0||_{L^1}$. Note that $\tilde{t}$ and $\tilde{R}$ are independent of x.

Thus, if initially $\operatorname{supp} u_0 \cap B_{R_1}(x_0) = \emptyset$, we obtain $\operatorname{supp} u(.,\tilde{t}) \cap B_{R_1-\frac{\tilde{R}}{2}}(x_0) = \emptyset$, since for any $x \in B_{R_1-\tilde{R}}(x_0)$ we have $B_{\tilde{R}}(x) \cap \operatorname{supp} u_0 = \emptyset$ and, therefore, for any $x \in B_{R_1-\tilde{R}}(x_0)$ application of the lemma was possible, yielding $\operatorname{supp} u(.,\tilde{t}) \cap B_{\frac{\tilde{R}}{2}}(x) = \emptyset$ for any $x \in B_{R_1-\tilde{R}}(x_0)$.

We intend to iterate this procedure, repeating the argument starting at times $\tilde{t}$, $2\tilde{t}, \ldots$ (and regarding the solution u as a solution with initial data $u(.,\tilde{t})$, $u(.,2\tilde{t}), \ldots$). For the kth step of the iteration, we assume that

$$B_{R_1-\frac{(k-1)\tilde{R}}{2}}(x_0) \cap \operatorname{supp} u(.,(k-1)\tilde{t}) = \emptyset . \tag{29}$$

We then show that this implies by the previous lemma that

$$B_{R_1-\frac{k\tilde{R}}{2}}(x_0) \cap \operatorname{supp} u(.,k\tilde{t}) = \emptyset .$$

Note that for $k = 1$, the assumption (29) required for the step is precisely the assumption (26) of our theorem; moreover, for $k > 1$ the result of step $k-1$ yields exactly the assumption (29) necessary for step k.

In the kth step, we apply the previous lemma to all balls of the form $B_{\tilde{R}}(x)$ with $x \in B_{R-\frac{(k+1)\tilde{R}}{2}}(x_0)$; the parameters of the previous lemma are chosen as follows: we use $u(.,(k-1)\tilde{t})$ in place of u_0 and x in place of x_0 and set $R := \tilde{R}$ as well as $t := \tilde{t}$. If applicable the previous lemma implies $\operatorname{supp} u(.,k\tilde{t}) \cap B_{\frac{\tilde{R}}{2}}(x) = \emptyset$ for any choice of $x \in B_{R-\frac{(k+1)\tilde{R}}{2}}(x_0)$; it is then immediate that $B_{R-\frac{k\tilde{R}}{2}}(x_0) \cap \operatorname{supp} u(.,k\tilde{t}) = \emptyset$.

It remains to verify the conditions of the previous lemma. As by assumption (29) we have $B_{R_1-\frac{(k-1)\tilde{R}}{2}}(x_0) \cap \operatorname{supp} u(.,(k-1)\tilde{t}) = \emptyset$, for any $x \in B_{R-\frac{(k+1)\tilde{R}}{2}}(x_0)$ it holds that $\operatorname{supp} u(.,(k-1)\tilde{t}) \cap B_{\tilde{R}}(x) = \emptyset$. Thus, the condition on the support of the initial data in the previous lemma is satisfied; it remains to verify the conditions on $\tilde{t}$. By our choice of $\tilde{R}$ (see (27)), application of the previous lemma with $t = \tilde{t}$ was in fact possible since $||u(.,(k-1)\tilde{t})||_{L^1} = ||u_0||_{L^1}$ which implies

$$\begin{aligned}\tilde{t} &= \min\left(\frac{c\tilde{R}^{\frac{d+6}{3}}}{\epsilon||u(.,(k-1)\tilde{t})||_{L^1}^{\frac{1}{3}}}, \frac{c}{\max(\sup(-\Delta\gamma),0)}, \frac{c\tilde{R}}{\sup|\nabla\gamma|}\right)\\ &\le \min\left(\frac{c\tilde{R}^{\frac{d+6}{3}}}{\epsilon S^{\frac{1}{3}}}, \frac{c}{\max(\sup(-\Delta\gamma),0)}, \frac{c\tilde{R}}{\sup|\nabla\gamma|}\right)\end{aligned}$$

by $S = \int A(u(.,(k-1)\tilde{t}))\,dx \le ||u(.,(k-1)\tilde{t})||_{L^1}$. Thus our induction step is complete.

We can perform $K := \lfloor \frac{\frac{1}{2}R_1}{\frac{1}{2}\tilde{R}} \rfloor$ iteration steps: after K steps, we still know that $\operatorname{supp} u(.,K\tilde{t}) \cap B_{R_1-K\frac{\tilde{R}}{2}}(x_0) = \emptyset$ holds which implies by our choice of K that $\operatorname{supp} u(.,K\tilde{t}) \cap B_{\frac{R_1}{2}}(x_0) = \emptyset$.

