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Upper bounds on waiting times for the thin-film equation: the case of weak slippage

Abstract We derive upper bounds on the waiting time of solutions to the thin-film equation in the regime of weak slippage $n \in [2, \frac{32}{11})$. In particular, we give sufficient conditions on the initial data for instantaneous forward motion of the free boundary. For $n \in (2, \frac{32}{11})$, our estimates are sharp, for $n = 2$, they are sharp up to a logarithmic correction term. Note that the case $n = 2$ corresponds - with a grain of salt - to the assumption of the Navier slip condition at the fluid-solid interface. We also obtain results in the regime of strong slippage $n \in (1, 2)$; however, in this regime we expect them not to be optimal. Our method is based on weighted backward entropy estimates, Hardy's inequality and singular weight functions; we deduce a differential inequality which would enforce blowup of the weighted entropy if the contact line were to remain stationary for too long.

1 Introduction

The thin-film equation has been the subject of intensive research during the last two decades. It describes the evolution of a thin liquid film gouverned by the force of surface tension for various slip conditions. The thin-film equation reads

$$u_t = -\nabla \cdot (u^n \nabla \Delta u)$$

where n is a positive real parameter and $\Omega \subset \mathbb{R}^d$. More general versions have also been considered, e.g. with u^n replaced by some nonnegative mobility function $f(u)$ (see e.g. [7,27]).

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For $n = 3$, the thin-film equation can formally be derived using a long-wave approximation of the Navier-Stokes equations with a no-slip boundary condition on the fluid-solid interface; see e.g. [38]. For the Navier slip condition (i.e. the effective boundary condition for laminar viscous flow on a rough surface, see the paper by Jäger and Mikelic [35]), the mobility function $f(u)$ is given (after rescaling) by $u^2 + u^3$ (see e.g. [30]). For $n = 1$, the thin-film equation arises as the lubrication approximation of the Hele-Shaw flow as rigorously proven by Giacomelli and Otto [26].

In order to prevent ill-posedness, one has to impose an additional boundary condition at the free boundary. One typically prescribes the contact angle of solutions. In the present paper, we shall only treat the case of zero contact angle solutions, which also the majority of the mathematical literature is concerned with (e.g. [2,4,5,6,11,14,17,18,19,23,24,25,28,29,31,32,33,34]); i.e. we formally require $|\nabla u| = 0$ on $\partial \operatorname{supp} u(.,t)$. For some results in the case of nonzero contact angle, see [12,36,39]. An overview of the known results for the thin-film equation can be found e.g. in the work of Ansini and Giacomelli [2] or the work of Giacomelli and Shishkov [29].

The thin-film equation may be regarded as a higher-order analogue of the porous medium equation

$$u_t = \nabla \cdot (u^{m-1} \nabla u) \ .$$

Like solutions of the porous medium equation, solutions of the thin-film equation feature the finite speed of support propagation property; see the papers by Bernis [5,6], Hulshof and Shishkov [34], Bertsch, Dal Passo, Garcke, and Grün [11], Grün [32].

Like solutions of the porous medium equation, solutions of the thin-film equation may also exhibit a waiting time phenomenon for certain initial data: if a droplet initially is “flat enough” near its boundary, the contact line of the droplet does not advance for some time before the droplet starts spreading. Sufficient conditions for the occurrence of a waiting time phenomenon have been deduced by Dal Passo, Giacomelli, and Grün [19]; Giacomelli and Grün [23] have derived quantitative lower bounds on waiting times. A formal analysis of the expected waiting time behaviour of solutions of the thin-film equation has been carried out by Blowey, King and Langdon [13]. Their analysis indicates that at least for $n \in (2,3)$, the results of [19] should be optimal: for initial data violating the flatness constraint of [19], one expects instantaneous propagation of the contact line. In particular, for “most” initial data instantaneous forward motion of the free boundary is expected.

However, up to now only *upper bounds on interface propagation* have been proven for the thin-film equation. With the exception of the self-similar solution, for no initial data instantaneous forward motion of the interface has been known to occur. Even worse, considering the evolution of any (non-self-similar) droplet for $d = 1$, it has not even been known whether e.g. the left interface of the droplet ever moves at all.

This lack of knowledge regarding free boundary behaviour has not been limited to the thin-film equation: to the best of our knowledge, no *lower bounds on free boundary propagation* have been available for any higher-order degenerate parabolic PDE. It is therefore of strong interest to prove upper bounds on waiting times and sufficient conditions for instantaneous propagation of the free boundary for such PDE.

The derivation of such estimates in the case of the thin-film equation is the central goal of the present paper. For the thin-film equation in the regime $2 < n < \frac{32}{11}$, we succeed in deriving sufficient conditions on the initial data for instantaneous forward motion of the interface. With a grain of salt, our conditions are the converse of the sufficient conditions for the occurrence of a waiting time phenomenon in [19]. Thus, both our conditions and the conditions in [19] are seen to be optimal. Moreover, in the regime $2 < n < \frac{32}{11}$ we succeed in proving upper bounds on waiting times for solutions to the thin-film equation. Our upper bounds coincide up to a constant factor with the reverse bounds given in [23]; therefore our bounds and the bounds in [23] are optimal up to a constant factor.

In the borderline case $n = 2$ we obtain similar results, which however are only optimal up to a logarithmic correction term. For $n < 2$ we still obtain some results, but they deteriorate quickly as n decreases and are supposedly no longer optimal; see section 6 for details.

Our results are valid in up to three spatial dimensions; in case $d > 1$ we additionally have to assume that the initial free boundary is locally of class C^4.

For second-order equations like the porous medium equation, lower bounds on interface propagation may be derived by the comparison principle; see e.g. [1,3,16].

For fourth-order equations, however, no comparison principle is available and one has to rely on integral estimates. Integral estimates have been used successfully to obtain *upper bounds on interface propagation* for higher-order degenerate parabolic PDE (see e.g. [5,6,11,19,23,32,34,37,40,41,42]). In particular, all the above-mentioned previous results on qualitative behaviour of solutions to the thin-film equation are based on such methods.

So far, there is only one technique known which relies on integral estimates to derive *lower bounds on free boundary propagation*, namely the technique by Chipot and Sideris [15] (see [20] for another application of the technique). Chipot and Sideris argue by contradiction. They test the PDE with a special cutoff and apply Hölder's inequality to the right-hand side of the resulting equation; this yields a differential inequality which entails finite-time blowup of the solution, giving the desired contradiction.

Unfortunately, a direct application of the approach by Chipot and Sideris to the case of the thin-film equation fails due to the structure of the fourth-order operator: when trying to derive the differential inequality, terms with negative sign and terms with indefinite sign appear which cannot be controlled.

The aforementioned analytical difficulties may be one reason why no lower bounds on free boundary propagation for higher-order degenerate parabolic PDE have been available so far.

The main idea of our new technique consists of proving certain new monotonicity formulas for solutions to the thin-film equation which enable us to proceed using the differential inequality technique of Chipot and Sideris. Our monotonicity formulas are weighted backward entropy inequalities with a singular weight. They read (for $d=1$)

$$\frac{d}{dt}\int u^{1+\alpha}|x-x_0|^\gamma\,dx \geq c\int u^{1+n+\alpha}|x-x_0|^{\gamma-4}\,dx \tag{1}$$

with certain $\alpha \in (-1,0)$, $\gamma < 0$. The formulas are valid as long as the support of the solution does not touch the singularity of the weight.

The derivation of our monotonicity formulas relies on new estimates for the evolution of weighted backward entropies. In these estimates, terms with positive sign and terms with negative sign are in competition. Choosing a singular weight which admits a small constant in Hardy's inequality, we are able to show that the positive terms dominate, thereby proving our monotonicity formula. In multiple space dimensions, a naive attempt to generalize the ideas from the one-dimensional case fails. Instead we need to design the weight carefully in such a way that the derivatives of the weight in directions tangent to $\partial\operatorname{supp} u_0$ are much smaller than the derivatives of the weight in direction perpendicular to $\partial\operatorname{supp} u_0$, resulting in an almost monotonicity formula.

