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OPTIMAL LOWER BOUNDS ON ASYMPTOTIC SUPPORT PROPAGATION RATES FOR THE THIN-FILM EQUATION

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Abstract. We derive lower bounds on asymptotic support propagation rates for strong solutions of the Cauchy problem for the thin-film equation. The bounds coincide up to a constant factor with the previously known upper bounds and thus are sharp. Our results hold in case of at most three spatial dimensions and $n \in (1, \frac{32}{11})$. The result is established using weighted backward entropy inequalities with singular weight functions to yield a differential inequality; combined with some entropy production estimates, the optimal rate of propagation is obtained. To the best of our knowledge, these are the first lower bounds on asymptotic support propagation rates for higher-order nonnegativity-preserving parabolic equations.

1. Introduction

The thin-film equation

$$u_t = -\nabla \cdot (u^n \nabla \Delta u) \ , \quad n \in \mathbb{R}^+,$$

is supposed to describe the evolution of a thin viscous liquid film driven by surface tension for various slip conditions. The case $n = 3$ corresponds to no-slip conditions imposed on the fluid-solid interface; the case $n = 2$ (or, more precisely, the case of u^n being replaced by the mobility function $u^2 + u^3$) corresponds to Navier slip conditions. For $n = 1$, the equation arises as the lubrication approximation of the Hele-Shaw flow.

In this paper, we are to prove a result on the qualitative behaviour of solutions of the thin-film equation in the case of complete wetting (i.e. if the solutions satisfy the zero contact-angle condition $\nabla u = 0$ at the free boundary): we shall prove a lower bound on support propagation for the Cauchy problem for large timescales, showing that the upper bounds on support propagation rates for solutions of the thin-film equation which have been obtained in the past are optimal for any initial data.

The existence results for solutions of the thin-film equation in the case of complete wetting are quite satisfactory; see the papers by Bernis and Friedman [5], Beretta, Bertsch, and Dal Passo [2], Bernis [4], Elliot and Garcke [13], Grün [17], Dal Passo, Garcke, and Grün [11], Dal Passo and Garcke [10], Grün [19]. On the other hand, to the best of the author's knowledge no uniqueness result for these solutions is available.

The literature on qualitative behaviour of solutions of the thin-film equation is vast; there are results on preservation of nonnegativity and strict positivity (see e.g. Bernis and Friedman [5], Beretta, Bertsch, and Dal Passo [2], Dal Passo, Garcke, and Grün [11]). Moreover, it is known that solutions of the thin-film equation are subject to the finite speed of support propagation property (see Bernis [3], Bernis [4], Hulshof and Shishkov [20], Bertsch, Dal Passo, Garcke, and Grün [6], Grün [18]). Sufficient conditions for the occurrence of a waiting time phenomenon have been derived by Dal Passo, Giacomelli, and Grün [12]; these estimates have been refined to yield quantitative lower bounds on waiting times by Giacomelli and Grün [16].

Upper bounds on support propagation rates for large times have been obtained by Bernis [3] for $d = 1$ and $n \in (0, 2)$, by Hulshof and Shishkov [20] for $d = 1$ and $n \in [2, 3)$, by Bertsch, Dal Passo, Garcke, and Grün [6] for $d \in \{2, 3\}$ and $n \in (\frac{1}{8}, 2)$, and by Grün [18] for $d \in \{2, 3\}$ and $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$. The scaling of these upper bounds with respect to time matches the scaling of the self-similar solution; thus, for general initial data these upper bounds are optimal. In the present work, we shall show that these upper bounds are in fact optimal for any initial data.

These finite speed of propagation results are either based on localized versions of the entropy estimates

$$\frac{d}{dt}\ \frac{1}{\alpha(1+\alpha)}\int u^{1+\alpha}\ dx \leq -c(n,\alpha)\int |\nabla u^{\frac{n+\alpha+1}{4}}|^4 + |D^2 u^{\frac{n+\alpha+1}{2}}|^2\ dx$$

which hold for $\alpha + n \in (\frac{1}{2}, 2)$, $\alpha \neq 0$, or on localized versions of the energy estimate

$$\frac{d}{dt}\ \frac{1}{2}\int |\nabla u|^2\ dx \leq -\int u^n |\nabla \Delta u|^2\ dx\ .$$

While there are many results providing upper bounds on the support of solutions, only few results have been available which yield lower bounds on support propagation rates of a solution. For the Cauchy problem, for $n = 1$ and $d = 1$ decay of solutions to the self-similar solutions has been obtained by Carrillo and Toscani [7]; this result implies an optimal lower bound on the support propagation rate. However, an extension of the work of Carrillo and Toscani to the case $n \neq 1$ currently seems out of reach.

This is in contrast to the situation of the second-order analogue of the thin-film equation, the porous medium equation. For the porous medium equation, it is known that any solution converges to the self-similar solution at some polynomial rate; see e.g. the papers by Kamin [21] [22], Friedman and Kamin [15], Vazquez [24], Aronson and Vazquez [1], Carrillo and Toscani [8]. Moreover, Chipot and Sideris [9] have shown that the upper bounds on support propagation rates for the porous medium equation are optimal for any initial data.

In [14] the author recently has derived upper bounds on waiting times for the thin-film equation for $n \in [2, \frac{32}{11})$, i.e. upper bounds on the time at which the support of a solution of the thin-film equation spreads beyond some given point x_0. However, in case x_0 has a large distance to the initial support, the scaling of the upper bound on the waiting time turns out to be not optimal, i.e. it does not coincide with the optimal bounds on support propagation.

In the present work, we prove optimal lower bounds on asymptotic support propagation rates for the thin film equation. As in [14], we first derive a weighted backward entropy estimate; however, to obtain optimal lower bounds on asymptotic support propagation rates a different choice of the (singular) weight function becomes necessary. An application of a variant of Hardy's inequality yields a differential inequality; using the known upper bounds on support propagation rates, we deduce that the support of the solution must reach the singularity of the weight after some time, as otherwise the differential inequality would force the weighted entropy to blow up. Finally, to obtain optimal lower bounds on asymptotic support propagation rates, we need to combine our technique with some entropy production estimates.

Throughout the paper, we use standard notation for Sobolev spaces. By $C_c^\infty(\Omega)$ we denote the space of smooth compactly supported functions. We abbreviate $I := [0,\infty)$. The notation $L^p_{loc}(I;X)$ is used for the set of all mappings $u : I \to X$ which belong to $L^p([0,T];X)$ for all $T > 0$.

2. Main results

In this section, we recall the definition of strong solutions of the thin-film equation from [11] and the properties of the strong solutions of the thin-film equation constructed in [19] to additionally satisfy the energy inequality; for the reader's convenience, we summarize the main results of the present work.

Dal Passo, Garcke and Grün introduced the following notion of strong solution for the thin-film equation [11]:

Definition 1. *Let $u_0 \in H^1(\mathbb{R}^d)$, $1 \le d \le 3$, be nonnegative and compactly supported and let $n \in (\frac{1}{8}, 2)$. A function $u \in L^\infty(I;H^1(\mathbb{R}^d))$ is called a strong solution of the Cauchy problem for the thin-film equation if the following conditions are satisfied:*

a) *$u \in H^1_{loc}\left(I;\left[W^{1,p}\right]'\right)$ for all $p > \frac{4d}{2d+n(2-d)}$*
b) *for some $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n)\setminus\{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I;L^2)$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I;L^4)$*
c) *for any $\xi \in C_c^\infty(\mathbb{R}^d \times I)$ we have*

$$\begin{aligned}
&\int_0^T \langle u_t, \xi\rangle \; dt\\
&= \int_0^T \int u^n \nabla u \cdot \nabla\Delta\xi \; dx \; dt + n\int_0^T \int u^{n-1}\nabla u \cdot D^2\xi \cdot \nabla u \; dx \; dt\\
&\quad + \frac{n}{2}\int_0^T \int u^{n-1}|\nabla u|^2 \Delta\xi \; dx \; dt + \frac{n(n-1)}{2}\int_0^T \int u^{n-2}|\nabla u|^2 \nabla u \cdot \nabla\xi \; dx \; dt
\end{aligned}$$

for all $T > 0$.

d) *u attains the initial data in the sense that $u(.,t) \to u_0$ in L^1 as $t \to 0$*

There is a stronger concept of strong solution characterized by the additional requirement that the Dirichlet energy be dissipated. Existence of such strong energy solutions (the author decided to use this name in order to distinguish this notion of solution from the weaker notion of strong solution defined above) of the thin-film equation has been shown by Bernis in the case of one spatial dimension [4]. In case $d = 2$ or $d = 3$, proving existence of these solutions is much more demanding. In this case the proof has been carried out by Grün [19].