We have by our choice of $\tilde{R}$ (see (27))

$$\left\lfloor \frac{\frac{1}{2}R_1}{\frac{1}{2}\tilde{R}} \right\rfloor \ge \frac{cR_1 \max(\sup(-\Delta\gamma),0)^{\frac{3}{d+6}}}{\left(\epsilon||u_0||_{L^1}^{\frac{1}{3}}\right)^{\frac{3}{d+6}}}\ .$$

Each iteration allows advancing in time by one time step $\tilde{t}$. Recalling (28), we see that $\operatorname{supp}(u(.,t_1)) \cap B_{\frac{1}{2}R_1}(x_0) = \emptyset$ for all

$$\begin{aligned}t_1 &\le \tilde{t}\cdot\left\lfloor \frac{\frac{1}{2}R_1}{\frac{1}{2}\tilde{R}} \right\rfloor\\ &\ge \min\left(\frac{c}{\max(\sup(-\Delta\gamma),0)}, \frac{c\tilde{R}}{\sup|\nabla\gamma|}\right)\cdot\frac{cR_1\max(\sup(-\Delta\gamma),0)^{\frac{3}{d+6}}}{\left(\epsilon||u_0||^{\frac{1}{3}}\right)^{\frac{3}{d+6}}}\\ &= \min\left(\frac{cR_1}{\epsilon^{\frac{3}{d+6}}\max(\sup(-\Delta\gamma),0)^{\frac{d+3}{d+6}}||u_0||_{L^1}^{\frac{1}{d+6}}}, \frac{cR_1}{\sup|\nabla\gamma|}\right)\ .\end{aligned}$$

This finishes the proof of our assertion. ∎

COROLLARY 2.21. *Let u, γ be a weak solution of* (1), (2) *with initial data $u_0 \in L^1$, where $0 \le u_0 \le 1$.*

Suppose $\operatorname{supp} u_0 \subset B_{R_0}(x_1)$. *Then there exists a constant* $C > 0$ *depending only on* d *such that* $\operatorname{supp} u(.,t) \subset B_{R(t)}(x_1)$ *for*

$$R(t) := R_0 + C\epsilon^{\frac{3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}} t^{\frac{3}{d+6}} + C\left(\sup|\nabla\gamma| + \epsilon^{\frac{3}{d+6}} |\max(\sup(-\Delta\gamma),0)|^{\frac{d+3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}}\right) t \ .$$

Proof. This corollary is derived applying the previous theorem to all balls of the form $B_{|x_0-x_1|-R_0}(x_0)$, where $x_0 \notin B_{R_0}(x_1)$ is arbitrary. Note that $\operatorname{supp} u_0 \cap B_{|x_0-x_1|-R_0}(x_0) = \emptyset$ for any such ball; thus, the previous theorem applies with x_0 and $R_1 := |x_0 - x_1| - R_0$, giving $\operatorname{supp} u(.,t) \cap B_{\frac{|x_0-x_1|-R_0}{2}}(x_0) = \emptyset$ for any $t \le t_1 = t_1(x_0)$. This especially implies $x_0 \notin \operatorname{supp} u(.,t)$ for $t \le t_1(x_0)$, where $t_1(x_0)$ is given by

$$t_1 = \min\left(\frac{c(|x_0-x_1|-R_0)^{\frac{d+6}{3}}}{\epsilon ||u_0||_{L^1}^{\frac{1}{3}}}, \frac{c(|x_0-x_1|-R_0)}{\epsilon^{\frac{3}{d+6}} |\max(\sup(-\Delta\gamma),0)|^{\frac{d+3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}}}, \frac{c(|x_0-x_1|-R_0)}{\sup|\nabla\gamma|}\right) .$$

Solving for $|x_0 - x_1| - R_0$ we see that we have $x_0 \notin \operatorname{supp} u(.,t)$ for all t subject to the condition

$$|x_0-x_1| - R_0 \ge C \max\left[\epsilon^{\frac{3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}} t^{\frac{3}{d+6}}, t\sup|\nabla\gamma|,\ \epsilon^{\frac{3}{d+6}} |\max(\sup(-\Delta\gamma),0)|^{\frac{d+3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}} t\right] .$$

We therefore have $\operatorname{supp} u(.,t) \subset B_{R(t)}(x_1)$ for

$$R(t) = R_0 + C\max\left[\epsilon^{\frac{3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}} t^{\frac{3}{d+6}}, \left(\sup|\nabla\gamma| + \epsilon^{\frac{3}{d+6}} |\max(\sup(-\Delta\gamma),0)|^{\frac{d+3}{d+6}} ||u_0||_{L^1}^{\frac{1}{d+6}}\right) t\right] .$$

This implies the corollary. ∎

We see that at some point a transition takes place between the short-time behavior of the front which is like $t^{\frac{3}{d+6}}$ and the large-time behavior which is like t. The occurrence of two different propagation rates is due to the different scaling behavior of the degenerate diffusion term and the advection term.

Remark 2.22. The quantitative bound on support propagation given in the previous theorem is presumably not optimal for large times. Our results do not depend on the particular structure of the advection velocity; only boundedness of the advection velocity field and boundedness of its divergence are required for our result. Taking into account the particular structure of the advection term in the chemotaxis model, one may hope to obtain improved bounds on asymptotic support propagation rates. This amounts to deriving a better estimate for the terms involving $\nabla\gamma \cdot \nabla\psi$ in formula (7). However this is beyond the scope of this paper.