Having derived the monotonicity formula (1) (or its multidimensional analogue), we argue by contradiction and estimate the right-hand side of (1) from below in terms of a constant multiple of $(\int u^{1+\alpha}|x-x_0|^\gamma\,dx)^{\frac{1+n+\alpha}{1+\alpha}}$ using Hölder's inequality. The resulting differential inequality then implies finite-time blowup of the weighted entropy $\int u^{1+\alpha}|x-x_0|^\gamma\,dx$, finally yielding a lower bound on interface propagation.

As a last point, let us mention some further (possible) applications of our new technique.

Combining our methods with entropy production estimates, our approach also allows for the derivation of optimal lower bounds on asymptotic support propagation rates for the thin-film equation; see the forthcoming paper [22].

The so-called Derrida-Lebowitz-Speer-Spohn equation is another example of a nonnegativity-preserving fourth-order parabolic equation; in contrast to the thin-film equation, it is nondegenerate. Using a nontrivial adaption of our method, we can show infinite speed of propagation for solutions [21]. In particular, this demonstrates that our method is not restricted to the thin-film equation, but also applicable to other nonlinear higher-order parabolic equations.

Finally, regarding further applications, our method might help in the analysis of the competition between convection and diffusion in the thin-film equation with convection due to gravity (see e.g. the paper by Giacomelli and Shishkov [29]); it might also aid in the analysis of the influence of additional second-order diffusion terms on solutions of the thin-film equation (see e.g. the papers by Bertozzi and Pugh [8,9,10]).

Throughout the paper, we use standard notation for Sobolev spaces. The space of smooth compactly supported functions is denoted by C_c^∞, while C^k denotes the normed space of k times continuously differentiable functions. The space of k times continuously differentiable functions will be denoted by C^k_{loc}. We abbreviate $I := [0,\infty)$. By $L^p_{loc}(I;X)$ we denote the space of all mappings $f : I \to X$ which belong to $L^p([0,T];X)$ for all $T > 0$.

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2 Main Results

In this section, we recall the definition of strong solutions of the thin-film equation which we shall work with; for the reader's convenience, we summarize the main results of the present work.

The strongest concept of solution for which global existence for fairly general initial data is known is characterized by the dissipation relation for the first-order energy

$$\frac{d}{dt}\int |\nabla u|^2 \, dx \leq -c(d,n)\int |\nabla \Delta u^{\frac{n+2}{2}}|^2 + u^{n-2}|\nabla u|^2|D^2u|^2 + |\nabla u^{\frac{n+2}{6}}|^6 \, dx$$

and the dissipation relations for the zeroth-order entropies

$$\frac{d}{dt}\frac{1}{\alpha}\int u^{1+\alpha} \, dx \leq -c(d,\alpha,n)\int |D^2 u^{\frac{1+\alpha+n}{2}}|^2 + |\nabla u^{\frac{1+\alpha+n}{4}}|^4 \, dx$$

for any $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n) \setminus \{0\}$.

Existence of such strong energy solutions (the author decided to use this name in order to distinguish this notion of solution from the weaker notion of strong solution in [18], which lacks the dissipation relation for the first-order energy) of the thin-film equation has been shown by Bernis [6] for the case of the one-dimensional Cauchy problem for $n \in (\frac{1}{2}, 3)$. In case $d = 2$ or $d = 3$, proving existence of these solutions is much more demanding. In this case the proof has been carried out by Grün [33] for $n \in \left(2 - \sqrt{\frac{8}{d+8}}, 3\right)$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain, a bounded $C^{1,1}$ domain, or let $\Omega = \mathbb{R}^d$. Let $u_0 \in H^1(\Omega)$ be nonnegative and have bounded support.

Definition 1 Set $I := [0,\infty)$. We call $u \in L^\infty(I; H^1(\Omega) \cap L^1(\Omega))$, $u \geq 0$, a strong energy solution for the thin-film equation if the following conditions are satisfied:

a) We have $\nabla u^{\frac{n+2}{6}} \in L^6(\Omega \times I)$, $u^{\frac{n-2}{2}} \nabla u \otimes D^2 u \in L^2(\Omega \times I)$, $u^{\frac{n}{2}} \nabla \Delta u \in L^2(\Omega \times I)$.
b) For all $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n)\setminus\{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I; L^2(\Omega))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I; L^4(\Omega))$.
c) It holds that $u \in H^1_{loc}\left(I; \left(W^{1,p}(\Omega)\right)'\right)$ for all $p > \frac{4d}{2d+n(2-d)}$.
d) For any $\xi \in L^2(I; W^{1,\infty}_0(\Omega))$ and any $T > 0$ it holds that

$$\int_0^T \langle u_t, \xi \rangle \, dt = \int_0^T \int_{\{u>0\}} u^n \nabla \Delta u \cdot \nabla \xi \, dx \, dt \; .$$

e) u attains its initial data u_0 in the sense that $\lim_{t\to 0} u(.,t) = u_0(.)$ in $L^1(\Omega)$.

Besides $d \leq 3$ and $n \in \left(2-\sqrt{\frac{8}{8+d}}, 3\right)$, the existence result due to Grün [33] required only that Ω be bounded and convex with smooth boundary or that $\Omega = \mathbb{R}^d$. Thus, for such Ω and any nonnegative initial data $u_0 \in H^1(\Omega)$ with bounded support, there exists a strong energy solution u of the thin-film equation.

The notion of "waiting time" refers to the phenomenon that depending on the initial data, it may happen that the free boundary of a solution of a degenerate parabolic equation does not advance in the neighbourhood of some point $x_0 \in \partial \operatorname{supp} u_0$ for some time T^*. More precisely, we define:

Definition 2 Let $u \in L^\infty(I; H^1(\Omega)) \cap H^1_{loc}(I; (W^{1,p}(\Omega))')$ be a solution of the thin-film equation and let $x_0 \in \partial \operatorname{supp} u_0$ be some point. We then define the waiting time T^* of u at x_0 as

$$T^* := \lim_{\epsilon \to 0} \inf\{t > 0 : \operatorname{supp} u(.,t) \cap B_\epsilon(x_0) \not\subset \operatorname{supp} u_0 \cap B_\epsilon(x_0)\} \ .$$

Note that the regularity $u \in L^\infty(I; H^1(\Omega)) \cap H^1_{loc}(I; (W^{1,p}(\Omega))')$ implies that $u \in C^0_{loc}(I; L^2(V))$ for any bounded open set $V \subset \Omega$ with smooth boundary; see e.g. Corollary 4 in [43]. Thus the essential support $\operatorname{supp} u(.,t)$ is well-defined for all $t \geq 0$.