Definition 2. *Let $u_0 \in H^1(\mathbb{R}^d)$, $1 \le d \le 3$, be nonnegative and compactly supported. We call $u \in L^\infty(I;H^1(\mathbb{R}^d))$ a strong energy solution of the Cauchy problem for the thin-film equation if the following conditions are satisfied:*

a) *we have* $\nabla u^{\frac{n+2}{6}} \in L^6(I;L^6)$, $u^{\frac{n-2}{2}}\nabla u \otimes D^2 u \in L^2(I;L^2)$, $u^{\frac{n}{2}}\nabla\Delta u \in L^2(I;L^2)$
b) *for some* $\alpha \in (\max\{-1,\frac{1}{2}-n\},2-n)\setminus\{0\}$, *we have* $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I;L^2)$ *and* $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I;L^4)$
c) *it holds that* $u \in H^1_{loc}\left(I;\left(W^{1,p}\right)'\right)$ *for all* $p > \frac{4d}{2d+n(d-2)}$
d) *for any* $\xi \in C_c^\infty(\mathbb{R}^d \times I)$ *it holds that*

$$\int_0^T \langle u_t, \xi\rangle \ dt = \int_0^T \int u^n \nabla\Delta u \cdot \nabla\xi \ dx \ dt$$

e) u *attains its initial data* u_0 *in the sense that* $\lim_{t\to 0} u(.,t) = u_0(.)$ *in* L^1

Note that the solutions constructed by Dal Passo, Garcke and Grün [11] and by Grün [19] satisfy the α entropy estimate for any $\alpha \in (\max\{-1,\frac{1}{2}-n\},2-n)$, not just for a particular α; thus for any nonnegative initial data $u_0 \in H^1(\mathbb{R}^d)$ with compact support, in case $n \in \left(2-\sqrt{\frac{8}{8+d}},3\right)$ there exists a strong energy solution u of the Cauchy problem for the thin-film equation which satisfies all α entropy estimates; in case $n \in (\frac{1}{8},2)$, there exists a strong solution satisfying all α entropy estimates.

The following upper bounds on asymptotic support propagation have been obtained by Bernis [3] for $n \in (0,2)$ and $d=1$, by Hulshof and Shishkov [20] for $n \in [2,3)$ and $d=1$, by Bertsch, Dal Passo, Garcke and Grün [6] for $d \in \{2,3\}$ and $n \in (\frac{1}{8},2)$, and by Grün [18] for $n \in \left(2-\sqrt{\frac{8}{8+d}},3\right)$ and $d \in \{2,3\}$:

Theorem 3. *Let* $u_0 \in H^1(\mathbb{R}^d)$, $1 \leq d \leq 3$, *be compactly supported. Let* u *be a strong solution of the Cauchy problem for the thin-film equation obtained by the procedure in* [11] *and* $n \in (\frac{1}{8},2)$ *or let* u *be a strong energy solution of the Cauchy problem for the thin-film equation and* $n \in \left(2-\sqrt{\frac{8}{8+d}},3\right)$. *Assume that* $\operatorname{supp} u_0 \subset B_{R_0}(x_0)$ *for some* $R_0 > 0$ *and some* $x_0 \in \mathbb{R}^d$. *Then for any* $t > 0$ *we have the estimate* $\operatorname{supp} u(.,t) \subset B_{R(t)}(x_0)$ *with*

$$R(t) := R_0 + C(n,d)||u_0||_{L^1}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} \ .$$

Our main result roughly provides the converse of this statement. It reads:

Theorem 4. *Let* $u_0 \in H^1(\mathbb{R}^d)$ *be nonnegative and compactly supported. Assume* $d \leq 3$ *and* $n \in (1,\frac{32}{11})$. *Let*

- $n \in \left(2-\sqrt{\frac{8}{8+d}},\frac{32}{11}\right)$ *and let* u *be a strong energy solution of the Cauchy problem for the thin-film equation satisfying all* α *entropy estimates for* $\alpha > -1$, *or*
- *let* $n \in (1,2)$ *and let* u *be a strong solution of the Cauchy problem constructed as in* [6].

Let $x_0 \in \mathbb{R}^d$ *be a point. Set*

$$T^* := \inf\left\{T \geq 0 : \inf_{0\leq t\leq T} \operatorname{dist}(x_0, \operatorname{supp} u(.,t)) = 0\right\}.$$

Then there exists a constant $C(d,n)$ depending only on d and n such that the following estimate holds:

$$T^* \leq C(d,n)\left[\operatorname{dist}(x_0, \operatorname{supp} u_0) + \operatorname{diam}(\operatorname{supp} u_0)\right]^{4+d\cdot n} ||u_0||_{L^1(\mathbb{R}^d)}^{-n}$$

Recall that the regularity $u \in L^\infty(I; H^1(\mathbb{R}^d)) \cap H^1_{loc}(I; [W^{1,p}(\mathbb{R}^d)]')$ implies $u \in C^0_{loc}(I; L^2(V))$ for any bounded open set $V \subset \mathbb{R}^d$ with smooth boundary (see e.g. Corollary 4 in [23]). Thus the essential support $\operatorname{supp} u(.,t)$ is well-defined for all $t \geq 0$.

Note that in the special case $n = 1$ and $d = 1$, a similar estimate could be inferred from the much stronger assertion by Carrillo and Toscani [7] who prove asymptotic decay of the solution to the self-similar solution in this case. However, to the best of our knowledge up to now no generalization of their result to $d > 1$ or $n \neq 1$ is available; moreover, for $d > 1$ such a convergence result would not imply our theorem.

For $n > 1.5$ the support of solutions to the thin-film equation is nondecreasing with respect to time, i.e. we have $\operatorname{supp} u(.,t_1) \subset \operatorname{supp} u(.,t_2)$ for all $t_1, t_2 \in I$ with $t_1 \leq t_2$. This has been proven for strong solutions constructed by the usual approximation procedure in case $d = 1$ in [2]; for strong solutions constructed as in [6] and $d \leq 3$ it follows by the considerations in [6] (though only the weaker assertion $\operatorname{supp} u_0 \subset \operatorname{supp} u(.,t)$ for $t > 0$ is explicitly stated in this paper). For strong energy solutions constructed as in [19] and $d \leq 3$, the estimate $\operatorname{supp} u(.,t_1) \subset \operatorname{supp} u(.,t_2)$ for $0 \leq t_1 \leq t_2$ is a consequence of the entropy estimates in [19] (though it is not stated explicitly in this paper). Given a solution with nondecreasing support, our previous theorem implies:

Corollary 5. *Let $u_0 \in H^1(\mathbb{R}^d)$ be nonnegative and compactly supported. Assume $1 \leq d \leq 3$ and $1.5 < n < \frac{32}{11}$. Let*

- *$n \in \left(1.5, \frac{32}{11}\right)$ and let u be a strong energy solution of the Cauchy problem for the thin-film equation satisfying all α entropy estimates for $\alpha > -1$, or*
- *let $n \in (1.5, 2)$ and let u be a strong solution of the Cauchy problem constructed as in [6].*

Suppose that $\operatorname{supp} u(.,t_1) \subset \operatorname{supp} u(.,t_2)$ holds for all $0 \leq t_1 \leq t_2$.

Let $x_s \in \operatorname{supp} u_0$ be some point. Then there exists a constant $c(d,n)$ depending only on n and d such that for any $t > 0$ with $R(t) > 0$ we have

$$B_{R(t)}(x_s) \subset \operatorname{supp} u(.,t)\ ,$$

where

$$R(t) := c(d,n)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} - \operatorname{diam}(\operatorname{supp} u_0)\ .$$

3. Proof of the main results

3.1. **Entropy estimates with optimal constants.** The following two Lemmas have been established in [14]:

Lemma 6. *Let $1 \leq d \leq 3$ and u be a strong energy solution of the thin-film equation on $\mathbb{R}^d$ which satisfies the α entropy estimate. Suppose that $\operatorname{supp} u_0$ is* *bounded. Assume $-1<\alpha\leq 0$ and $1\leq n<3$. Defining $b:=n+\alpha$, the formula*

$$\begin{aligned}
&\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)\psi(.)\;dx-\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)\psi(.)\;dx\\
&=\left(b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int u^{b-1}|\nabla u|^2\Delta\psi\;dx\;dt+b\int_{t_1}^{t_2}\int u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u\;dx\;dt\\
&\quad-\frac{1}{b+1}\int_{t_1}^{t_2}\int u^{b+1}\Delta^2\psi\;dx\;dt+(n-b)\int_{t_1}^{t_2}\int u^{b-1}|D^2u|^2\psi\;dx\;dt\\
&\quad+\left(b-\frac{n}{2}\right)(b-1)(2-b)\int_{t_1}^{t_2}\int u^{b-3}|\nabla u|^4\psi\;dx\;dt\\
&\quad+(2n-3b)(b-1)\int_{t_1}^{t_2}\int u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\;\psi\;dx\;dt\\
&\quad+\left(\frac{n}{2}-b\right)(b-1)\int_{t_1}^{t_2}\int u^{b-2}|\nabla u|^2\Delta u\;\psi\;dx\;dt
\end{aligned}\tag{1}$$

holds for any $\psi\in C_c^\infty(\mathbb{R}^d)$, a.e. $t_2>t_1>0$ and a.e. t_2 in case $t_1=0$.

Lemma 7. *For any u with $u^{\frac{b+1}{2}}\in H^2$ and $u^{\frac{b+1}{4}}\in W^{1,4}$, we have*

$$\begin{aligned}
&\int u^{b-1}|D^2u|^2\psi\;dx+(b-1)\int u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\;\psi\;dx+\int u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u\;dx\\
&=\int u^{b-1}|\Delta u|^2\psi\;dx+(b-1)\int u^{b-2}|\nabla u|^2\Delta u\;\psi\;dx+\int u^{b-1}|\nabla u|^2\Delta\psi\;dx
\end{aligned}$$

for any $\psi\in C_c^\infty(\mathbb{R}^d)$.

We set $b:=n+\alpha$ and introduce the following conditions:

(H1) Assume that $1\leq b\leq 2$.
(H2) Suppose that $\frac{n}{2}\leq b\leq n$.
(H3) Assume that $n-1<b$.
(H4) Suppose that the inequality

$$(n-b)\left(b-\frac{n}{2}\right)(b-1)(2-b)\geq\frac{1}{4}\left[\left(\frac{5n}{2}-4b\right)(b-1)\right]^2$$

is satified.