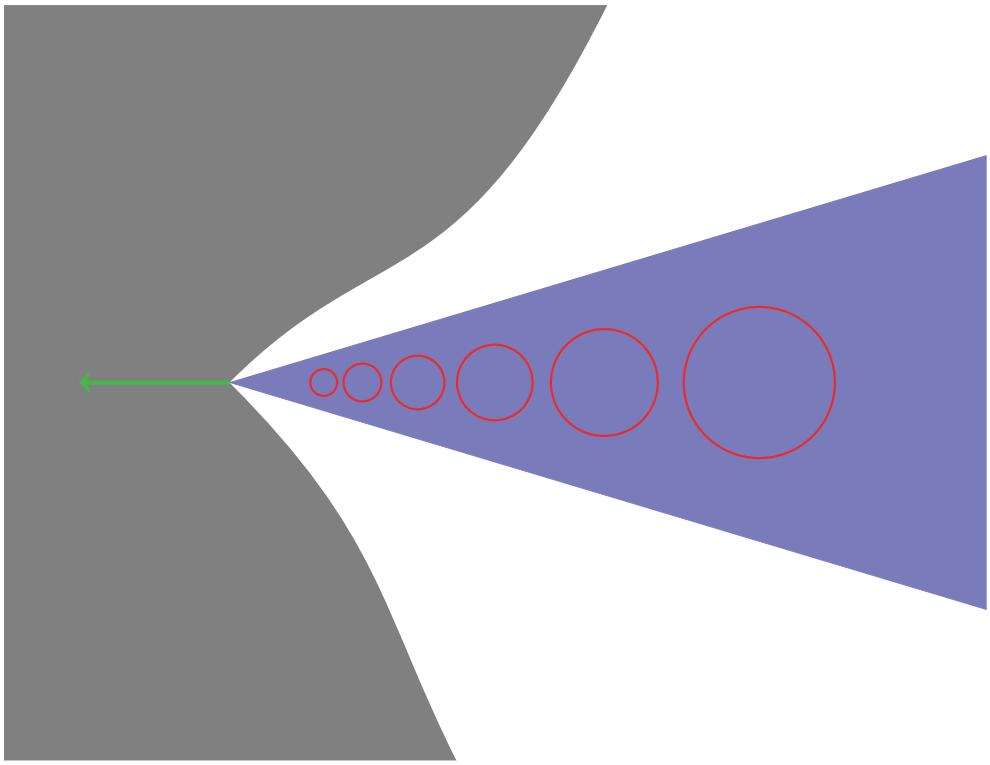


FIG. 1. *The situation of Theorem* 2.23: *We have a cone (blue) with axis* $-\nabla\gamma(x_0, 0)$ *and apex* x_0 *which only touches the support of* u_0 *(gray); therefore for any point* x *on the axis of the cone there exists a ball centered on it with radius proportional to* $|x - x_0|$ *which has an empty intersection with the support of* u_0. *Several such balls are depicted (red). The vector* $\nabla\gamma(x_0, 0)$ *is drawn in green and originates at* x_0.

A sufficient criterion for the appearance of a waiting time and even support shrinking can also be derived from Lemma 2.10.

THEOREM 2.23. *Let* u, γ *be a weak solution of* (1), (2) *with initial data* $u_0 \in L^1$, *where* $0 \le u_0 \le 1$.

Given a point $x_0 \in \partial \operatorname{supp} u_0$ *with* $\nabla\gamma(x_0, 0) \neq 0$, *assume that there exists a neighborhood* U *of* x_0 *and a cone* B *with vertex* x_0 *and axis* $-\nabla\gamma(x_0, 0)$ *such that* $(U \cap B) \cap \operatorname{supp} u_0 = \emptyset$. *Suppose in addition*

$$\lim_{r\to 0} r^{-3} \fint_{B_r(x_0)} A(u_0(x))dx = 0 \ . \tag{30}$$

Then there exist $T^* > 0$ *and* $c > 0$ *such that for any* $0 < t \le T^*$ *we have* $\operatorname{supp} u(., t) \cap B_{ct}(x_0) = \emptyset$.

The situation of Theorem 2.23 is illustrated in Figure 1.

Proof. Note that the conditions of the theorem imply that there exists a constant $\alpha > 0$ and a neighborhood $\tilde{U}$ of x_0 such that for all points $x \in \tilde{U} \cap \{x_0 - \tau\nabla\gamma(x_0, 0) : \tau > 0\}$ we have

$$B_{2\alpha|x_0 - x|}(x) \cap \operatorname{supp} u_0 = \emptyset \ . \tag{31}$$

The value of $\alpha > 0$ depends on the opening of the cone and tends to zero as the opening of the cone tends to zero; α cannot exceed $\frac{1}{2}$.

Setting $v := \nabla\gamma(x_0, 0)$, the general idea is that for small $\tau > 0$ the balls $B_{2\alpha|(x_0-\tau v)-x_0|}(x_0 - \tau v)$ have empty intersection with $\operatorname{supp} u_0$ and near x_0 and for small times the function u gets advected approximately with speed $v = \nabla\gamma(x_0, 0)$; therefore at time τ we may hope to obtain a (smaller) ball around x_0 which has empty intersection with $\operatorname{supp} u(., \tau)$.

Fix x_0 and α; from now on, α may be treated as a constant. Choose $c_0 > 0$ small such that

$$c_0 \leq \min\left(c_S, \frac{1}{2C^*}\right) \alpha|\nabla\gamma(x_0, 0)| \tag{32}$$

holds true, where the constants c_S, C^* are the constants in Lemma 2.10. Select $\delta > 0$ such that for $|x - x_0| < \delta$ and $|t| < \delta$ we have

$$|\nabla\gamma(x, t) - \nabla\gamma(x_0, 0)| < c_0 \ . \tag{33}$$

This is possible by Lemma 2.17. Decreasing δ if necessary, we can ensure that $\delta \leq \frac{\epsilon}{c_0+|v|}$.