In the one-dimensional case, our main result reads as follows:

Theorem 1 *Let $d = 1$ and $x_0 \in \mathbb{R}$. Let u be a strong energy solution of the Cauchy problem for the thin-film equation with compactly supported nonnegative initial data $u_0 \in H^1(\mathbb{R})$.*

a) Suppose $n \in \left(2, \frac{32}{11}\right)$. Assume $\operatorname{supp} u_0 \cap (-\infty, x_0) = \emptyset$. Then there exist constants $\alpha \in (-1, -\frac{1}{2})$ with $\alpha + n < 2$ and $C > 0$ which depend only on n such that the quantity $T := \inf\{t > 0 : (-\infty, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset\}$ is bounded by

$$T \leq C(n) \inf_{\epsilon > 0} \epsilon^{4-\frac{n}{1+\alpha}} \left[\int_{\mathbb{R}} u_0^{1+\alpha} |x - x_0 + \epsilon|^{-2} \ dx\right]^{-\frac{n}{1+\alpha}} \ .$$

b) Suppose $n \in \left(2, \frac{32}{11}\right)$. Assume $\operatorname{supp} u_0 \cap (x_0 - \delta, x_0) = \emptyset$ for some $\delta > 0$ and $x_0 \in \partial \operatorname{supp} u_0$. Then there exist constants $\alpha \in (-1, -\frac{1}{2})$ and $C > 0$ which depend only on n and satisfy $\alpha + n < 2$ such that the waiting time T^* at x_0 is bounded by

$$T^* \leq C(n) \left[\limsup_{\epsilon \to 0} \fint_{(x_0, x_0+\epsilon)} \left[\frac{1}{\epsilon^{\frac{4}{n}}} u_0\right]^{1+\alpha} \ dx\right]^{-\frac{n}{1+\alpha}} \ .$$

c) Assume that $n = 2$. Let $x_0 \in \partial \operatorname{supp} u_0$ be a point such that $(x_0 - \delta, x_0) \cap \operatorname{supp} u_0 = \emptyset$ holds for some $\delta > 0$. Then the waiting time T^* at x_0 is bounded by

$$T^* \leq C \left[\limsup_{\epsilon \to 0} \frac{1}{|\log \epsilon|^{\frac{5}{4}}} \int_{(x_0+\epsilon, \infty)} u_0^{\frac{1}{2}} \ |x - x_0|^{-2} \ dx\right]^{-4} \ .$$

Note that $T^* = 0$ entails instantaneous forward motion of the interface near x_0; therefore our theorem includes both upper bounds on waiting times and sufficient conditions for immediate forward motion of the contact line.

As a corollary, one obtains easily:

Corollary 1 *Suppose $d = 1$. Let u be a strong energy solution of the Cauchy problem for the thin-film equation with compactly supported nonnegative initial data $u_0 \in H^1(\mathbb{R})$. Let a point $x_0 \in \partial \operatorname{supp} u_0$ be given such that $\operatorname{supp} u_0 \cap (x_0 - \delta, x_0) = \emptyset$ holds for some $\delta > 0$.*

a) Let $n \in \left(2, \frac{32}{11}\right)$. If

$$u_0(x) \geq \tilde{S} \cdot (x - x_0)_+^{\frac{4}{n}}$$

is satisfied for all $x \in B_\epsilon(x_0)$ for some $\epsilon > 0$, then the waiting time T^* at x_0 is bounded from above by

$$T^* \leq C(n) \tilde{S}^{-n} \ .$$

b) Let $n \in \left(2, \frac{32}{11}\right)$. If

$$\lim_{x \searrow x_0} \frac{u_0(x)}{(x - x_0)_+^{\frac{4}{n}}} = \infty$$

holds, then the interface at x_0 starts moving forward instantaneously.

c) Let $n = 2$. If

$$u_0(x) \geq \tilde{S} \cdot (x - x_0)_+^2 |\log(x - x_0)_+|^{\frac{1}{2}}$$

is satisfied for all $x \in B_\epsilon(x_0)$ for some $\epsilon > 0$, then the waiting time T^* at x_0 is bounded from above by

$$T^* \leq C \tilde{S}^{-2} \ .$$

d) Let $n = 2$. If we have

$$\lim_{x \searrow x_0} \frac{u_0(x)}{(x - x_0)_+^2 \, |\log(x - x_0)_+|^{\frac{1}{2}}} = \infty \ ,$$

then the interface at x_0 starts moving forward instantaneously.

Of course, analogous assertions hold in the mirrored case $x_0 \in \partial \operatorname{supp} u_0$, $\operatorname{supp} u_0 \cap (x_0, x_0 + \delta) = \emptyset$. Note that the bounds on the waiting time do not depend on the size of the neighbourhood in which the growth condition is satisfied.

In the case of several spatial dimensions, we obtain the following result:

Theorem 2 *Let u be a strong energy solution of the thin-film equation on some domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with nonnegative initial data $u_0 \in H^1(\Omega)$. Assume that $\operatorname{supp} u_0$ is bounded. Let M denote the closure of some domain with boundary of class C^4. Suppose that $\operatorname{supp} u_0 \subset M$. Assume that there exists $x_0 \in \partial M \cap \partial \operatorname{supp} u_0 \cap \Omega$.*

Define H to be the tangent plane of ∂M at x_0 and let P denote the orthogonal projection onto H. Abbreviate $\operatorname{dist}_C(x, x_0) := \max(|Px - x_0|, \operatorname{dist}(x, H))$.

a) Let $n \in \left(2, \frac{32}{11}\right)$*. Then there exists a constant* $\alpha \in (-1, -\frac{1}{2})$ *depending only on* n *with* $n+\alpha < 2$ *such that the following holds: Provided that we have*

$$W := \limsup_{r\to 0} \limsup_{h\to 0} \fint_{\{x:\mathrm{dist}_C(x,x_0)<r, \mathrm{dist}(x,\partial M)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx > 0 \ ,$$

the waiting time T^* *of* u *at* x_0 *is bounded from above by*

$$T^* \leq C(d,n) W^{-\frac{n}{1+\alpha}} \ .$$

b) Let $n = 2$*. If we have*

$$\limsup_{h\to 0} \fint_{\{x:\mathrm{dist}_C(x,x_0)<\frac{1}{|\log h|}, \mathrm{dist}(x,\partial M)<h\}} \left[\frac{1}{|\log h|^{14+2d} h^2} u_0\right]^{\frac{1}{2}} dx > 0 \ ,$$

then u *has no waiting time at* x_0*.*

The following corollary follows easily:

Corollary 2 *Let* u *be a strong energy solution of the thin-film equation on a domain* $\Omega \subset \mathbb{R}^d$*,* $d \leq 3$*, with nonnegative initial data* $u_0 \in H^1(\Omega)$*. Assume that* $\operatorname{supp} u_0$ *is bounded. Let* $x_0 \in \partial \operatorname{supp} u_0$ *be some point with the property that in some neighbourhood of* x_0*,* $\operatorname{supp} u_0$ *is the closure of a* C^4 *domain.*

a) Suppose $n \in \left(2, \frac{32}{11}\right)$*. Provided that there exist constants* $r > 0$*,* $\tilde{S} > 0$*, such that for any* $x \in B_r(x_0) \cap \operatorname{supp} u_0$ *we have*

$$u_0(x) \geq \tilde{S} \cdot \mathrm{dist}(x, \partial \operatorname{supp} u_0)^{\frac{4}{n}} \ ,$$

the waiting time T^* *of* u *at* x_0 *is bounded from above by*

$$T^* \leq C(d,n) \tilde{S}^{-n} \ .$$

b) Suppose that $n \in \left(2, \frac{32}{11}\right)$*. Set* $A := (\operatorname{supp} u_0)^\circ$*. If we have*

$$\lim_{A \ni x \to x_0} \frac{u_0(x)}{\mathrm{dist}(x, \partial \operatorname{supp} u_0)^{\frac{4}{n}}} = \infty \ ,$$

then the interface at x_0 *starts moving forward instantaneously.*

c) Let $n = 2$*. Provided that there exist constants* $r > 0$*,* $\tilde{S} > 0$ *such that for any* $x \in B_r(x_0) \cap \operatorname{supp} u_0$ *we have*

$$u_0(x) \geq \tilde{S} \cdot |\log \mathrm{dist}(x, \partial \operatorname{supp} u_0)|^{14+2d} \cdot \mathrm{dist}(x, \partial \operatorname{supp} u_0)^2 \ ,$$

u *has no waiting time at* x_0*.*

Remark 1 Note that for $n = 2$, the growth condition on u_0 known to be sufficient for the nonexistence of a waiting time is a bit stricter in the multidimensional case than in the one-dimensional case. This is likely due to a limitation of our technique. The author is not entirely sure whether the one-dimensional result represents the optimal growth condition either. However, the condition is of course optimal up to some logarithmic factor.