The set of $(b,n)\in\mathbb{R}\times\mathbb{R}$ for which (H1) to (H4) are satisfied is depicted below. The set of points for which (H1), (H2) and (H4) hold at the same time corresponds to the red area. All points below the yellow line satisfy (H3). The yellow line intersects the boundary of the red area at $b\approx 1.92$, $n\approx 2.92$. For all points below the green line, $\gamma=-d$ is an admissible choice in condition (H5) of Lemma 11 below. The green line intersects the boundary of the red area at $b=1$, $n=1$.

In [14], the following lemma has been seen to be a consequence of Lemma 6, Lemma 7, and Young's inequality:

Lemma 8. *Let $n\in[1,3)$, $\alpha\in(-1,0]$, and let u be a strong energy solution of the thin-film equation on $\mathbb{R}^d$, $d\leq 3$, with initial data $u_0\in H^1(\Omega)$. Assume that* $\operatorname{supp} u_0$ *is bounded. Suppose that u satisfies the α entropy estimate. Set $b:=n+\alpha$ and assume that (H1) to (H4) are satisfied.*

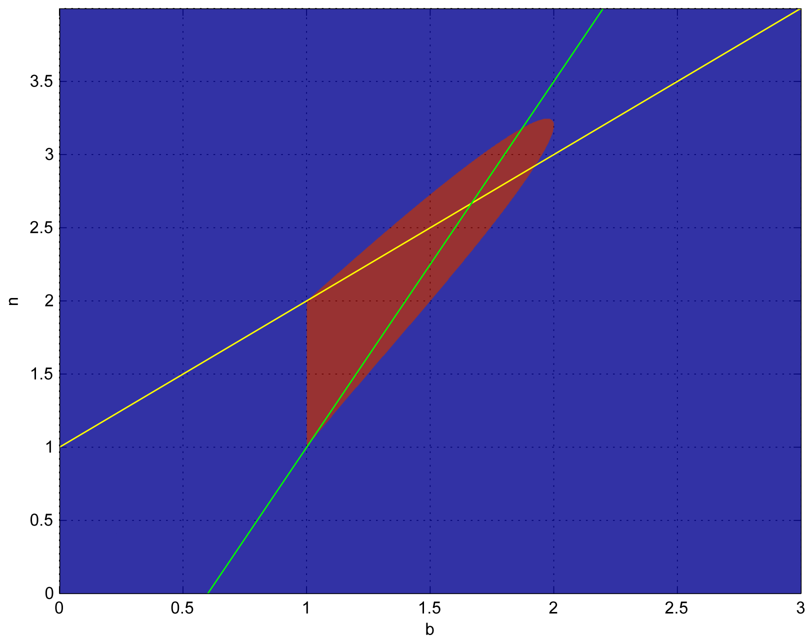

FIGURE 1. Admissible values for b and n.

Let $\psi \in C_c^4(\mathbb{R}^d)$. Assume $\psi \geq 0$. Then for a.e. $t_1, t_2 \in [0,\infty)$ with $t_2 \geq t_1$ and for a.e. $t_2 \in [0,\infty)$ in case $t_1 = 0$ we have

$$
\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&\geq \left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int u^{b-1}|\nabla u|^2 \Delta\psi\ dx\ dt \\
&\quad + \left(\frac{4}{3}b - \frac{1}{3}n\right) \int_{t_1}^{t_2} \int u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u\ dx\ dt \\
&\quad - \frac{1}{b+1} \int_{t_1}^{t_2} \int u^{b+1} \Delta^2\psi\ dx\ dt\ .
\end{aligned}
$$

However, for $d > 1$ existence of strong energy solutions is only known for $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$. Thus, for $d > 1$ and n slightly larger than 1 only existence of strong solutions is guaranteed. We therefore need to extend Lemma 8 to the case of strong solutions:

Lemma 9. *Let $d \leq 3$ and $n \in [1,2)$. The assertion of Lemma 8 also holds if u is not a strong energy solution, but a strong solution of the thin-film equation constructed by the procedure in* [11].

The remainder of the current subsection is dedicated to the approximation argument necessary for establishing the lemma; it may be skipped on first reading.

Proof. We use the notation from [6]. First a solution u of the thin-film equation on the bounded domain $\Omega_W := B_W(0)$ is constructed as the limit of the solutions $u_{\delta\sigma}$ of the problem

$$\begin{aligned} \frac{d}{dt}u_{\delta\sigma} &= -\nabla\cdot(m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}) \quad \text{on } \Omega_W\times[0,\infty), \\ \vec{n}\cdot\nabla u_{\delta\sigma} &= \vec{n}\cdot\nabla\Delta u_{\delta\sigma} = 0 \text{ on } \partial\Omega_W\times[0,\infty)\ , \\ u_{\delta\sigma}(.,0) &= u_0+\delta^{\theta_1}+\sigma^{\theta_2}\ , \end{aligned}$$

with $\theta_1,\theta_2>0$ and

$$m_{\delta\sigma}(v) := \frac{v^{n+s}}{\delta v^n+v^s+\sigma v^{n+s}}\ .$$

Let $T>0$; by the finite speed of propagation result, if W has been chosen large enough (depending on T) we see that $\operatorname{supp} u$ cannot touch $\partial\Omega_W$ before time T. Therefore extending u to $\mathbb{R}^d\times[0,T)$ by setting it to zero outside of $\Omega_W\times[0,T)$, we obtain a solution of the Cauchy problem for the thin-film equation in the time interval $[0,T)$. Finally, we pass to the limit $W\to\infty$; a subsequence then converges to a solution of the Cauchy problem on the time interval $[0,\infty)$.

Fix $W>0$; from now on we write Ω instead of Ω_W. Choosing s sufficiently large, as shown in [6] there exists a bounded solution $u_{\delta\sigma}$ which is strictly positive for a.e. $t>0$ and satisfies the energy estimate

$$\int_\Omega |\nabla u_{\delta\sigma}(.,t)|^2\ dx + \int_0^t\int_{\Omega_T} m_{\delta\sigma}(u_{\delta\sigma})|\nabla\Delta u_{\delta\sigma}|^2\ dx\ dt \le \int_\Omega |\nabla u_0|^2\ dx\ .$$

Moreover, this solution satisfies $u_{\delta\sigma}\in H^1_{loc}(I;(H^1(\Omega))')$; for a.e. $t>0$ we know $\nabla\Delta u_{\delta\sigma}\in L^2(\Omega)$ and for any $\phi\in L^2(I;H^1(\Omega))$ the solution satisfies

$$\int_0^t \left\langle \frac{d}{dt}u_{\delta\sigma},\phi\right\rangle\ dt = \int_0^t\int_\Omega m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}\cdot\nabla\phi\ dx\ dt\ . \tag{2}$$

For a proof of this claim see [6].

We now proceed as in the derivation of the entropy inequalities in [6]. Set

$$g^\alpha_{\delta\sigma}(v) := \frac{\delta}{\alpha+n-s}v^{\alpha+n-s}+\frac{1}{\alpha}v^\alpha+\frac{\sigma}{\alpha+n}v^{\alpha+n}$$

and

$$\begin{aligned} G^\alpha_{\delta\sigma}(v) :=& \frac{\delta}{(\alpha+n-s)(\alpha+n-s+1)}v^{\alpha+n-s+1}+\frac{1}{\alpha(\alpha+1)}v^{\alpha+1} \\ &+\frac{\sigma}{(\alpha+n)(\alpha+n+1)}v^{\alpha+n+1}\ . \end{aligned}$$

We have $G'_{\delta\sigma}(v)=g_{\delta\sigma}(v)$ and $g'_{\delta\sigma}(v)=\frac{v^{n+\alpha-1}}{m_{\delta\sigma}(v)}$. Using the boundedness of $u_{\delta\sigma}$ it is easily seen that $g^\alpha_{\delta\sigma}(u_{\delta\theta}+\epsilon)\in L^\infty([0,\infty);H^1(\Omega))$. Thus we may test the equation (2) with $\psi\cdot g^\alpha_{\delta\sigma}(u_{\delta\theta}+\epsilon)$, where $\psi\in C^\infty_c(\Omega)$, to obtain for a.e. $t_2>t_1>0$ and a.e. $t_2>0$ in case $t_1=0$ (for the rearrangements involving the term with the time derivative, see [6])

$$\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot\psi\,dx\Big|_{t_1}^{t_2}\\
=&\int_{t_1}^{t_2}\int_\Omega \psi\cdot\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&+\int_{t_1}^{t_2}\int_\Omega g_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\ .
\end{aligned}$$

Integrating by parts yields (recall that for a.e. $t>0$ we have $\inf_x u(x,t)>0$ and $u(.,t)\in H^3_{loc}(\Omega)$; thus integration by parts is possible)

$$\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot\psi\,dx\Big|_{t_1}^{t_2}\\
=&-\int_{t_1}^{t_2}\int_\Omega \psi\cdot\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})D^2u_{\delta\sigma}:D^2u_{\delta\sigma}\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega \frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})\nabla\psi\cdot D^2u_{\delta\sigma}\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega \psi\cdot\Bigg[\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m'_{\delta\sigma}(u_{\delta\sigma})+(n+\alpha-1)\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-2}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})\\
&\qquad\qquad-\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)^2}\cdot m'_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m_{\delta\sigma}(u_{\delta\sigma})\Bigg]\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega \Bigg[\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})+g_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m'_{\delta\sigma}(u_{\delta\sigma})\Bigg]\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega g_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m_{\delta\sigma}(u_{\delta\sigma})D^2u_{\delta\sigma}:D^2\psi\,dx\,dt\\
=:&\,I+II+III+IV+V\ .
\end{aligned}$$

Now we can pass to the limit $\epsilon\to 0$ since we know that $u_{\delta\sigma}^{\frac{n+\alpha+1}{2}}\in L^2([0,\infty);H^2(\Omega))$ and $u_{\delta\sigma}^{\frac{n+\alpha+1}{4}}\in L^4([0,\infty);W^{1,4}(\Omega))$ (see [6]) from which we deduce that $u_{\delta\sigma}^{\frac{n+\alpha-1}{2}}D^2u_{\delta\sigma}\in L^2(\Omega\times I)$ and that $u_{\delta\sigma}^{\frac{n+\alpha-3}{4}}\nabla u_{\delta\sigma}\in L^4(\Omega\times I)$.