For the reader's convenience, we recapitulate Lemma 2.10.

LEMMA 2.10. *Suppose that there exist positive constants M_{adv}, M_{div}, R', R, s, and a constant vector $v \in \mathbb{R}^d$ such that the following conditions are satisfied:*

(H1) $0 < R' < R \leq \frac{\epsilon}{M_{adv}+|v|}$;

(H2) $|\nabla\gamma - v| \leq M_{adv}$ *for all* (x, t) *such that* $|x - x_0| \leq R + |v|s$ *and* $t \leq s$;

(H3) $\Delta\gamma \geq -M_{div}$ *for all* (x, t) *such that* $|x - x_0| \leq R + |v|s$ *and* $t \leq s$;

(H4) $S(R) < \infty$.

Then there exist positive universal constants c_S, C^* (i.e., constants depending only on d) such that the following holds:

$$u(., t) \equiv 0 \text{ on } B_{R'-C^*M_{adv}t}(x_0 + tv)$$
$$\text{for any}$$
$$t \in \left[0, \min\left(s, \frac{c_S}{M_{div}}, \frac{c_S R'}{M_{adv}}, c_S\epsilon^{-\frac{4}{3}}\left(G_s(0)(R - R')^{-d-8} + \epsilon^{-1}S(R)\right)^{-\frac{1}{3}}\right)\right].$$

For $\tau > 0$ with $\tau < \tau_0$ (where τ_0 will be specified in (34) below), set $x := x_0 - \tau\nabla\gamma(x_0, 0)$ and let $r := |x - x_0|$. We now seek to apply Lemma 2.10 to balls with center x and $R := \frac{\delta}{2}$, $R' := \alpha r$, $M_{adv} := c_0$. Note that x and r depend on τ.

For M_{div} we use the universal upper bound on $-\Delta\gamma$ from Lemma 2.18. Condition (H3) is immediate by our choice of M_{div} (as this bound holds globally, it does not matter that we have not yet chosen s).

Set

$$\tau_0 := \min\left(\delta, \frac{c_S}{M_{div}}, \frac{\delta}{4|v|}, \frac{\delta}{4\alpha|v|}\right) \ . \tag{34}$$

Since we have chosen δ so as to satisfy $\delta \leq \frac{\epsilon}{c_0+|v|}$, using $M_{adv} = c_0$, $R = \frac{\delta}{2}$, $R' = \alpha r = \alpha\tau|v|$, and $\tau \leq \tau_0 \leq \frac{\delta}{4\alpha|v|}$ condition (H1) is seen to hold.

We set s in Lemma 2.10 to equal τ_0.

Recall that we have $\vec{v} = \nabla\gamma(x_0, 0)$ and $M_{adv} = c_0$. Our choice of τ_0 in (34) ensures that $|v|s = |v|\tau_0 \leq \frac{\delta}{4}$ which implies $B_R(x_0 + tv) \subset B_{\frac{3\delta}{4}}(x_0)$ for $t \leq s$. Thus by (33) we see that condition (H2) is satisfied.

It remains to check (H4). This is straightforward using (10) and noting that $B_{2R'}(x) \cap \operatorname{supp} u_0 = \emptyset$ and $\int A(u_0(.))\, dx < \infty$.

We would like to set $t = \tau$ in Lemma 2.10: this being possible, we would obtain $u(.,\tau) \equiv 0$ on $B_{\alpha|\nabla\gamma(x_0,0)|\tau - C^* c_0 \tau}(x + \vec{v}\tau)$ and by our choice of c_0 (see (32)) and our definition of x and $\vec{v}$ (see just below the recapitulation of Lemma 2.10) we would have

$$B_{\frac{1}{2}\alpha|\nabla\gamma(x_0,0)|\tau}(x_0) \subset B_{\alpha|\nabla\gamma(x_0,0)|\tau - C^* c_0 \tau}(x + \tau v)\ .$$

It remains to show that it is possible to set $t = \tau$ in Lemma 2.10 for any $\tau \leq \min(\tau_0, \tau_1)$, where $\tau_1 > 0$ is a constant depending on the solution u and on x_0 and α which is to be selected below (see (40) below); then the assertion of the lemma is proven for $T^* := \min(\tau_0, \tau_1)$.

By our choice of c_0 (see (32)), x, r, M_{adv}, and δ (see (33)) the inequality $\tau \leq \frac{c_S \alpha \tau |\nabla\gamma(x_0,0)|}{c_0} = \frac{c_S \alpha r}{c_0} = \frac{c_S \alpha r}{M_{adv}} = \frac{c_S R'}{M_{adv}}$ is satisfied.

Additionally we have $\tau \leq \frac{c_S}{M_{div}}$ for any $\tau \leq \tau_0$ by our choice of τ_0 (see (34)).

It remains to show that

$$\tau \leq c_S \epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)(R - R')^{-d-8} + \epsilon^{-1} S(R) \right)^{-\frac{1}{3}} \tag{35}$$

for any $\tau \leq \min(\tau_0, \tau_1)$. Recall that we have chosen $s = \tau_0$ in Lemma 2.10.