The proof for the multidimensional case also applies to the one-dimensional situation, thereby providing upper bounds on waiting times for solutions of the thin-film equation on domains $\Omega \neq \mathbb{R}$. However, as the proof is much more technical, we prefer to give the proof in the case of the one-dimensional Cauchy problem separately.

In the regime of strong slippage $n \in (1,2)$, we obtain:

Theorem 3 *Let $d = 1$, $n \in (1,2)$ and let $u_0 \in L^1(\mathbb{R})$ be nonnegative and compactly supported. Let u be a solution of the Cauchy problem for the thin-film equation with weak initial trace u_0 constructed as in [17] as the limit of a certain approximating sequence u_δ. Suppose additionally that this approximating sequence satisfies $u_\delta(.,0) \to u_0$ strongly in $L^1(\mathbb{R})$. Let $x_0 \in \partial \operatorname{supp} u_0$ be the point with $(-\infty, x_0) \cap \operatorname{supp} u_0 = \emptyset$.*

Then there exist constants $\alpha \in (-\frac{1}{2}, 0)$ and $\beta \in (0,2)$ depending only on n such that the following assertion holds: if there exists $\tau > 0$ with

$$\limsup_{\epsilon \to 0} \fint_{(x_0, x_0+\epsilon)} \left[u_0(x)\epsilon^{-\beta+\tau}\right]^{1+\alpha} \, dx > 0 \ ,$$

then u has no waiting time at x_0, i.e. $\inf\{t > 0 : \operatorname{supp} u(.,t) \cap (-\infty, x_0) \neq \emptyset\} = 0$.

As n approaches 2, the constant α tends to $-\frac{1}{2}$ and the constant β tends to 2. For n approaching 1, both α and β tend to 0.

See the end of Section 6 at the end of the paper for a discussion of our results.

3 Evolution of the Weighted Entropy

Our technique is based on weighted "backward" entropy estimates, i.e. estimates on the evolution of certain weighted entropies $\int u^{1+\alpha}\psi \, dx$ for $-1 < \alpha < 0$. For our approach to work we need the precise equation for the evolution of the entropy, which cannot be found in the literature. Therefore we first derive this equation, starting with the weak formulation of the thin-film equation and making use of the additional regularity of strong energy solutions.

Lemma 1 *Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be an arbitrary domain. Let u be a strong energy solution of the thin-film equation. Suppose that* $\operatorname{supp} u_0$ *is bounded.*

Assume $-1 < \alpha \leq 0$ *and* $1 \leq n < 3$*. Defining* $b := n + \alpha$*, the formula*

$$
\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
=&\left(b - \frac{1}{2}n\right) \int_{t_1}^{t_2} \int u^{b-1}|\nabla u|^2 \Delta\psi + b \int_{t_1}^{t_2} \int u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u \\
&- \frac{1}{b+1} \int_{t_1}^{t_2} \int u^{b+1} \Delta^2 \psi + (n-b) \int_{t_1}^{t_2} \int u^{b-1} |D^2 u|^2 \psi \\
&+ \left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int u^{b-3} |\nabla u|^4 \psi \\
&+ (2n - 3b)(b-1) \int_{t_1}^{t_2} \int u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi \\
&+ \left(\frac{n}{2} - b\right)(b-1) \int_{t_1}^{t_2} \int u^{b-2} |\nabla u|^2 \Delta u\ \psi
\end{aligned}
\tag{2}
$$

holds for any $\psi \in C_c^\infty(\Omega)$ *and a.e.* $t_1, t_2 > 0$ *and a.e.* $t_2 > 0$ *in case* $t_1 = 0$.

Note that we have chosen a representation of the right-hand side which involves (at least for $d = 1$ and weights ψ satisfying $\psi, \psi_{xx}, \psi_{xxxx} \geq 0$) only a single term of unknown sign.

The proof of the formula (2) is very technical; for the reader's convenience, we here only give a formal derivation. A rigorous derivation can be found in the appendix.

We formally insert $\phi := u^\alpha \psi$ with $\psi \in C_c^\infty(\Omega)$ as a test function in the thin-film equation. Then we perform repeated formal integrations by parts; more precisely, we integrate by parts in all terms that do not have a definite sign (with the exception of terms of the form $\int u^{b-1} \nabla u \cdot D^2 u \cdot \nabla u\ \psi\ dx$ and $\int u^{b-1} |\nabla u|^2 \Delta u\ \psi\ dx$, which are left unchanged) until no more such integrations by parts are possible. This yields

$$
\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}\psi \, dx \Big|_{t_1}^{t_2} \\
=&\int_{t_1}^{t_2}\int u^b \nabla\Delta u\cdot\nabla\psi + \alpha\int_{t_1}^{t_2}\int u^{b-1}\nabla\Delta u\cdot\nabla u\ \psi \\
=&-b\int_{t_1}^{t_2}\int u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi - \int_{t_1}^{t_2}\int u^b D^2u : D^2\psi \\
&+(n-b)\int_{t_1}^{t_2}\int u^{b-1}|D^2u|^2\psi \\
&+(n-b)(b-1)\int_{t_1}^{t_2}\int u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi + (n-b)\int_{t_1}^{t_2}\int u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi \\
=&\frac{1}{2}b(b-1)\int_{t_1}^{t_2}\int u^{b-2}|\nabla u|^2\nabla u\cdot\nabla\psi + \frac{1}{2}b\int_{t_1}^{t_2}\int u^{b-1}|\nabla u|^2\Delta\psi \\
&+b\int\int u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u + \int_{t_1}^{t_2}\int u^b\nabla u\nabla\Delta\psi \\
&+(n-b)\int_{t_1}^{t_2}\int u^{b-1}|D^2u|^2\psi + (n-b)(b-1)\int_{t_1}^{t_2}\int u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi \\
&-\frac{1}{2}(n-b)\int_{t_1}^{t_2}\int u^{b-1}|\nabla u|^2\Delta\psi - \frac{1}{2}(n-b)(b-1)\int_{t_1}^{t_2}\int u^{b-2}|\nabla u|^2\nabla u\cdot\nabla\psi \\
=&\left(\frac{1}{2}(n-b)-\frac{1}{2}b\right)(b-1)(b-2)\int\int u^{b-3}|\nabla u|^4\psi \\
&+\left(-\frac{1}{2}b+\frac{1}{2}(n-b)\right)(b-1)\Bigg(2\int_{t_1}^{t_2}\int u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi \\
&\qquad\qquad\qquad + \int_{t_1}^{t_2}\int u^{b-2}|\nabla u|^2\Delta u\ \psi\Bigg) \\
&+(n-b)\int_{t_1}^{t_2}\int u^{b-1}|D^2u|^2\psi + \left(b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int u^{b-1}|\nabla u|^2\Delta\psi \\
&+b\int_{t_1}^{t_2}\int u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u \\
&-\frac{1}{b+1}\int_{t_1}^{t_2}\int u^{b+1}\Delta^2\psi + (n-b)(b-1)\int_{t_1}^{t_2}\int u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi \\
=&\left(b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int u^{b-1}|\nabla u|^2\Delta\psi + b\int_{t_1}^{t_2}\int u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u \\
&-\frac{1}{b+1}\int\int u^{b+1}\Delta^2\psi \\
&+(n-b)\int_{t_1}^{t_2}\int u^{b-1}|D^2u|^2\psi + \left(b-\frac{n}{2}\right)(b-1)(2-b)\int_{t_1}^{t_2}\int u^{b-3}|\nabla u|^4\psi \\
&+(2n-3b)(b-1)\int_{t_1}^{t_2}\int u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi \\
&+\left(\frac{n}{2}-b\right)(b-1)\int_{t_1}^{t_2}\int u^{b-2}|\nabla u|^2\Delta u\ \psi\ ,
\end{aligned}
$$

which is just the desired formula.