Convergence of term I can be shown as follows: Suppose $\epsilon<1$. In case $v\le\epsilon$ we have

$$\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)}\cdot m_{\delta\sigma}(v)\le C(n,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-1}}{\epsilon^s}\cdot v^s\le C(n,\alpha,\delta,\sigma)v^{n+\alpha-1}$$

where we have used the fact that s is large (in particular larger than n and larger than $n+\alpha-1$). In case $v>\epsilon$ we obtain

$$\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)}\cdot m_{\delta\sigma}(v)\le C(n,\alpha,\delta,\sigma)\frac{v^{n+\alpha-1}}{m_{\delta\sigma}(v)}m_{\delta\sigma}(v)\le C(n,\alpha,\delta,\sigma)v^{n+\alpha-1}\ .$$

Thus, we get convergence of term I as $\epsilon\to 0$ using dominated convergence (since pointwise convergence holds a.e. due to the strict positivity of $u_{\delta\sigma}$ a.e.).

Term II can be treated similarly.

To show convergence of term IV, we only need to show the bound $|g_{\delta\sigma}(v+\epsilon)\cdot m'_{\delta\sigma}(v)|\leq C(n,\alpha,\delta,\sigma)v^{n+\alpha-1}$ (as a corresponding bound for the first term in brackets has already been derived); then again convergence follows by the dominated convergence theorem. Let $\epsilon<1$. We obtain in case $v\leq\epsilon$

$$|g_{\delta\sigma}(v+\epsilon)\cdot m'_{\delta\sigma}(v)|\leq C(n,\alpha,\delta,\sigma)\epsilon^{\alpha+n-s}\cdot v^{s-1}\leq C(n,\alpha,\delta,\sigma)v^{\alpha+n-1}$$

since $\alpha+n-s\leq\alpha$ and $\alpha+n-1\leq s-1$ (s being large). For $v>\epsilon$ we get

$$\begin{aligned}|g_{\delta\sigma}(v+\epsilon)\cdot m'_{\delta\sigma}(v)|\leq& C(n,\alpha,\delta,\sigma)(v^{\alpha+n-s}+v^{\alpha+n})\frac{v^{n+s-1}}{v^n+v^{n+s}}\\ \leq& C(n,\alpha,\delta,\sigma)v^{\alpha+n-1}\ .\end{aligned}$$

Regarding term V, for $\epsilon\leq 1$ we deduce in case $v\leq\epsilon$ that

$$|g_{\delta\sigma}(v+\epsilon)\cdot m_{\delta\sigma}(v)|\leq C(n,\alpha,\delta,\sigma)\epsilon^{\alpha+n-s}v^s\leq C(n,\alpha,\delta,\sigma)v^{\alpha+n}\ .$$

In case $v>\epsilon$ we have

$$\begin{aligned}|g_{\delta\sigma}(v+\epsilon)\cdot m_{\delta\sigma}(v)|\leq& C(n,\alpha,\delta,\sigma)(v^{\alpha+n-s}+v^{\alpha+n})\frac{v^{n+s}}{v^n+v^{n+s}}\\ \leq& C(n,\alpha,\delta,\sigma)v^{\alpha+n}\ .\end{aligned}$$

Again by dominated convergence, the term V converges.

It remains to prove convergence of term III. Using dominated convergence, the convergence of this term is established as soon as we have shown the appropriate estimates. Assume $\epsilon\leq 1$. For $v\leq\epsilon$ we get

$$\begin{aligned}&\Bigg|\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)}\cdot m'_{\delta\sigma}(v)+(n+\alpha-1)\frac{(v+\epsilon)^{n+\alpha-2}}{m_{\delta\sigma}(v+\epsilon)}\cdot m_{\delta\sigma}(v)\\ &\quad-\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)^2}\cdot m'_{\delta\sigma}(v+\epsilon)\cdot m_{\delta\sigma}(v)\Bigg|\\ \leq& C(n,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-1}}{\epsilon^s}\cdot v^{s-1}+C(n,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-2}}{\epsilon^s}\cdot v^s\\ &\quad+C(n,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-1}}{\epsilon^{2s}}\cdot\epsilon^{s-1}\cdot v^s\\ \leq& C(n,\alpha,\delta,\sigma)v^{n+\alpha-2}\ .\end{aligned}$$

On the other hand, for $v>\epsilon$ we obtain

$$\begin{aligned}&\Bigg|\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)}\cdot m'_{\delta\sigma}(v)+(n+\alpha-1)\frac{(v+\epsilon)^{n+\alpha-2}}{m_{\delta\sigma}(v+\epsilon)}\cdot m_{\delta\sigma}(v)\\ &\quad-\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)^2}\cdot m'_{\delta\sigma}(v+\epsilon)\cdot m_{\delta\sigma}(v)\Bigg|\\ \leq& C(n,\alpha,\delta,\sigma)\frac{v^{n+\alpha-1}}{m(v)}\cdot\frac{v^{n+s-1}}{v^n+v^s+v^{n+s}}+C(n,\alpha,\delta,\sigma)\frac{v^{n+\alpha-2}}{m_{\delta\sigma}(v)}\cdot m_{\delta\sigma}(v)\\ &\quad+C(n,\alpha,\delta,\sigma)\frac{v^{n+\alpha-1}}{m(v)^2}\cdot\frac{v^{n+s-1}}{v^n+v^s+v^{n+s}}\cdot m(v)\\ \leq& C(n,\alpha,\delta,\sigma)v^{n+\alpha-2}\ .\end{aligned}$$

Summing up, we have shown that (recall that $b := n + \alpha$)

$$
\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}) \cdot \psi \; dx \bigg|_{t_1}^{t_2} \\
=& -\int_{t_1}^{t_2} \int_\Omega \psi \cdot u_{\delta\sigma}^{n+\alpha-1} D^2 u_{\delta\sigma} : D^2 u_{\delta\sigma} \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{n+\alpha-1} \nabla\psi \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega \psi \cdot (n+\alpha-1) u_{\delta\sigma}^{n+\alpha-2} \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega \left[u_{\delta\sigma}^{n+\alpha-1} + g_{\delta\sigma}(u_{\delta\sigma}) \cdot m'_{\delta\sigma}(u_{\delta\sigma}) \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) D^2 u_{\delta\sigma} : D^2\psi \; dx \; dt \\
=& -\int_{t_1}^{t_2} \int_\Omega \psi \cdot u_{\delta\sigma}^{b-1} D^2 u_{\delta\sigma} : D^2 u_{\delta\sigma} \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} \nabla\psi \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega \psi \cdot (n+\alpha-1) u_{\delta\sigma}^{b-2} \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega \left[u_{\delta\sigma}^{b-1} + \frac{n}{\alpha} u_{\delta\sigma}^{b-1} \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega \frac{1}{\alpha} u_{\delta\sigma}^{b} D^2 u_{\delta\sigma} : D^2\psi \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m'_{\delta\sigma}(u_{\delta\sigma}) - \frac{n}{\alpha} u_{\delta\sigma}^{b-1} \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \; dx \; dt \\
& - \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) - \frac{1}{\alpha} u_{\delta\sigma}^{b} \right] D^2 u_{\delta\sigma} : D^2\psi \; dx \; dt \; .
\end{aligned}
$$

Recall that u is strictly positive and bounded for a.e. $t>0$ and that $u(.,t)\in H^3_{loc}(\Omega)$ for a.e. $t>0$. Several integrations by parts therefore yield

$$
\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma})\cdot\psi\,dx\Big|_{t_1}^{t_2}\\
=&-\int_{t_1}^{t_2}\int_\Omega \psi\cdot u_{\delta\sigma}^{b-1}|D^2u_{\delta\sigma}|^2\,dx\,dt\\
&+\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{b-1}|\nabla u_{\delta\sigma}|^2\Delta\psi\,dx\,dt\\
&+\frac{b-1}{2}\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{b-2}|\nabla u_{\delta\sigma}|^2\nabla u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\\
&-(b-1)\int_{t_1}^{t_2}\int_\Omega \psi\cdot u_{\delta\sigma}^{b-2}\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&+\frac{b}{2\alpha}\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{b-1}|\nabla u_{\delta\sigma}|^2\Delta\psi\,dx\,dt\\
&+\frac{b(b-1)}{2\alpha}\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{b-2}|\nabla u_{\delta\sigma}|^2\nabla u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\\
&+\frac{b}{\alpha}\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{b-1}\nabla u_{\delta\sigma}\cdot D^2\psi\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&+\frac{1}{\alpha}\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{b}\nabla u_{\delta\sigma}\cdot\nabla\Delta\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega\left[g_{\delta\sigma}(u_{\delta\sigma})\cdot m_{\delta\sigma}'(u_{\delta\sigma})-\frac{n}{\alpha}u_{\delta\sigma}^{b-1}\right]\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega\left[g_{\delta\sigma}(u_{\delta\sigma})\cdot m_{\delta\sigma}(u_{\delta\sigma})-\frac{1}{\alpha}u_{\delta\sigma}^{b}\right]D^2u_{\delta\sigma}:D^2\psi\,dx\,dt\,.
\end{aligned}
$$