We can estimate for $T = \tau \leq \tau_0 \leq \frac{\delta}{4\alpha|v|}$

$$\begin{aligned}
&\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)(R - R')^{-d-8} + \epsilon^{-1} S(R) \right)^{-\frac{1}{3}} \\
&\quad \geq c\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)\delta^{-d-8} + \epsilon^{-1} \sup_{\tilde{r} \in (R';R]} (\tilde{r} - R')^{-6-d} \int_{B_{\tilde{r}}(x)} A(u(.,0))dx \right)^{-\frac{1}{3}},
\end{aligned}$$

where we have used that $R' = \alpha r = \alpha|v|\tau \leq \frac{\delta}{4} = \frac{R}{2}$ (see (34)). Furthermore

$$\begin{aligned}
&\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)\delta^{-d-8} + \epsilon^{-1} \sup_{\tilde{r} \in (R';R]} (\tilde{r} - R')^{-6-d} \int_{B_{\tilde{r}}(x)} A(u(.,0))dx \right)^{-\frac{1}{3}} \\
&\quad = \epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)\delta^{-d-8} + \epsilon^{-1} \sup_{\tilde{r} \in (2R';R]} (\tilde{r} - R')^{-6-d} \int_{B_{\tilde{r}}(x)} A(u(.,0))dx \right)^{-\frac{1}{3}},
\end{aligned}$$

where we have made use of the fact that $B_{2R'}(x) \cap \operatorname{supp} u_0 = \emptyset$ which holds by (31) since $R' = \alpha r = \alpha|x - x_0|$. Therefore we have the estimate

$$\begin{aligned}
&\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)(R - R')^{-d-8} + \epsilon^{-1} S(R) \right)^{-\frac{1}{3}} \\
&\quad \geq c\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)\delta^{-d-8} + \epsilon^{-1} \sup_{\tilde{r} \in (2R';R]} \tilde{r}^{-6-d} \int_{B_{\tilde{r}}(x)} A(u(.,0))dx \right)^{-\frac{1}{3}}
\end{aligned}$$

as $2R' < \tilde{r}$ implies $R' < \frac{1}{2}\tilde{r}$, i.e., $\frac{1}{2}\tilde{r} \leq (\tilde{r} - R') \leq \tilde{r}$.

Recall that $|x - x_0| = r = \frac{R'}{\alpha}$ (see the definitions just below the recapitulation of Lemma 2.10); it follows that $B_{\tilde{r}}(x) \subset B_{\frac{1+\alpha}{\alpha}\tilde{r}}(x_0)$ in case $\tilde{r} \geq R'$. Thus

$$\begin{aligned}
&\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)(R - R')^{-d-8} + \epsilon^{-1} S(R) \right)^{-\frac{1}{3}} \\
&\quad \geq c\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)\delta^{-d-8} + \epsilon^{-1} \sup_{\tilde{r} \in (2R';R]} \tilde{r}^{-6-d} \int_{B_{\frac{1+\alpha}{\alpha}\tilde{r}}(x_0)} A(u(.,0))dx \right)^{-\frac{1}{3}}.
\end{aligned}$$

We therefore obtain

(36)

$$\epsilon^{-\frac{4}{3}}\left(G_{\tau_0}(0)(R-R')^{-d-8}+\epsilon^{-1}S(R)\right)^{-\frac{1}{3}}$$
$$\geq c\epsilon^{-\frac{4}{3}}\left(G_{\tau_0}(0)\delta^{-d-8}+\epsilon^{-1}\sup_{\tilde{r}\in(2(\alpha+1)r;\frac{\alpha+1}{\alpha}R]}\tilde{r}^{-6-d}\int_{B_{\tilde{r}}(x_0)}A(u(.,0))dx\right)^{-\frac{1}{3}},$$

where we have performed a change of variables, made use of the fact $R'=\alpha r$, and have treated α as a constant.

Fix $c_2>0$ so small that

$$c_1c_2^{-\frac{1}{3}}|\nabla\gamma(x_0,0)|\geq 1 \tag{37}$$

for some constant $c_1>0$ which is to be determined and which depends only on d and on α. Choose r_1 small such that

$$\sup_{\tilde{r}\in(0;r_1]}\tilde{r}^{-3-d}\int_{B_{\tilde{r}}(x_0)}A(u(.,0))dx<c_2\ . \tag{38}$$

This is possible by our assumption (30). Now select r_2 small such that

$$\sup_{\tilde{r}\in[r_1;\frac{\alpha+1}{\alpha}R]}\tilde{r}^{-6-d}\int_{B_{\tilde{r}}(x_0)}A(u(.,0))dx<c_2r_2^{-3}\ . \tag{39}$$

We now let τ_1 be the largest constant satisfying both

$$\tau_1\leq\frac{\min(r_1,r_2)}{|v|} \tag{40}$$

and

$$G_{\tau_0}(0)\delta^{-d-6}\leq c_2\left(|v|\tau_1\right)^{-3}\ . \tag{41}$$

Note that this choice implies that for $\tau\leq\tau_1$ the inequality $r=|v|\tau\leq\min(r_1,r_2)$ is satisfied.