4 The Case of One Spatial Dimension

We now proceed to the proof of our main results in the case of one spatial dimension. While already containing the main new ideas, the proofs in the one-dimensional case are much less technical than in the multidimensional case; for the reader's convenience, we have therefore chosen to present them separately.

Our first goal is to simplify the equation (2) by showing nonnegativity of the sum of the last four terms in (2) using Young's inequality. We set $b := n + \alpha$ and introduce the following conditions:

(H1) Assume that $1 \leq b \leq 2$.
(H2) Suppose that $\frac{n}{2} \leq b \leq n$.
(H3) Assume that $n - 1 < b$.
(H4) Suppose that the inequality

$$(n-b)\left(b-\frac{n}{2}\right)(b-1)(2-b) \geq \frac{1}{4}\left[\left(\frac{5n}{2}-4b\right)(b-1)\right]^2$$

is satified.

The set of $(b, n) \in \mathbb{R} \times \mathbb{R}$ for which (H1) to (H4) are satisfied is depicted in Section 6. We obtain the following lemma:

Lemma 2 *Let $n \in [1, 3)$, $\alpha \in (-1, 0]$, and let u be a strong energy solution of the thin-film equation on a domain $\Omega \subset \mathbb{R}$ with initial data $u_0 \in H^1(\Omega)$. Assume that $\operatorname{supp} u_0$ is bounded. Set $b := n+\alpha$ and assume that (H1) to (H4) are satisfied. Let $\psi \in C_c^4(\Omega)$. Assume $\psi \geq 0$. Then for a.e. $t_1, t_2 \in [0, \infty)$ with $t_2 \geq t_1$ and for a.e. $t_2 \in [0, \infty)$ in case $t_1 = 0$ we have*

$$\begin{aligned}&\int \frac{1}{1+\alpha} u^{1+\alpha}(., t_2)\psi(.)\, dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(., t_1)\psi(.)\, dx\\ &\geq \left(2b - \frac{1}{2}n\right)\int_{t_1}^{t_2}\int u^{b-1}|u_x|^2\psi_{xx} - \frac{1}{b+1}\int_{t_1}^{t_2}\int u^{b+1}\psi_{xxxx}\ .\end{aligned}$$

Proof Lemma 1 implies that for $\psi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}&\int \frac{1}{1+\alpha} u^{1+\alpha}(., t_2)\psi(.)\, dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(., t_1)\psi(.)\, dx\\ &= \left(2b - \frac{1}{2}n\right)\int_{t_1}^{t_2}\int u^{b-1}|u_x|^2\psi_{xx} - \frac{1}{b+1}\int_{t_1}^{t_2}\int u^{b+1}\psi_{xxxx}\\ &\quad + (n-b)\int_{t_1}^{t_2}\int u^{b-1}|u_{xx}|^2\psi + \left(b-\frac{n}{2}\right)(b-1)(2-b)\int_{t_1}^{t_2}\int u^{b-3}|u_x|^4\psi\\ &\quad + \left(\frac{5}{2}n - 4b\right)(b-1)\int_{t_1}^{t_2}\int u^{b-2}|u_x|^2 u_{xx}\ \psi\end{aligned}$$

for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$. By (H1), (H2), and (H4), Young's inequality shows that the sum of the last three terms is nonnegative. By an approximation argument, the inequality carries over to the case $\psi \in C_c^4(\Omega)$. □

An application of Hardy's inequality to the right-hand side of the estimate obtained in the previous lemma and a careful choice of the test function ψ will enable us to derive upper bounds on waiting times for the thin-film equation for $n \in [2, \frac{32}{11})$.

Lemma 3 (Hardy's inequality) *For $v \in H^1(\mathbb{R})$ with $\operatorname{supp} v \subset\subset \mathbb{R} \setminus \{0\}$ and any $\psi \in C^\infty_{loc}(\mathbb{R} \setminus \{0\})$ with $\psi_{xx} > 0$ on $\mathbb{R} \setminus \{0\}$ the inequality*

$$\int v^2 \psi_{xx}\ dx \leq 4 \int |v_x|^2 \frac{|\psi_x|^2}{\psi_{xx}}\ dx$$

holds.

Proof We calculate

$$\int v^2 \psi_{xx}\ dx = -2 \int v v_x \psi_x\ dx \leq 2 \left(\int |v_x|^2 \frac{|\psi_x|^2}{\psi_{xx}}\ dx \right)^{\frac{1}{2}} \left(\int v^2 \psi_{xx}\ dx \right)^{\frac{1}{2}} .$$

The desired inequality follows. □

Lemma 4 *Let $d = 1$, $n \in [1, 3)$, $\alpha \in (-1, 0]$ and let u be a strong energy solution of the Cauchy problem (i.e. $\Omega = \mathbb{R}$) for the thin-film equation with compactly supported initial data $u_0 \in H^1(\Omega)$. Assume that conditions (H1), (H2), (H3), (H4) (preceding Lemma 2) are satisfied. Set $b := n + \alpha$. Given $\gamma \leq -1$, suppose furthermore that*

(H5) The condition

$$\left(2b - \frac{1}{2} n\right) \frac{\gamma - 3}{(b+1)^2 (\gamma - 2)} - \frac{1}{b+1} \geq \tau$$

is satisfied for some $\tau > 0$.

Let $x_1 \notin \operatorname{supp} u_0$, $\epsilon > 0$, $T > 0$. Define $K := \overline{\bigcup_{t \in [0,T]} \operatorname{supp} u(.,t)}$. Then in case

$$T \geq \frac{1}{n\tau} \left[\int_{K \setminus (x_1 - \epsilon, x_1 + \epsilon)} |x - x_1|^{\gamma + 4 \frac{1+\alpha}{n}}\ dx \right]^{\frac{n}{1+\alpha}} \left[\int u_0^{1+\alpha} |x - x_1|^\gamma\ dx \right]^{-\frac{n}{1+\alpha}}$$

we have $\operatorname{dist}(x_1, K) < \epsilon$.

Proof By our assumptions, Lemma 2 is applicable.

We argue by contradiction. Suppose that $B_\epsilon(x_1) \cap K = \emptyset$. Hardy's inequality (Lemma 3) applied with $\psi = \frac{d^2}{dx^2} |x - x_1|^\gamma$ reads

$$\int v^2 {|x - x_1|^\gamma}_{xxxx}\ dx \leq \frac{4(\gamma - 2)}{\gamma - 3} \int |\nabla v|^2 {|x - x_1|^\gamma}_{xx}\ dx .$$

We now use $\psi := |x-x_1|^\gamma$ as a test function in Lemma 2; this is possible since ψ is C^∞ on some neighborhood of the compact set K. Rewriting $\int u^{b-1}|u_x|^2\psi_{xx} = \frac{4}{(b+1)^2}\int |(u^{\frac{b+1}{2}})_x|^2\psi_{xx}$ and using the previous inequality we therefore obtain

$$\begin{aligned}
&\int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_1|^\gamma\,dx - \int \frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_1|^\gamma\,dx\\
\geq&\left(\left(2b-\frac{1}{2}n\right)\frac{\gamma-3}{(b+1)^2(\gamma-2)}-\frac{1}{b+1}\right)\int_{t_1}^{t_2}\int u^{b+1}|x-x_1|^\gamma{}_{xxxx}\,dx\\
=&\left(\left(2b-\frac{1}{2}n\right)\frac{\gamma-3}{(b+1)^2(\gamma-2)}-\frac{1}{b+1}\right)\\
&\quad\cdot\gamma(\gamma-1)(\gamma-2)(\gamma-3)\int_{t_1}^{t_2}\int u^{b+1}|x-x_1|^{\gamma-4}\,dx\\
\geq&\tau\int_{t_1}^{t_2}\int u^{b+1}|x-x_1|^{\gamma-4}\,dx
\end{aligned}$$

where in the last step we have used condition (H5) and $\gamma\leq -1$. Note that this estimate is precisely the desired monotonicity formula (1).