Further integrations by parts yield

$$
\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma})\cdot\psi\,dx\Big|_{t_1}^{t_2}\\
=&\frac{n-b}{\alpha}\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-1}|D^2u_{\delta\sigma}|^2\psi\,dx\,dt+\frac{1}{\alpha}\left(b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-1}|\nabla u_{\delta\sigma}|^2\Delta\psi\,dx\,dt\\
&+\frac{b}{\alpha}\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-1}\nabla u_{\delta\sigma}\cdot D^2\psi\cdot\nabla u_{\delta\sigma}\,dx\,dt-\frac{1}{\alpha(b+1)}\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b+1}\Delta^2\psi\,dx\,dt\\
&+\frac{1}{\alpha}\left(b-\frac{n}{2}\right)(b-1)(2-b)\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-3}|\nabla u_{\delta\sigma}|^4\psi\,dx\,dt\\
&+\frac{1}{\alpha}(2n-3b)(b-1)\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-2}\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla u_{\delta\sigma}\;\psi\,dx\,dt\\
&+\frac{1}{\alpha}\left(\frac{n}{2}-b\right)(b-1)\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-2}|\nabla u_{\delta\sigma}|^2\Delta u_{\delta\sigma}\;\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega\left[g_{\delta\sigma}(u_{\delta\sigma})\cdot m'_{\delta\sigma}(u_{\delta\sigma})-\frac{n}{\alpha}u_{\delta\sigma}^{b-1}\right]\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega\left[g_{\delta\sigma}(u_{\delta\sigma})\cdot m_{\delta\sigma}(u_{\delta\sigma})-\frac{1}{\alpha}u_{\delta\sigma}^{b}\right]D^2u_{\delta\sigma}:D^2\psi\,dx\,dt\,.
\end{aligned}
$$

Multiplying the equation by α, conditions (H1) to (H4) in connection with Lemma 7 now imply (for details see [14]) that

$$
\begin{aligned}
&\alpha\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}(.,t_2))\cdot\psi\,dx-\alpha\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}(.,t_1))\cdot\psi\,dx\\
\geq&\left(\frac{2}{3}b-\frac{1}{6}n\right)\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-1}|\nabla u_{\delta\sigma}|^2\Delta\psi\,dx\,dt\\
&+\left(\frac{4}{3}b-\frac{1}{3}n\right)\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b-1}\nabla u_{\delta\sigma}\cdot D^2\psi\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&-\frac{1}{b+1}\int_{t_1}^{t_2}\int u_{\delta\sigma}^{b+1}\Delta^2\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega\left[\alpha g_{\delta\sigma}(u_{\delta\sigma})\cdot m'_{\delta\sigma}(u_{\delta\sigma})-nu_{\delta\sigma}^{b-1}\right]\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega\left[\alpha g_{\delta\sigma}(u_{\delta\sigma})\cdot m_{\delta\sigma}(u_{\delta\sigma})-u_{\delta\sigma}^{b}\right]D^2u_{\delta\sigma}:D^2\psi\,dx\,dt
\end{aligned}\tag{3}
$$

for a.e. $t_2>t_1>0$ and a.e. $t_2>0$ in case $t_1=0$.

We now pass to the limit $\delta\to 0$, then to the limit $\sigma\to 0$. The first three terms on the right-hand side are seen to converge since $u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}\to u^{\frac{\alpha+n+1}{2}}$ strongly in $L^2_{loc}(I;H^1(\Omega))$ (see [6]).

For a.e. $t_1,t_2>0$ the terms on the left-hand side converge: Considering the entropy inequalities for $u_{\delta\sigma}$ for $\alpha+\nu$ and $\alpha-\nu$ (where $\nu>0$ is sufficiently small), we deduce that

$$\begin{aligned}
&\sup_t \int_\Omega \delta u_{\delta\sigma}^{\alpha-\nu+n-s+1}(x,t)\ dx\\
\leq &C(n,\alpha,\nu,s,\Omega)\delta(\delta^{\theta_1}+\sigma^{\theta_2})^{\alpha-\nu+n-s+1}+\sup_t C(n,\alpha,\nu,s)\int_\Omega u_{\delta\sigma}^{1+\alpha-\nu}(x,t)\ dx\\
&+C(n,\alpha,\nu,s)\int_\Omega u_0^{1+\alpha-\nu}\ dx+\sigma C(n,\alpha,\nu,s)\int_\Omega (u_0+\delta^{\theta_1}+\sigma^{\theta_2})^{n+\alpha+1-\nu}\ dx
\end{aligned}$$

and that

$$\begin{aligned}
&\sup_t \int_\Omega \sigma u_{\delta\sigma}^{\alpha+n+1+\nu}(x,t)\ dx\\
\leq &C(n,\alpha,\nu,s,\Omega)\delta(\delta^{\theta_1}+\sigma^{\theta_2})^{\alpha+\nu+n-s+1}+\sup_t C(n,\alpha,\nu,s)\int_\Omega u_{\delta\sigma}^{1+\alpha+\nu}(x,t)\ dx\\
&+C(n,\alpha,\nu,s)\int_\Omega (u_0+\delta^{\theta_1}+\sigma^{\theta_2})^{1+\alpha+\nu}\ dx+\sigma C(n,\alpha,\nu,s)\int_\Omega (u_0+\delta^{\theta_1}+\sigma^{\theta_2})^{n+\alpha+1+\nu}\ dx\ .
\end{aligned}$$

Using the strong convergence of $u_{\delta\sigma}^{n+\alpha+1}$ in L^1, we deduce that the terms on the left-hand side converge to the desired limit for a.e. $t_1,t_2>0$ since by the previous considerations, the first and the last term in the definition of $G_{\delta\sigma}(u_{\delta\sigma})$ vanish in the limit: For the first term, we have for every $\mu>0$

$$\int_\Omega \delta u_{\delta\sigma}^{\alpha+n-s+1}(x,t)\ dx\leq \delta\mu^{\alpha+n-s+1}|\Omega|+\mu^\nu\int_\Omega \delta u_{\delta\sigma}^{\alpha-\nu+n-s+1}(x,t)\ dx\ .$$

The latter integral being bounded uniformly, setting $\mu:=\delta^\beta$ for $\beta>0$ small enough the convergence to zero of the term on the left-hand side follows. Regarding the last term in the definition of $G_{\delta\sigma}(u_{\delta\sigma})$, for every $\mu>0$ we have

$$\int_\Omega \sigma u_{\delta\sigma}^{\alpha+n+1}(x,t)\ dx\leq \sigma\mu^{\alpha+n+1}|\Omega|+\mu^{-\nu}\int_\Omega \sigma u_{\delta\sigma}^{\alpha+n+1+\nu}(x,t)\ dx\ .$$

As the last integral is bounded uniformly, setting $\mu:=\sigma^{-\beta}$ for $\beta>0$ small enough the convergence to zero of the term on the left-hand side follows.

It remains to prove that the last two terms on the right-hand side in (3) converge to zero. We compute

$$\begin{aligned}
&|\alpha g_{\delta\sigma}(v)m_{\delta\sigma}(v)-v^{n+\alpha}|\\
\leq&\left|v^\alpha\frac{v^{n+s}}{\delta v^n+v^s+\sigma v^{n+s}}-v^{n+\alpha}\right|+\delta C(n,\alpha,s)v^{n+\alpha-s}\frac{v^{n+s}}{\delta v^n+v^s+\sigma v^{n+s}}\\
&+\sigma C(n,\alpha,s)v^{n+\alpha}\frac{v^{n+s}}{\delta v^n+v^s+\sigma v^{n+s}}\\
\leq&C(n,\alpha,s)\frac{\delta v^n+\sigma v^{n+s}}{\delta v^n+v^s+\sigma v^{n+s}}v^{n+\alpha}\ .
\end{aligned}$$

Using this estimate and Young's inequality we get for all $\mu > 0$

$$\begin{aligned}&\left|\int_{t_1}^{t_2}\int_\Omega \left[\alpha g_{\delta\sigma}(u_{\delta\sigma})\cdot m_{\delta\sigma}(u_{\delta\sigma}) - u_{\delta\sigma}^b\right] D^2 u_{\delta\sigma} : D^2\psi \ dx\ dt\right|\\
\leq&\frac{\mu}{2}\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{b-1}|D^2 u_{\delta\sigma}|^2 \ dx\ dt\\
&+\frac{1}{2\mu}\int_{t_1}^{t_2}\int_\Omega C(n,\alpha,s)\left(\frac{\delta u_{\delta\sigma}^n + \sigma u_{\delta\sigma}^{n+s}}{\delta u_{\delta\sigma}^n + u_{\delta\sigma}^s + \sigma u_{\delta\sigma}^{n+s}}\right)^2 u_{\delta\sigma}^{n+\alpha+1}|D^2\psi|^2 \ dx\ dt\ .\end{aligned}$$

By Vitali's convergence theorem, the second term on the right-hand side tends to zero; the first integral is known to be bounded uniformly. Since $\mu > 0$ is arbitrary, the term on the left-hand side converges to zero.