Let us estimate using (36)

$$\epsilon^{-\frac{4}{3}}\left(G_{\tau_0}(0)(R-R')^{-d-6}+\epsilon^{-1}S(R)\right)^{-\frac{1}{3}}$$
$$\geq c\epsilon^{-\frac{4}{3}}\Bigg(G_{\tau_0}(0)\delta^{-d-8}+\epsilon^{-1}\sup_{\tilde{r}\in(2(\alpha+1)r;r_1]}\tilde{r}^{-6-d}\int_{B_{\tilde{r}}(x_0)}A(u(.,0))dx$$
$$+\epsilon^{-1}\sup_{\tilde{r}\in[r_1;\frac{\alpha+1}{\alpha}R]}\tilde{r}^{-6-d}\int_{B_{\tilde{r}}(x_0)}A(u(.,0))dx\Bigg)^{-\frac{1}{3}}$$
$$\geq c\epsilon^{-\frac{4}{3}}\Bigg(G_{\tau_0}(0)\delta^{-d-8}+\epsilon^{-1}r^{-3}\sup_{\tilde{r}\in(2(\alpha+1)r;r_1]}\tilde{r}^{-3-d}\int_{B_{\tilde{r}}(x_0)}A(u(.,0))dx$$
$$+\epsilon^{-1}\sup_{\tilde{r}\in[r_1;\frac{\alpha+1}{\alpha}R]}\tilde{r}^{-6-d}\int_{B_{\tilde{r}}(x_0)}A(u(.,0))dx\Bigg)^{-\frac{1}{3}},$$

where again we have treated α as a constant. This yields using $r \le \min(r_1, r_2)$ (see (40)) and (38), (39)

$$\begin{aligned}
&\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)(R-R')^{-d-6} + \epsilon^{-1} S(R)\right)^{-\frac{1}{3}} \\
&\quad \ge c\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)\delta^{-d-8} + \epsilon^{-1} r^{-3} c_2 + \epsilon^{-1} r_2^{-3} c_2\right)^{-\frac{1}{3}} \\
&\quad \ge c\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)\delta^{-d-8} + \epsilon^{-1} r^{-3} c_2\right)^{-\frac{1}{3}} \ .
\end{aligned}$$

Thus, for $\tau \le \tau_1$ we obtain by (41) and $(|v|\tau_1)^{-3} \le |v|^{-3}\tau^{-3} = r^{-3}$

$$\begin{aligned}
&\epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)(R-R')^{-d-8} + \epsilon^{-1} S(R)\right)^{-\frac{1}{3}} \\
&\quad \ge c c_2^{-\frac{1}{3}} r = c c_2^{-\frac{1}{3}} |\nabla\gamma(x_0, 0)|\tau \ .
\end{aligned}$$

If we take c_1 to equal the product of the c in the last expression and c_S, by our choice of c_2 (see (37)) the condition (35)

$$\tau \le c_S \epsilon^{-\frac{4}{3}} \left(G_{\tau_0}(0)(R-R')^{-d-8} + \epsilon^{-1} S(R)\right)^{-\frac{1}{3}}$$

is satisfied. This finishes our proof. ∎

Remark 2.24. The proof of the theorem shows that the condition

$$\liminf_{r\to 0} r^{-3-d} \int A(u_0(x))\ dx \le c_2$$

for some $c_2 > 0$ small would suffice; however, c_2 depends on many quantities, including the opening of the cone in the theorem. We have therefore chosen to state the theorem in the above form.

Remark 2.25. Essentially, the theorem says that the condition

$$\lim_{r\to 0} r^{-3-d} \int_{B_r(x_0)} A(u)\ dx = 0$$

implies that locally at x_0 the diffusive spreading of the support of u initially has zero speed. Since the advective transport has locally almost constant speed (since $\nabla\gamma$ is continuous), it is the advective transport which initially dictates the local behavior of the interface $\partial \operatorname{supp} u(., t)$.

Remark 2.26. The condition in the theorem involving the cone is especially fulfilled if x_0 lies at the boundary of a C^1 manifold with boundary, $\nabla\gamma(x_0)$ points into the manifold (i.e., the dot product with the outward unit normal vector is negative), and $\operatorname{supp} u_0$ is locally contained in this manifold.

Let us use the abbreviation $\operatorname{supp}_c u_0 := \operatorname{Conv}(\operatorname{supp} u_0)$ for the convex hull of the support of u_0. By $\operatorname{supp} u_0$ we mean the closure of the set of all Lebesgue points of u_0 for which $u_0 > 0$.

LEMMA 2.27. *For $x_0 \notin (\operatorname{supp}_c u_0)^\circ$, either the half-line given by $x_0 + \lambda\nabla\gamma(x_0, 0)$, $\lambda > 0$, has nonzero intersection with $(\operatorname{supp}_c u_0)^\circ$ or $u_0 \equiv 0$. Thus by convexity of $\operatorname{supp}_c u_0$, for $x_0 \in \partial \operatorname{supp}_c u_0$ we have $x_0 + \lambda\nabla\gamma(x_0, 0) \in (\operatorname{supp}_c u_0)^\circ$ for all sufficiently small $\lambda > 0$ and all nonvanishing u_0.*

Proof. We have

$$\nabla\gamma(x_0, 0) = (\nabla\mathcal{B} * u_0)(x_0) = \int \nabla\mathcal{B}(x_0 - y) u_0(y) dy \ .$$

The closure of the set of vectors w with the property that the half-line starting in x_0 with direction w meets $\operatorname{supp}_c u_0$ is a closed convex cone[1] C. With $\nabla\mathcal{B}(x)$ pointing in direction $-x$, we see that for a.e. y with $u_0(y) > 0$ the vector $\nabla\mathcal{B}(x_0 - y)$ points from x_0 towards y and thus $\nabla\mathcal{B}(x_0 - y) \in C$. $\nabla\gamma(x_0, 0)$ being a positive linear combination of such vectors, we have $\nabla\gamma(x_0, 0) \in C$.