Now notice that due to Hölder's inequality one can estimate

$$\begin{aligned}
&\int u^{1+\alpha}|x-x_1|^\gamma\,dx\\
\leq&\left(\int u^{b+1}|x-x_1|^{\gamma-4}\,dx\right)^{\frac{1+\alpha}{b+1}}\left(\int_{\operatorname{supp} u}|x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right)^{\frac{n}{b+1}}
\end{aligned}$$

where we have used the definition $b=\alpha+n$. Putting these estimates together, using (H3) which implies $\alpha>-1$ we arrive at the differential inequality

$$\begin{aligned}
&\int u^{1+\alpha}(.,t_2)|x-x_1|^\gamma\,dx - \int u^{1+\alpha}(.,t_1)|x-x_1|^\gamma\,dx\\
\geq&(1+\alpha)\tau\left[\int_{K\cap(\mathbb{R}\setminus B_\epsilon(x_1))}|x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{-\frac{n}{1+\alpha}}\\
&\qquad\cdot\int_{t_1}^{t_2}\left(\int u^{1+\alpha}|x-x_1|^\gamma\,dx\right)^{\frac{1+b}{1+\alpha}}\,dt
\end{aligned}$$

where we have the assumption $B_\epsilon(x_1)\cap K=\emptyset$.

The solution of the differential equation $\frac{d}{dt}z(t)=q\cdot[z(t)]^m$ is given by $z(t)=[z(0)^{1-m}-(m-1)\cdot q\cdot t]^{\frac{1}{1-m}}$. Using the comparison principle, we therefore obtain blow-up of the quantity $\int u^{1+\alpha}(.,t)|x-x_1|^\gamma\,dx$ by no later than

$$T^*=\frac{1}{n\tau}\left[\int_{K\cap(\mathbb{R}\setminus B_\epsilon(x_1))}|x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{\frac{n}{1+\alpha}}\left[\int u_0^{1+\alpha}|x-x_1|^\gamma\,dx\right]^{-\frac{n}{1+\alpha}}.$$

As $\int u(.,t)\,dx=\int u_0\,dx<\infty$ and $\alpha\in(-1,0)$ as well as $\gamma\leq -1$, by Hölder's inequality we see that $\int_{\mathbb{R}\setminus B_\epsilon(x_1)}u^{1+\alpha}|x-x_1|^\gamma\,dx$ must remain bounded. Therefore we have obtained the desired contradiction. □

We are now in position to prove the main theorem in the one-dimensional case.

Proof (Theorem 1) Assertion a) is an easy consequence of the previous lemma: we choose $b := \frac{9}{20}n + \frac{12}{20}$, i.e. $\alpha = -\frac{11}{20}n + \frac{12}{20}$, and $\gamma := -2$. Condition (H3) is then equivalent to $\frac{11}{20}n < \frac{32}{20}$, i.e. $n < \frac{32}{11}$. Condition (H1) is seen to be satisfied for any $n \in [1,3)$. Condition (H2) is satisfied for $n \in (\frac{12}{11}, 12)$. Condition (H5) is also satisfied for some $\tau = \tau(n)$ since for $\gamma = -2$ it is equivalent to $\frac{5}{4}(2b - \frac{1}{2}n) - (b+1) > 0$ which in turn is equivalent to $5(\frac{8}{20}n + \frac{24}{20}) - 4(\frac{9}{20}n + \frac{32}{20}) > 0$, i.e. $40n + 120 - 36n - 128 > 0$. The latter condition reduces to $4n > 8$, i.e. $n > 2$. It remains to check condition (H4). This condition now reads

$$\begin{aligned}
&\left(\frac{11}{20}n - \frac{12}{20}\right)\left(-\frac{1}{20}n + \frac{12}{20}\right)\left(\frac{9}{20}n - \frac{8}{20}\right)\left(\frac{28}{20} - \frac{9}{20}n\right) \\
\geq &\frac{1}{4}\left[\left(\frac{14}{20}n - \frac{48}{20}\right)\left(\frac{9}{20}n - \frac{8}{20}\right)\right]^2 .
\end{aligned}$$

Using a computer algebra program (or doing the calculations by hand), one can check that (H5) is therefore equivalent to

$$\frac{3(n-2)(9n-8)(n(188-57n)-48)}{80\ 000} \geq 0 .$$

Calculating the roots of the third polynomial factor in this expression, Condition (H3) is therefore satisfied as long as $n \in [2, \frac{2(47+5\sqrt{61})}{57})$, i.e. especially for $n \in [2,3)$.

Now that we have checked the assumptions of Lemma 4, to finish the proof of assertion a) we apply this Lemma with $x_1 := x_0 - \epsilon$, where ϵ is the parameter of the lemma. Evaluating the first integral in the estimate from Lemma 4, we obtain that

$$\begin{aligned}
&\inf\{t : \operatorname{supp} u(.,t) \cap (-\infty, x_0) \neq 0\} \\
\leq &\inf\{t : \operatorname{supp} u(.,t) \cap (x_1 - \epsilon, x_1 + \epsilon) \neq 0\} \\
\leq &\frac{1}{n\tau}\left[\int_{\mathbb{R}\setminus(x_1-\epsilon,x_1+\epsilon)} |x - x_1|^{\gamma+4\frac{1+\alpha}{n}}\ dx\right]^{\frac{n}{1+\alpha}} \left[\int u_0^{1+\alpha}|x - x_1|^{\gamma}\ dx\right]^{-\frac{n}{1+\alpha}} \\
\leq &C(n)\left[-\frac{2}{1+\gamma+4\frac{1+\alpha}{n}}|\epsilon|^{1+\gamma+4\frac{1+\alpha}{n}}\right]^{\frac{n}{1+\alpha}} \left[\int u_0^{1+\alpha}|x - x_0 + \epsilon|^{\gamma}\ dx\right]^{-\frac{n}{1+\alpha}} ,
\end{aligned}$$

where we have used the fact that $\alpha < -\frac{1}{2}$, $\gamma = -2$, $n > 2$, which implies $\gamma + 4\frac{1+\alpha}{n} < -1$. This proves assertion a) since $\epsilon > 0$ was arbitrary.

Note that we could prove the theorem for the slightly larger range $n \in (2, \frac{2}{9}(10 + \sqrt{10}))$ (instead of $n \in (2, \frac{32}{11})$) using either a computer algebra system or more tedious computations to solve the inequalities $(H1)$ to $(H5)$. See Section 6 below for a plot of the set of admissible pairs (n, b).

Assertion b) is shown just as assertion a), the only difference being that we estimate $\int_{\mathbb{R}} u_0^{1+\alpha}|x - x_0 + \epsilon|^{-2}\ dx \geq \frac{1}{4}\epsilon^{-2}\int_{B_\epsilon(x_0)} u_0^{1+\alpha}\ dx$ and pass to the limit $\epsilon \to 0$.