We have

$$\begin{aligned}&|\alpha g_{\delta\sigma}(v) m'_{\delta\sigma}(v) - n v^{n+\alpha-1}|\\
\leq&\left|v^\alpha\frac{(\delta v^n + v^s + \sigma v^{n+s})(n+s)v^{n+s-1} - v^{n+s}(n\delta v^{n-1} + s v^{s-1} + (n+s)\sigma v^{n+s-1})}{(\delta v^n + v^s + \sigma v^{n+s})^2} - n v^{n+\alpha-1}\right|\\
&+C(n,s,\alpha)(\delta v^{n+\alpha-s} + \sigma v^{n+\alpha})\\
&\qquad\cdot\frac{(\delta v^n + v^s + \sigma v^{n+s})(n+s)v^{n+s-1} + v^{n+s}(n\delta v^{n-1} + s v^{s-1} + (n+s)\sigma v^{n+s-1})}{(\delta v^n + v^s + \sigma v^{n+s})^2}\\
\leq&C(n,s,\alpha)\frac{\delta v^n + \sigma v^{n+s}}{\delta v^n + v^s + \sigma v^{n+s}} v^{n+\alpha-1}\ .\end{aligned}$$

We therefore obtain by Young's inequality

$$\begin{aligned}&\left|\int_{t_1}^{t_2}\int_\Omega \left[\alpha g_{\delta\sigma}(u_{\delta\sigma})\cdot m'_{\delta\sigma}(u_{\delta\sigma}) - n u_{\delta\sigma}^{b-1}\right]\nabla u_{\delta\sigma}\cdot D^2 u_{\delta\sigma}\cdot\nabla\psi \ dx\ dt\right|\\
\leq&\mu\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{n+\alpha-1}|D^2 u_{\delta\sigma}|^2 \ dx\ dt + \mu\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{n+\alpha-3}|\nabla u_{\delta\sigma}|^4 \ dx\ dt\\
&+\frac{C(n,s,\alpha)}{\mu^3}\int_{t_1}^{t_2}\int_\Omega\left(\frac{\delta u_{\delta\sigma}^n + \sigma u_{\delta\sigma}^{n+s}}{\delta u_{\delta\sigma}^n + u_{\delta\sigma}^s + \sigma u_{\delta\sigma}^{n+s}}\right)^4 u_{\delta\sigma}^{n+\alpha+1}|\nabla\psi|^4 \ dx\ dt\end{aligned}$$

and conclude that the term on the left-hand side tends to zero using Vitali's theorem, the uniform bounds on $u_{\delta\sigma}$, and the arbitrariness of μ.

By the finite speed of support propagation property, our estimates survive the passage to the limit $W \to \infty$, i.e. $\Omega_W \to \mathbb{R}^d$. □

3.2. **Suboptimal estimates on support propagation rate.** We now derive a differential inequality for the weighted entropy $\int u^{1+\alpha}(.,t)|x - x_0|^\gamma \ dx$ in order to obtain a first lower bound on support propagation, which however is not yet optimal.

We need the following version of Hardy's inequality:

Lemma 10 (Hardy's inequality)**.** *For $v \in H^1(\mathbb{R}^d)$ with $\operatorname{supp} v \subset\subset \mathbb{R}^d \setminus \{0\}$ and any $\psi \in C^\infty(\mathbb{R}^d \setminus \{0\})$ with $\Delta\psi > 0$ on $\mathbb{R}^d \setminus \{0\}$ the inequality*

$$\int v^2\Delta\psi \ dx \leq 4\int\left|\frac{\nabla\psi}{|\nabla\psi|}\cdot\nabla v\right|^2\frac{|\nabla\psi|^2}{\Delta\psi}\ dx$$

holds.

Proof. Integration by parts and Hölder's inequality give

$$\int v^2\Delta\psi\ dx = -2\int v\nabla v\cdot\nabla\psi\ dx \leq 2\left(\int v^2\Delta\psi\ dx\right)^{\frac{1}{2}}\left(\int \frac{1}{\Delta\psi}|\nabla v\cdot\nabla\psi|^2\ dx\right)^{\frac{1}{2}}.$$

The inequality now follows easily. □

Combining the results of the previous subsection with Hardy's inequality, we shall prove the following lemma:

Lemma 11. *Let $u_0 \in H^1(\mathbb{R}^d)$, $1 \leq d \leq 3$, be compactly supported. Let u be a strong energy solution of the Cauchy problem for the thin-film equation with initial data u_0 and $n \in \left(2-\sqrt{\frac{8}{8+d}}, 3\right)$ or let u be a strong solution of the thin-film equation constructed as in [11] and $n \in [1,2)$.*

Suppose that conditions (H1) to (H4) of Lemma 8 are satisfied and assume that u satisfies the α entropy estimate. Given $\gamma \leq -\frac{1}{2}$, suppose furthermore that

(H5) *The condition*

$$\frac{2b-\frac{1}{2}n}{(b+1)^2}\cdot\frac{(\gamma-4+d)(\gamma+\frac{d-4}{3})}{(\gamma-2)(\gamma-2+d)} - \frac{1}{b+1} \geq \tau$$

is satisfied for some $\tau > 0$.

(H6) *We have $\gamma \leq -d$.*

(H7) *It holds that $\gamma + d + 4\frac{1+\alpha}{n} > 0$.*

Let $x_0 \in \mathbb{R}^d \setminus \operatorname{supp} u_0$ be some point.

Let $t_0 > 0$. Suppose that

$$t_0 \geq \epsilon[\operatorname{diam}(\operatorname{supp} u_0) + \operatorname{dist}(x_0, \operatorname{supp} u_0)]^{4+n\cdot d}||u_0||_{L^1}^{-n}$$

holds for some $\epsilon > 0$. Define

$$T^* := \inf\left\{T \geq 0 : \inf_{0\leq t\leq T}\operatorname{dist}(x_0, \operatorname{supp} u(.,t)) = 0\right\}.$$

Then there exists a constant $C(d,n,\alpha,\epsilon) > 0$ such that the estimate

$$T^* \leq \max\left(2t_0,\ C(d,n,\alpha,\gamma,\epsilon)||u_0||_{L^1}^{\frac{4+4\alpha+n(d+\gamma)}{-\gamma+\alpha\cdot d}}\left[\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)|x-x_0|^\gamma\ dx\right]^{-\frac{4+n\cdot d}{-\gamma+\alpha\cdot d}}\right)$$

holds.

Proof. By our assumptions, Lemma 8 and/or Lemma 9 (depending on n) are applicable.

Let $T < T^*$. The function $|x-x_0|^\gamma$ is smooth in some neighbourhood of the set $\bigcup_{t\in[0,T]}\operatorname{supp} u(.,t)$. By the FSOP estimate Theorem 3, $|x-x_0|^\gamma$ coincides on $\bigcup_{t\in[0,T]}\operatorname{supp} u(.,t)$ with a smooth compactly supported function.

Thus, we may insert $|x-x_0|^\gamma$ in Lemma 8 or Lemma 9. This yields for $t_0 \le t_1 \le t_2 < T^*$

$$\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\ge &\left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int u^{b-1}|\nabla u|^2 \Delta |x-x_0|^\gamma \, dx \, dt \\
&+ \left(\frac{4}{3}b - \frac{1}{3}n\right) \int_{t_1}^{t_2} \int u^{b-1} \nabla u \cdot D^2|x-x_0|^\gamma \cdot \nabla u \, dx \, dt \\
&- \frac{1}{b+1} \int_{t_1}^{t_2} \int u^{b+1} \Delta^2 |x-x_0|^\gamma \, dx \, dt \ .
\end{aligned}$$

We calculate

$$\begin{aligned}
&D^2|x-x_0|^\gamma = D(\gamma |x-x_0|^{\gamma-2}(x-x_0)) \\
=&(\gamma(\gamma-2)|x-x_0|^{\gamma-4}(x-x_0)\otimes(x-x_0) + \gamma|x-x_0|^{\gamma-2} Id) \ .
\end{aligned}$$

Combining the previous formulas, we obtain

$$\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\ge &\gamma \cdot (\gamma - 2 + d) \cdot \left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int u^{b-1}|\nabla u|^2 |x-x_0|^{\gamma-2} \, dx \, dt \\
&+ \gamma(\gamma-1) \left(\frac{4}{3}b - \frac{1}{3}n\right) \int_{t_1}^{t_2} \int u^{b-1} \left| \frac{x-x_0}{|x-x_0|} \cdot \nabla u \right|^2 |x-x_0|^{\gamma-2} \, dx \, dt \\
&+ \gamma \left(\frac{4}{3}b - \frac{1}{3}n\right) \int_{t_1}^{t_2} \int u^{b-1} \nabla u \cdot \left(Id - \frac{x-x_0}{|x-x_0|} \otimes \frac{x-x_0}{|x-x_0|} \right) \cdot \nabla u \ |x-x_0|^{\gamma-2} \, dx \, dt \\
&- \gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d) \frac{1}{b+1} \int_{t_1}^{t_2} \int u^{b+1} |x-x_0|^{\gamma-4} \, dx \, dt \ .
\end{aligned}$$

We now rewrite

$$|\nabla u|^2 = \nabla u \cdot \left(Id - \frac{x-x_0}{|x-x_0|} \otimes \frac{x-x_0}{|x-x_0|} \right) \cdot \nabla u + \left| \frac{x-x_0}{|x-x_0|} \cdot \nabla u \right|^2 \ .$$

Thus we obtain

$$\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\ge &\gamma(3\gamma - 4 + d) \left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int u^{b-1} \left| \frac{x-x_0}{|x-x_0|} \cdot \nabla u \right|^2 |x-x_0|^{\gamma-2} \, dx \, dt \\
&+ \gamma(\gamma+d) \left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int u^{b-1} \nabla u \cdot \left(Id - \frac{x-x_0}{|x-x_0|} \otimes \frac{x-x_0}{|x-x_0|} \right) \cdot \nabla u \ |x-x_0|^{\gamma-2} \, dx \, dt \\
&- \gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d) \frac{1}{b+1} \int_{t_1}^{t_2} \int u^{b+1} |x-x_0|^{\gamma-4} \, dx \, dt \ .
\end{aligned}$$