Additionally, we see that $\nabla\gamma(x_0, 0)$ cannot lie at the boundary of the cone: let $b \in \partial C$; since the cone is convex, we have a vector a_b with $(d-b)\cdot a_b \geq 0$ for all $d \in C$, where equality can only hold if $d \in \partial C$. Since $\nabla\mathcal{B}(x_0 - y) \in C^\circ$ for $y \in (\operatorname{supp}_c u_0)^\circ$ and thus $(\nabla\mathcal{B}(x_0 - y) - b)\cdot a_b > 0$ for a.e. y (since $\partial \operatorname{supp}_c u_0$ has zero Lebesgue measure; see the last paragraph of the proof), we have $\big((\nabla\mathcal{B} * u_0)(x_0) - b\int u_0\ dx\big)\cdot a_b > 0$ which implies $\nabla\gamma(x_0, 0) \neq \int u_0\ dx \cdot b$. Since $b \in \partial C$ was arbitrary, we conclude $\nabla\gamma(x_0, 0) \notin \partial C$.

If the half-line starting in x_0 with direction $\nabla\gamma(x_0, 0)$ would only touch $\operatorname{supp}_c u_0$, we would obtain a contradiction since this would imply that we would have $\nabla\gamma(x_0, 0) \in \partial C$: for some vectors w in the neighborhood of $\nabla\gamma(x_0)$, the half-lines starting in x_0 with direction w would fail to meet $\operatorname{supp}_c u_0$ as the latter set is convex. As we have excluded the possibility that $\nabla\gamma(x_0, 0) \in \partial C$, the half-line starting in x_0 with direction $\nabla\gamma(x_0, 0)$ necessarily has to meet the interior of $\operatorname{supp}_c u_0$. As $x_0 \in \partial \operatorname{supp}_c u_0$, the assertion of the lemma follows using the fact that $\operatorname{supp}_c u_0$ is convex.

In the course of the proof we have used that the boundary of the convex set $\operatorname{supp}_c u_0$ has vanishing Lebesgue measure. This property is easily established: if $\operatorname{supp}_c u_0$ does not contain any interior point, it is contained in a hyperplane and thus $u_0 \equiv 0$ a.e. If it does contain an interior point x, the property follows easily by integrating $\chi_{\partial \operatorname{supp}_c u_0}$ using spherical coordinates centered around x and noting that any half-line starting in x meets ∂C in at most one point. ∎

Putting these results together, we obtain the following corollary.

COROLLARY 2.28. *Let u, γ be a weak solution of* (1), (2) *with initial data $u_0 \in L^1$, where $0 \leq u_0 \leq 1$.*

Suppose $x_0 \in \partial \operatorname{Conv}(\operatorname{supp} u_0)$; then in the case

$$\lim_{r\to 0} r^{-3} \fint_{B_r(x_0)} A(u_0(x))dx = 0$$

the support of u shrinks immediately near x_0; more precisely there exist $T^* > 0$, $c > 0$ such that for every $0 < t \leq T^*$ we have $\operatorname{supp} u(.,t) \cap B_{ct}(x_0) = \emptyset$.

Proof. This is immediate from the previous theorems, since we may choose a hyperplane running through x_0 and tangent to $\operatorname{supp}_c u_0$; by Lemma 2.27 and convexity of $\operatorname{supp}_c u_0$, $-\nabla\gamma(x_0, 0)$ points away from $\operatorname{supp}_c u_0$. ∎

3. Concluding remarks and possible extensions. We have obtained a sufficient condition for advection-induced support shrinking in a degenerate chemotaxis model; as a by-product, we have shown finite speed of support propagation for the model.

Our methods can be adapted to the case of the degenerate charge-transport model by Diaz, Galiano, and Jüngel [10], resulting in a new sufficient criterion for support shrinking. The calculations are mainly analogous to the ones presented above.

Our method can be applied to the subcritical Keller–Segel model

$$u_t = \Delta u^m + \operatorname{div}(u\nabla(V * u))\ , \tag{42}$$

[1] Here, by "cone" we mean a set C with the property that $v \in C$ implies $\lambda v \in C$ for all $\lambda \geq 0$.

with $\Omega = \mathbb{R}^d$, $V(x) := -c_d|x|^{2-d}$, $d > 1$, $m > 2 - \frac{2}{d}$, again yielding sufficient conditions for initial backward motion of the free boundary. Note that solutions to this model are bounded in the case of bounded initial data; see, e.g., [19] and the references therein.

More precisely, using our approach we could prove the following theorems.

THEOREM. *Let u be a weak solution of* (42) *with initial data $u_0 \in L^1 \cap L^\infty$. Suppose that u_0 has finite second moment. Let $x_0 \in \mathbb{R}^d$, $(\nabla V * u_0)(x_0) \neq 0$.*

Assume that there exists a neighborhood U of x_0 and a cone B with vertex x_0 and axis $(\nabla V * u_0)(x_0)$ such that $(B \cap U) \cap \operatorname{supp} u_0 = \emptyset$. Suppose furthermore that

$$\lim_{r\to 0} r^{-2+\frac{m-3}{m-1}} \fint_{B_r(x_0)} u_0^{m+1} dx = 0 \ . \tag{43}$$

Then there exist $T^* > 0$ and $c > 0$ such that $\operatorname{supp} u(.,t) \cap B_{ct}(x_0) = \emptyset$ for $0 < t < T^*$.