Assertion c) is also a consequence of the previous lemma: for $n = 2$ and $\alpha = -\frac{1}{2}$, conditions (H1) to (H4) (see Lemma 2) are readily verified. Inserting $n = 2$ and $\alpha = -\frac{1}{2}$ into (H5) and multiplying the resulting inequality by $(b+1)^2$, we see that for $\gamma < 0$ the condition (H5) is equivalent to

$$4(\gamma - 3) - 5(\gamma - 2) \leq 5\tau(\gamma - 2) \ .$$

Thus, for $\gamma \geq -2$ the condition (H5) is satisfied for sure if

$$-2 - \gamma \leq -20\tau \ .$$

This implies that we can choose $\tau := \frac{2+\gamma}{20}$. Fix some $\tilde{T} > 0$. By the finite speed of support propagation property which holds for strong energy solutions [34], we may assume that for $t \leq \tilde{T}$ we have $\operatorname{supp} u(.,t) \subset B_{R_1}(x_0)$ for some R_1 depending on u_0 and $\tilde{T}$. Set $x_1 := x_0 - \epsilon$ with $\epsilon < \min(R_1, \frac{\delta}{2})$. Lemma 4 now asserts that $(x_0 - 2\epsilon, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset$ for some $t \in (0,T)$, where

$$T \leq \frac{1}{4\tau} \left[\int_{B_{2R_1}(x_1) \setminus B_\epsilon(x_1)} |x - x_1|^{\gamma+1} \ dx \right]^4 \left(\int u_0^{\frac{1}{2}} |x - x_1|^\gamma \ dx \right)^{-4} ,$$

if the expression on the right-hand side does not exceed $\tilde{T}$ (for $t > \tilde{T}$ the assumption $\operatorname{supp} u(.,t) \subset B_{R_1}(x_0)$ which we used may be invalid). Using the fact that

$$\int_{B_{2R_1}(x_1) \setminus B_\epsilon(x_1)} |x - x_1|^{\gamma+1} \ dx \leq (2R_1)^{2+\gamma} \int_{B_{2R_1}(x_1) \setminus B_\epsilon(x_1)} |x - x_1|^{-1} \ dx \ ,$$

we obtain $(x_0 - 2\epsilon, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset$ for some $t \in (0,T)$, where

$$T \leq \frac{1}{4\tau} \left[c R_1^{2+\gamma} \log \frac{2R_1}{\epsilon} \right]^4 \left(\int u_0^{\frac{1}{2}} |x - x_0 + \epsilon|^\gamma \ dx \right)^{-4} ,$$

if the expression on the right-hand side does not exceed $\tilde{T}$. We now set $\gamma := -2 - \frac{20}{\log \epsilon}$ which implies $\tau = -\frac{1}{\log \epsilon}$ and obtain $(x_0 - 2\epsilon, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset$ for some $t \in (0,T)$, where

$$T \leq C R_1^{8+4\gamma} \left(\frac{\epsilon^{-\frac{20}{\log \epsilon}}}{|\log(2R_1)| \cdot |\log \epsilon|^{\frac{1}{4}} + |\log \epsilon|^{\frac{5}{4}}} \int u_0^{\frac{1}{2}} |x - x_0 + \epsilon|^{-2} \ dx \right)^{-4} ,$$

if the expression on the right again does not exceed $\tilde{T}$. Here we have used $\tau := \frac{2+\gamma}{20}$.

Note that $\epsilon^{-\frac{20}{\log \epsilon}} = e^{-20}$. We now let $\epsilon \to 0$. Using $8 + 4\gamma = -\frac{80}{\log \epsilon} \to 0$ and $|\log R_1| \leq |\log \epsilon|$ for ϵ small enough as well as $|x - x_0|^{-2} \leq 4|x - x_0 + \epsilon|^{-2}$ for $x > x_0 + \epsilon$, we get

$$T^* \leq C \liminf_{\epsilon \to 0} \left(\frac{1}{|\log \epsilon|^{\frac{5}{4}}} \int_{(x_0+\epsilon, \infty)} u_0^{\frac{1}{2}} |x - x_0|^{-2} \ dx \right)^{-4}$$

if the expression on the right-hand side does not exceed $\tilde{T}$. As $\tilde{T} > 0$ was arbitrary, by choosing $\tilde{T}$ large enough the assertion c) of the theorem is obtained. □

5 The Case of Several Spatial Dimensions

In the multidimensional case, the arguments are similar in spirit; however, they are technically much more involved.

For the proof of the multidimensional analogue of Lemma 2 we need the following lemma, which allows for partially replacing $\int u^{b-1}|D^2u|^2\psi$ by $\int u^{b-1}|\Delta u|^2\psi$ on the right-hand side of equation (2).

Lemma 5 *Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be an arbitrary domain and $b \geq 1$. For any u with $u^{\frac{b+1}{2}} \in H^2(\Omega)$ and $u^{\frac{b+1}{4}} \in W^{1,4}(\Omega)$, we have*

$$\int u^{b-1}|D^2u|^2\psi + (b-1)\int u^{b-2}\nabla u \cdot D^2u \cdot \nabla u\ \psi + \int u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u$$
$$= \int u^{b-1}|\Delta u|^2\psi + (b-1)\int u^{b-2}|\nabla u|^2\Delta u\ \psi + \int u^{b-1}|\nabla u|^2\Delta\psi$$

for any $\psi \in C_c^\infty(\Omega)$.

Proof Using the fact that $\operatorname{supp}\psi \subset\subset \Omega$, for smooth strictly positive u we calculate

$$\int u^{b-1}|D^2u|^2\psi + (b-1)\int u^{b-2}\nabla u \cdot D^2u \cdot \nabla u\ \psi + \int u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u$$
$$= -\int u^{b-1}\nabla\Delta u \cdot \nabla u\ \psi - \int u^{b-1}\nabla u \cdot D^2u \cdot \nabla\psi + \int u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u$$
$$= \int u^{b-1}|\Delta u|^2\psi + (b-1)\int u^{b-2}|\nabla u|^2\Delta u\ \psi + \int u^{b-1}\Delta u\nabla u \cdot \nabla\psi$$
$$- \int u^{b-1}\nabla u \cdot D^2u \cdot \nabla\psi + \int u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u$$
$$= \int u^{b-1}|\Delta u|^2\psi + (b-1)\int u^{b-2}|\nabla u|^2\Delta u\ \psi$$
$$- 2\int u^{b-1}\nabla u \cdot D^2u \cdot \nabla\psi - (b-1)\int u^{b-2}|\nabla u|^2\nabla\psi \cdot \nabla u$$
$$= \int u^{b-1}|\Delta u|^2\psi + (b-1)\int u^{b-2}|\nabla u|^2\Delta u\ \psi + \int u^{b-1}|\nabla u|^2\Delta\psi\ .$$

For strictly positive u with $u^{\frac{b+1}{2}} \in H^2$, the formula is seen to hold by approximation.

The formula carries over to the case of nonnegative u with $u^{\frac{b+1}{2}} \in H^2(\Omega)$ and $u^{\frac{b+1}{4}} \in W^{1,4}(\Omega)$ by considering $f_\delta(u)$ (for the definition of f_δ see formula (24) in the proof of Lemma 1) and passing to the limit $\delta \to 0$: $u^{\frac{b+1}{4}}$ (and therefore u) is continuous, thus the set A_δ is open (A_δ being defined as in the proof of Lemma 1). We have $f_\delta(u) \equiv \delta$ on some neighbourhood of $\Omega \setminus A_\delta$ and we have $u \in H^2(A_\delta) \cap W^{1,4}(A_\delta)$ which implies $f_\delta(u) \in H^2(A_\delta)$. Thus, we have $f_\delta(u) \in H^2(\Omega)$; moreover, $f_\delta(u) \geq \delta$. Therefore the formula holds with $f_\delta(u)$ in place of u.

We then pass to the limit $\delta \to 0$; the limit is calculated using the convergence properties (27) and (29), whose proof only required the regularity $u^{\frac{b+1}{2}} \in H^2(\Omega)$, $u^{\frac{b+1}{4}} \in W^{1,4}(\Omega)$, and the convergence property $||u - f_\delta(u)||_{L^\infty(\Omega)} \to 0$. □

Recall the abbreviation $b := n + \alpha$ and the definitions (H1) to (H4):

(H1) Assume that $1 \leq b \leq 2$.
(H2) Suppose that $\frac{n}{2} \leq b \leq n$.
(H3) Assume that $n - 1 < b$.
(H4) Suppose that the inequality

$$(n-b)\left(b-\frac{n}{2}\right)(b-1)(2-b) \geq \frac{1}{4}\left[\left(\frac{5n}{2}-4b\right)(b-1)\right]^2$$

is satified.