We may drop the second term on the right-hand side since it is nonnegative by (H6) and (H2) . Thus we get

$$\begin{aligned}&\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx-\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx\\ \geq&4\frac{\gamma(\gamma+\frac{d-4}{3})}{(b+1)^2}\left(2b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int\left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u^{\frac{b+1}{2}}\right|^2|x-x_0|^{\gamma-2}\ dx\ dt\\ &-\gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d)\frac{1}{b+1}\int_{t_1}^{t_2}\int u^{b+1}|x-x_0|^{\gamma-4}\ dx\ dt\ .\end{aligned}$$

An application of Hardy's inequality (Lemma 10) with $\psi=|x-x_0|^{\gamma-2}$ yields

$$\begin{aligned}&\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx-\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx\\ \geq&\gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d)\\ &\cdot\left[\frac{2b-\frac{1}{2}n}{(b+1)^2}\cdot\frac{(\gamma-4+d)(\gamma+\frac{d-4}{3})}{(\gamma-2)(\gamma-2+d)}-\frac{1}{b+1}\right]\int_{t_1}^{t_2}\int u^{b+1}|x-x_0|^{\gamma-4}\ dx\ dt\ .\end{aligned}\tag{4}$$

By (H5) we obtain

$$\begin{aligned}&\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx-\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx\\ \geq&\tau\int_{t_1}^{t_2}\int u^{b+1}|x-x_0|^{\gamma-4}\ dx\ dt\ .\end{aligned}$$

Hölder's inequality gives

$$\begin{aligned}&\int u^{1+\alpha}(.,t)|x-x_0|^\gamma\ dx\\ \leq&\left(\int_{\operatorname{supp}u(.,t)}|x-x_0|^{\gamma+4\frac{1+\alpha}{n}}\ dx\right)^{\frac{n}{b+1}}\left(\int u^{b+1}(.,t)|x-x_0|^{\gamma-4}\ dx\right)^{\frac{1+\alpha}{b+1}}\ .\end{aligned}$$

Let $y\in\operatorname{supp}u_0$. By Theorem 3 we know that $\operatorname{supp}u(.,t)\subset B_{R(t)}(y)$, where $R(t)=\operatorname{diam}(\operatorname{supp}u_0)+C(n,d)||u_0||_{L^1}^{\frac{n}{4+d\cdot n}}t^{\frac{1}{4+d\cdot n}}$. Rearranging the previous inequality, we obtain

$$\begin{aligned}&\int u^{b+1}(.,t)|x-x_0|^{\gamma-4}\ dx\\ \geq&\left(\int_{B_{R(t)+\operatorname{dist}(y,x_0)}(x_0)}|x-x_0|^{\gamma+4\frac{1+\alpha}{n}}\ dx\right)^{-\frac{n}{1+\alpha}}\cdot\left(\int u^{1+\alpha}(.,t)|x-x_0|^\gamma\ dx\right)^{\frac{b+1}{1+\alpha}}\ .\end{aligned}$$

Using (H7) and (4), we therefore arrive at the differential inequality

$$\begin{aligned}&\frac{1}{1+\alpha}\int u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx-\frac{1}{1+\alpha}\int u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx\\ \geq&c(n,\alpha,d,\gamma)\int_{t_1}^{t_2}[R(t)+\operatorname{dist}(y,x_0)]^{-\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}}\cdot\left(\int u^{1+\alpha}(.,t)|x-x_0|^\gamma\ dx\right)^{\frac{b+1}{1+\alpha}}\ dt\ .\end{aligned}$$

Taking into account our assumption on t_0, we obtain for all t_2, t_1 with $t_2 \geq t_1$ and $t_1, t_2 \in [t_0, T)$

$$\frac{1}{1+\alpha}\int u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx - \frac{1}{1+\alpha}\int u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx$$
$$\geq c(n,\alpha,d,\gamma,\epsilon)\int_{t_1}^{t_2}\left[||u_0||_{L^1}^{\frac{n}{4+n\cdot d}}t^{\frac{1}{4+n\cdot d}}\right]^{-\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}}\cdot\left(\int u^{1+\alpha}(.,t)|x-x_0|^\gamma\ dx\right)^{\frac{b+1}{1+\alpha}}\ dt\ .$$

The solution of the differential equation $\frac{d}{dt}z(t) = b(t)\cdot(z(t))^p$, $z(t_0) = a$, is given by $z(t) = (a^{1-p} - (p-1)\int_{t_0}^t b(t)\ dt)^{\frac{1}{1-p}}$; in particular the solution blows up as soon as the term in brackets becomes zero. Set $p := \frac{b+1}{1+\alpha}$. Using the comparison principle, we see that blowup of the weighted entropy $\int u^{1+\alpha}(.,t)|x-x_0|^\gamma\ dx$ occurs before or at time T if the condition

$$c(n,\alpha,d,\gamma,\epsilon)\int_{t_0}^{T}\left[||u_0||_{L^1}^{\frac{n}{4+n\cdot d}}t^{\frac{1}{4+n\cdot d}}\right]^{-\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}}\ dt$$
$$\geq\left(\int u^{1+\alpha}(.,t_0)|x-x_0|^\gamma\ dx\right)^{-\frac{n}{1+\alpha}}$$

is satisfied. This condition is implied by

$$c(n,\alpha,d,\gamma,\epsilon)||u_0||_{L^1}^{-\frac{n}{4+n\cdot d}\cdot\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}}\cdot t^{\frac{\alpha\cdot n\cdot d-\gamma\cdot n}{(1+\alpha)(4+n\cdot d)}}\Big|_{t=t_0}^{t=T}$$
$$\geq\left(\int u^{1+\alpha}(.,t_0)|x-x_0|^\gamma\ dx\right)^{-\frac{n}{1+\alpha}}$$

(note that the exponent at t is nonnegative since $\alpha \in (-1,0]$ and since $\gamma \leq -d$) which in turn in case $T \geq 2t_0$ is implied by

$$T^{\frac{\alpha\cdot n\cdot d-\gamma\cdot n}{(1+\alpha)(4+n\cdot d)}}$$
$$\geq C(n,\alpha,d,\gamma,\epsilon)||u_0||_{L^1}^{\frac{n}{4+n\cdot d}\cdot\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}}\cdot\left(\int u^{1+\alpha}(.,t_0)|x-x_0|^\gamma\ dx\right)^{-\frac{n}{1+\alpha}}\ .$$

This proves our lemma since blowup of the entropy cannot occur before T^* as we have

$$\int u^{1+\alpha}(.,t)|x-x_0|^\gamma\ dx$$
$$\leq\left(\int u(.,t)\ dx\right)^{1+\alpha}\left(\int_{|x-x_0|\geq\operatorname{dist}(x_0,\operatorname{supp}u(.,t))}|x-x_0|^{-\frac{\gamma}{\alpha}}\ dx\right)^{-\alpha}$$

(note that the right-hand side is finite for $t < T^*$ since $\int u(.,t)\ dx = ||u_0||_{L^1}$ and since the second integral is finite due to $\alpha \in (-1,0)$ and $\gamma \leq -d$; for $\alpha = 0$ a similar estimate holds). □

3.3. **Estimate on entropy production.** Our next aim is to bound the entropy $\int u(.,t_0)\ dx$ from below; the estimate for this quantity will provide the starting point for an application of the results from the previous subsection.

Lemma 12. *Assume $d \leq 3$ and $n \geq 1$. Let $u_0 \in H^1(\mathbb{R}^d)$ be compactly supported. Let u either be a strong solution of the Cauchy problem for the thin-film equation constructed as in* [11] *and $n \in [1,2)$ or let u be a strong energy solution of the*

Cauchy problem for the thin-film equation and $n \in \left(2-\sqrt{\frac{8}{8+d}},3\right)$. *Assume that* u *satisfies the* α *entropy estimate, where* $\alpha \in (-1,0)$. *Set* $R_0 := \operatorname{diam}[\operatorname{supp} u_0]$. *For any* $\epsilon > 0$ *the following assertion holds: If the condition*

$$t_0 \geq \epsilon R_0^{4+n\cdot d} ||u_0||_{L^1}^{-n}$$

is satisfied, then there exists a constant $c(d,n,\alpha,\epsilon) > 0$ *such that*

$$\int u^{1+\alpha}(.,t_0)\ dx \geq c(d,n,\alpha,\epsilon)||u_0||_{L^1}^{1+\alpha-\frac{\alpha\cdot n\cdot d}{4+n\cdot d}} t_0^{\frac{-\alpha\cdot d}{4+n\cdot d}}\ .$$

Proof. By the α entropy inequality we have

$$\int u^{1+\alpha}(.,t_0)\ dx \geq \int u_0^{1+\alpha}\ dx + c(\alpha,n)\int_0^{t_0}\int |\nabla u^{\frac{1+n+\alpha}{4}}|^4\ dx\ dt$$

We have $\operatorname{diam}[\operatorname{supp} u(.,t)] \leq C(d,n)(R_0 + ||u_0||_{L^1}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}})$; this is a consequence of Theorem 3. Moreover we have $\int u(.,t)\ dx = ||u_0||_{L^1}$. This implies by the Poincare-Sobolev inequality (note that $4 > d$ which implies that any L^p norm of $u^{\frac{1+n+\alpha}{4}}$ may be estimated in terms of the L^4 norm of $\nabla u^{\frac{1+n+\alpha}{4}}$) that

$$\begin{aligned}
||u_0||_{L^1}^{1+n+\alpha} =& \left(\int u(.,t)\ dx\right)^{1+n+\alpha} \\
\leq& C(d,n,\alpha)\left(R_0 + ||u_0||_{L^1}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right)^{4+(n+\alpha)\cdot d}\int |\nabla u^{\frac{1+n+\alpha}{4}}|^4\ dx\ .
\end{aligned}$$