As the interaction between bacteria again is only attractive, we obtain the corollary below.

COROLLARY. *Let u be a weak solution of* (42) *with initial data $u_0 \in L^1 \cap L^\infty$. Assume that u_0 has finite second moment. Suppose $x_0 \in \partial \operatorname{Conv}(\operatorname{supp} u_0)$; then in the case*

$$\lim_{r\to 0} r^{-2+\frac{m-3}{m-1}} \fint_{B_r(x_0)} u_0^{m+1} dx = 0 \ ,$$

the support of u shrinks immediately near x_0; more precisely there exist $T^* > 0$, $c > 0$ such that for every $0 < t \leq T^*$ we have $\operatorname{supp} u(.,t) \cap B_{ct}(x_0) = \emptyset$.

Remark. Recall that in the case of vanishing advection, the condition ensuring the existence of a waiting time phenomenon is

$$\lim_{r\to 0} r^{-2-\frac{4}{m-1}} \fint_{B_r(x_0)} u^{m+1} dx < \infty$$

which again is a stronger condition than the one stated above.

In case of bounded initial data, Kim and Yao [19] have proven the following theorem on finite speed of propagation.

THEOREM. *Let u be a weak solution of* (42) *with bounded and compactly supported initial data u_0. If $\operatorname{supp} u(.,t) \subset B_R(0)$ holds for some $t \geq 0$, then $\operatorname{supp} u(.,t+h) \subset B_{R+Ch^{\frac{1}{2}}}(0)$ holds for any $h \in (0,1)$, where C depends on m, d, u_0.*

However, the constant C depends on the L^∞ norm of u_0 and blows up as the L^∞ norm of the initial data blows up.

Kim and Yao have also proven an L^∞ regularization theorem for solutions of the subcritical Keller–Segel model (Theorem 1.6b in [19]). For every $T > 0$ their result in particular yields a bound on $\int_0^T ||u(.,t)||^\beta_{L^\infty(\mathbb{R}^d)}\, dt$ for some $\beta(d,m) > 1$, the bound only depending on d, m, T, and $||u_0||_{L^1}$.

Using this regularization theorem and our method, we can prove finite speed of propagation also in the case of unbounded initial data. Note that in order to do so, in the analogue of Lemma 2.11 we must allow for time dependent M_{div} and M_{adv}. Using the fact that the equation $\Delta(V*u) = u$ and the estimate $||\nabla V * u||_{L^\infty} \leq C(d)||u||_{L^\infty} + C(d)||u||_{L^1}$ hold, by the L^∞ regularization theorem we see that M_{div} and M_{adv} at least are known to belong to $L^\beta([0,T])$ for every $T > 0$ for some $\beta(d,m) > 1$. For the proof of the analogue of Lemma 2.11 we now need to change our test functions slightly by replacing the terms $C^* M_{adv} t$ and $-C^- M_{div} t$ in formula (15) by $C^* \int_0^t M_{adv}(\tau)\, d\tau$ and $-C^- \int_0^t M_{div}(\tau)\, d\tau$, respectively. It is easily verified that then conditions analogous

to the properties T1 to T6 are satisfied by our test function. Our method now applies with minor modifications. We therefore obtain the following theorem.

THEOREM. *Let $d \geq 3$. Let u be a weak solution of* (42) *with initial data $u_0 \in L^1 \cap L^p$ for some $p > 1$. Assume that u_0 has finite second moment. Suppose that the weak solution u has been constructed as a limit of solutions u_δ with regularized initial data $\rho_\delta * u_0$. Assume that there exist some $x_0 \in \mathbb{R}^d$ and $R_0 > 0$ such that $B_{R_0}(x_0) \cap \operatorname{supp} u_0 = \emptyset$. Then there exists a continuous nonincreasing function $R : [0, \infty) \to [0, R_0]$ with $R(0) = R_0$ and $B_{R(t)}(x_0) \cap \operatorname{supp} u(., t) = \emptyset$ for any $t \in [0, \infty)$. The function R depends only on d, m, $||u_0||_{L^1}$, p, and R_0.*

If additionally we have $\operatorname{supp} u_0 \subset B_{\tilde{R}_0}(0)$, then there exists a continuous nondecreasing function $\tilde{R} : [0, \infty) \to (0, \infty)$ with $\tilde{R}(0) = \tilde{R}_0$ and $\operatorname{supp} u(., t) \subset B_{\tilde{R}(t)}(0)$ for every $t > 0$.

Note that the L^∞ regularization theorem due to Kim and Yao allows for the construction of a weak solution (in a sufficiently weak sense) for initial data $u_0 \in L^1 \cap L^p$ with finite second moment by considering solutions u_δ with initial data $\rho_\delta * u_0$ and passing to the limit $\delta \to 0$ (along some subsequence).

We see that in contrast to the result by Kim and Yao, our finite speed of propagation result also applies to unbounded initial data; we can even deal with certain initial data for which all superlevel sets are dense in some open set, as we only require $u_0 \in L^1 \cap L^p$ for some $p > 1$. Kim and Yao need to assume boundedness of the initial data since the supersolutions which they use for comparison are bounded.

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