Putting these inequalities together, we obtain

$$\begin{aligned}
\int u^{1+\alpha}(.,t_0)\ dx \geq& c(d,n,\alpha)||u_0||_{L^1}^{1+n+\alpha}\int_0^{t_0}\left(R_0 + ||u_0||_{L^1}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right)^{-4-(n+\alpha)\cdot d}\ dt \\
\geq& c(d,n,\alpha,\epsilon)||u_0||_{L^1}^{1+n+\alpha}\int_{\frac{t_0}{2}}^{t_0}\left(||u_0||_{L^1}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right)^{-4-(n+\alpha)\cdot d}\ dt \\
\geq& c(d,n,\alpha,\epsilon)||u_0||_{L^1}^{1+\alpha-\frac{\alpha\cdot n\cdot d}{4+n\cdot d}} t_0^{\frac{-\alpha\cdot d}{4+n\cdot d}}
\end{aligned}$$

where in the second step we have used the assumption $t_0 \geq \epsilon R_0^{4+n\cdot d}||u_0||_{L^1}^{-n}$ and in the third step we have used the fact that $\frac{-\alpha\cdot d}{4+n\cdot d} > 0$. □

3.4. **Optimality of estimates on asymptotic support propagation rates.** We are now in position to prove the main result of this paper.

Proof of Theorem 4. Define $r := \operatorname{dist}(x_0, \operatorname{supp} u_0) + \operatorname{diam}(\operatorname{supp} u_0)$. Set $t_0 := r^{4+n\cdot d}||u_0||_{L^1}^{-n}$. For $\alpha \in (-1,0)$, we obtain by Lemma 12

$$\int u^{1+\alpha}(.,t_0)\ dx \geq c(d,n,\alpha)||u_0||_{L^1}^{1+\alpha} r^{-\alpha\cdot d}\ .$$

Let $y \in \operatorname{supp} u_0$. We now know by Theorem 3 that $\operatorname{supp} u(.,t_0) \subset B_{R(t_0)}(y)$, where $R(t_0) = \operatorname{diam}(\operatorname{supp} u_0) + C(d,n)||u_0||_{L^1}^{\frac{n}{4+d\cdot n}} t_0^{\frac{1}{4+d\cdot n}}$. Putting these considerations together, we obtain in case $\gamma < 0$

$$\int u^{1+\alpha}(.,t_0)|x-x_0|^\gamma\ dx \geq c(d,n)r^\gamma\int u^{1+\alpha}(.,t_0)\ dx \geq c(d,n,\alpha)||u_0||_{L^1}^{1+\alpha} r^{-\alpha\cdot d} r^\gamma\ .$$

If we could find $\alpha \in (-1,0]$ and $\gamma < 0$ such that Lemma 11 were applicable (with $\epsilon = 1$), we would obtain the estimate

$$T^* \leq \max\left(2t_0, C(d,n,\alpha,\gamma)||u_0||_{L^1}^{\frac{4+4\alpha+n(d+\gamma)}{-\gamma+\alpha\cdot d}} \left[||u_0||_{L^1}^{1+\alpha} r^{-\alpha\cdot d} r^{\gamma}\right]^{-\frac{4+n\cdot d}{-\gamma+\alpha\cdot d}}\right)$$

which gives

$$T^* \leq C(d,n,\alpha,\gamma)||u_0||_{L^1}^{-n} r^{4+d\cdot n} \ .$$

Since α and γ only depend on n and d, the result would be established.

Thus it remains to find admissible values for α and γ. We first treat the case $n \in (1.5, 2)$. Set $b := \frac{49}{40} + \frac{11}{20}(n - \frac{3}{2}) = \frac{11}{20}n + \frac{16}{40}$ and $\gamma := -d$. This implies $\alpha = \frac{16}{40} - \frac{9}{20}n$. Conditions (H1), (H2), (H3), (H6), (H7) are immediate. Condition (H4) is equivalent to

$$\frac{18n-16}{40} \cdot \frac{2n+16}{40} \cdot \frac{22n-24}{40} \cdot \frac{64-22n}{40} \geq \frac{1}{4}\left(\frac{12n-64}{40}\right)^2 \left(\frac{22n-24}{40}\right)^2$$

which is equivalent to (since $n \geq 1.5$)

$$4(18n-16)(2n+16)(64-22n) \geq (12n-64)^2(22n-24) \ .$$

Factorization (e.g. using a computer algebra system) leads to

$$64(2-n)(99n^2 - 176n + 256) \geq 0 \ .$$

As the second polynomial factor is immediately seen to be strictly positive, condition (H4) is satisfied since $n \leq 2$. Finally, condition (H5) is satisfied for some $\tau > 0$ if the inequality

$$\left(2b - \frac{n}{2}\right) \cdot \frac{(-d-4+d)(-d+\frac{d-4}{3})}{(-d-2)(-d-2+d)} > b + 1$$

holds. Simplifying, this inequality becomes

$$\frac{24n+32}{40} \cdot \frac{4}{3} > \frac{22n+56}{40} \ ,$$

which is satisfied for $n = 2$ and for $n = 1.5$. Thus the inequality is satisfied in the whole interval $(1.5, 2]$ (as the difference of both sides of the inequality is an affine function).

We now deal with the case $n \in (1, 1.5]$. We set $b := \frac{49}{40} + \frac{9}{20}(n - \frac{3}{2}) = \frac{9}{20}n + \frac{22}{40}$ and $\gamma := -d$. This implies $\alpha = \frac{22}{40} - \frac{11}{20}n$. Conditions (H1), (H2), (H3), (H6) and (H7) are verified immediately. Condition (H4) is now equivalent to

$$\frac{22n-22}{40} \cdot \frac{22-2n}{40} \cdot \frac{18n-18}{40} \cdot \frac{58-18n}{40} \geq \frac{1}{4}\left(\frac{28n-88}{40}\right)^2 \left(\frac{18n-18}{40}\right)^2 \ ,$$

which is equivalent to

$$4 \cdot 22 \cdot (22-2n) \cdot (58-18n) \geq (28n-88)^2 \cdot 18 \ .$$

Rearranging the latter inequality, we obtain

$$11 \cdot (11-n) \cdot (29-9n) \geq 9 \cdot (7n-22)^2 \ .$$

The last condition is seen to be equivalent to

$$-342n^2 + 1364n - 847 \geq 0 \ .$$

This condition is true for all $n \in [1, 1.5]$ since for all such n we have $-342n^2 + 1364n - 847 \geq (1364 - 1.5 \cdot 342)n - 847 \geq 1364 - 1.5 \cdot 342 - 847 = 4$. It remains to verify (H5). Condition (H5) is seen to be satisfied for some $\tau > 0$ if

$$\frac{16n+44}{40} \cdot \frac{4}{3} > \frac{18n+62}{40} \ .$$

This inequality holds for $n = 1.5$; for $n = 1$, equality holds. Thus, since the difference of the functions on both sides of the inequality is an affine function, the inequality holds for every $n \in (1, 1.5]$.

Finally, we treat the case $n \in [2, \frac{32}{11})$ (one could prove the theorem for the slightly larger range $n \in \left[2, \frac{2}{9}(10+\sqrt{10})\right)$ e.g. by verifying the conditions (H1) to (H7) using a computer algebra system). In this parameter range, we are fine with the choice $b := \frac{9}{20}n + \frac{12}{20}$, i.e. $\alpha = -\frac{11}{20}n + \frac{12}{20}$, and $\gamma := -d$. For these choices, conditions (H1) to (H4) have been verified in [14]. Conditions (H6) and (H7) are immediate. It remains to check (H5). We see that (H5) is satisfied for some $\tau > 0$ if the inequality

$$\left(2b - \frac{n}{2}\right) \cdot \frac{(-d-4+d)(-d+\frac{d-4}{3})}{(-d-2)(-d-2+d)} > b+1$$

holds. Using our choice of b, this condition becomes

$$\frac{8n+24}{20} \cdot \frac{4}{3} > \frac{9n+32}{20} \ .$$

For $n = 3$, this condition is satisfied; for $n = 2$, it is also satisfied. Both sides of the inequality being affine functions, the inequality holds for all $n \in [2, 3]$. This finishes our proof. □

4. Conclusion

We have shown that the upper bounds on asymptotic support propagation rates for the thin-film equation by Bernis [3], by Hulshof and Shishkov [20], by Bertsch, Dal Passo, Garcke and Grün [6] and by Grün [18] are optimal for any initial data: we have derived lower bounds on asymptotic support propagation rates which coincide with these upper bounds up to a constant factor for any solution of the thin-film equation.

To the best of our knowledge, this is the first nontrivial result on large-time behaviour of solutions of the Cauchy problem for the thin-film equation for $n \neq 1$.

While we have shown that for large times solutions to the thin-film equation display support spreading at the rate suggested by the behaviour of the corresponding self-similar solution, one may hope to prove polynomial decay of any solution to the self-similar solution as done by Carrillo and Toscani [7] in case $n = 1$. However, proving the latter assertion for $n \neq 1$ currently seems out of reach: for $n \neq 1$, an entropy useful for proving decay to the self-similar solution cannot have such a simple structure as in the case $n = 1$, since for $n \neq 1$ there is no explicit formula available for the self-similar solution.

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