We now state and prove the multidimensional analogue of Lemma 2.

Lemma 6 *Let $n \in [1,3)$, $\alpha \in (-1,0]$, and let u be a strong energy solution of the thin-film equation on a domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with initial data $u_0 \in H^1(\Omega)$. Assume that* $\operatorname{supp} u_0$ *is bounded. Set $b := n + \alpha$ and assume that (H1) to (H4) are satisfied.*

Let $\psi \in C_c^4(\Omega)$. Assume $\psi \geq 0$. Then for a.e. $t_1, t_2 \in [0,\infty)$ with $t_2 \geq t_1$ and for a.e. $t_2 \in [0,\infty)$ in case $t_1 = 0$ we have

$$\begin{aligned}&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\ &\geq \left(\frac{2}{3}b - \frac{1}{6}n\right)\int_{t_1}^{t_2}\int u^{b-1}|\nabla u|^2 \Delta\psi + \left(\frac{4}{3}b - \frac{1}{3}n\right)\int_{t_1}^{t_2}\int u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u \\ &\quad - \frac{1}{b+1}\int_{t_1}^{t_2}\int u^{b+1}\Delta^2\psi\ .\end{aligned}$$

Proof Assume for the moment that $\psi \in C_c^\infty(\Omega)$. Recall that by Lemma 5 we have

$$\begin{aligned}&\int u^{b-1}|D^2u|^2\psi + (b-1)\int u^{b-2}\nabla u \cdot D^2 u \cdot \nabla u\ \psi + \int u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u \\ &= \int u^{b-1}|\Delta u|^2\psi + (b-1)\int u^{b-2}|\nabla u|^2 \Delta u\ \psi + \int u^{b-1}|\nabla u|^2\Delta\psi\ .\end{aligned}$$

Since by (H2) and (H3) it holds that $-1 < \alpha \leq 0$, formula (2) states that

$$\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&= \left(b - \frac{1}{2}n\right) \int_{t_1}^{t_2} \int u^{b-1} |\nabla u|^2 \Delta\psi + b \int_{t_1}^{t_2} \int u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u \\
&\quad - \frac{1}{b+1} \int_{t_1}^{t_2} \int u^{b+1} \Delta^2 \psi + (n-b) \int_{t_1}^{t_2} \int u^{b-1} |D^2 u|^2 \psi \\
&\quad + \left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int u^{b-3} |\nabla u|^4 \psi \\
&\quad + (2n - 3b)(b-1) \int_{t_1}^{t_2} \int u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi \\
&\quad + \left(\frac{n}{2} - b\right)(b-1) \int_{t_1}^{t_2} \int u^{b-2} |\nabla u|^2 \Delta u\ \psi\ .
\end{aligned}$$

We now multiply the formula from Lemma 5 by $\frac{1}{3}(n-b)$ and add it to this equation, resulting in

$$\begin{aligned}
&\int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&= \left(b - \frac{1}{2}n + \frac{n-b}{3}\right) \int_{t_1}^{t_2} \int u^{b-1} |\nabla u|^2 \Delta\psi \\
&\quad + \left(b - \frac{n-b}{3}\right) \int_{t_1}^{t_2} \int u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u - \frac{1}{b+1} \int_{t_1}^{t_2} \int u^{b+1} \Delta^2 \psi \\
&\quad + \frac{2}{3}(n-b) \int_{t_1}^{t_2} \int u^{b-1} |D^2 u|^2 \psi + \frac{1}{3}(n-b) \int_{t_1}^{t_2} \int u^{b-1} |\Delta u|^2 \psi \\
&\quad + \left(\frac{2}{3} + \frac{1}{3}\right)\left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int u^{b-3} |\nabla u|^4 \psi \\
&\quad + \left(\frac{5}{3}n - \frac{8}{3}b\right)(b-1) \int_{t_1}^{t_2} \int u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi \\
&\quad + \left(\frac{5}{6}n - \frac{4}{3}b\right)(b-1) \int_{t_1}^{t_2} \int u^{b-2} |\nabla u|^2 \Delta u\ \psi\ .
\end{aligned}$$

We now see that the expressions

$$\begin{aligned}
&\frac{2}{3}(n-b) \int u^{b-1} |D^2 u|^2 \psi + \frac{2}{3}\left(b - \frac{n}{2}\right)(b-1)(2-b) \int u^{b-3} |\nabla u|^4 \\
&\quad + \left(\frac{5}{3}n - \frac{8}{3}b\right)(b-1) \int u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi
\end{aligned}$$

and

$$\frac{1}{3}(n-b)\int u^{b-1}|\Delta u|^2\psi+\frac{1}{3}\left(b-\frac{n}{2}\right)(b-1)(2-b)\int u^{b-3}|\nabla u|^4$$
$$+\left(\frac{5}{6}n-\frac{4}{3}b\right)(b-1)\int u^{b-2}|\nabla u|^2\Delta u\;\psi$$

are nonnegative: Young's inequality implies nonnegativity of these terms if $n-b\geq 0$, $1\leq b\leq 2$, $b\geq \frac{n}{2}$ and

$$(n-b)\left(b-\frac{n}{2}\right)(b-1)(2-b)\geq \frac{1}{4}\left(\frac{5}{2}n-4b\right)^2(b-1)^2$$

are satisfied. These conditions however were precisely part of our assumptions. This proves the lemma for $\psi\in C_c^\infty(\Omega)$.

For $\psi\in C_c^4(\Omega)$, we consider the mollifications $\rho_\delta*\psi$ which belong to $C_c^\infty(\Omega)$; passing to the limit $\delta\to 0$, we obtain $\rho_\delta*\psi\to\psi$ in $C_c^4(\Omega)$. Using the regularity of u and dominated convergence, this is sufficient for passing to the limit in all expressions of our inequality. □

We now derive upper bounds on waiting times for the thin-film equation in the case of several spatial dimensions. For $d>1$, an additional difficulty arises: The attempt to use $|x|^\gamma$ as a weight function fails as the constant in front of the positive terms in the weighted entropy estimate is no longer large enough to ensure that the positive terms dominate the negative term. This problem is resolved using a localized test function adapted to the shape of the initial support, which approximately reduces the situation to the one-dimensional one.

For the next two lemma, we assume that we are given a point $x_0\in\partial\operatorname{supp}u_0$ such that there exists a C^4 domain, whose closure we denote by M, with the property that in some neighbourhood of x_0 the set $\operatorname{supp}u_0$ is contained in M; moreover, we require $x_0\in\partial M$. The tangent plane of the manifold ∂M in x_0 will be denoted by H. Without loss of generality, we may assume that $x_0=0$ and $H=\{x\in\mathbb{R}^d : x_d=0\}$. In this case, ∂M is locally given as the graph of a function $\xi : H\to\mathbb{R}$. We define another function $\tilde\xi$ to be equal to ξ in some neighbourhood Z_r of x_0, but require the graph of $\tilde\xi$ to move away from M as one moves away from x_0.

Our weight function ψ takes the form $|x_d-\tilde\xi(x_1,\dots,x_{d-1})+\delta|^\gamma\cdot\phi^2$, where ϕ is a cutoff supported in the neighbourhood Z_{3r} of x_0. The singularities of our weight function ψ lie on a curve which corresponds to the graph of $\tilde\xi$ shifted downwards by δ.

As $\tilde\xi$ is nonconstant, we shall see that additional terms involving derivatives of $\tilde\xi$ arise during the derivation of the multidimensional analogue of the monotonicity formula (1). If r^{-1} is large enough (in comparison to these derivatives of $\tilde\xi$), these terms can be absorbed and we obtain the almost monotonicity formula (7) below (i.e. a monotonicity formula with some inhomogeneity). By decreasing r one can always enforce this condition; however, decreasing r increases the undesireable inhomogeneity in the almost monotonicity formula (7).

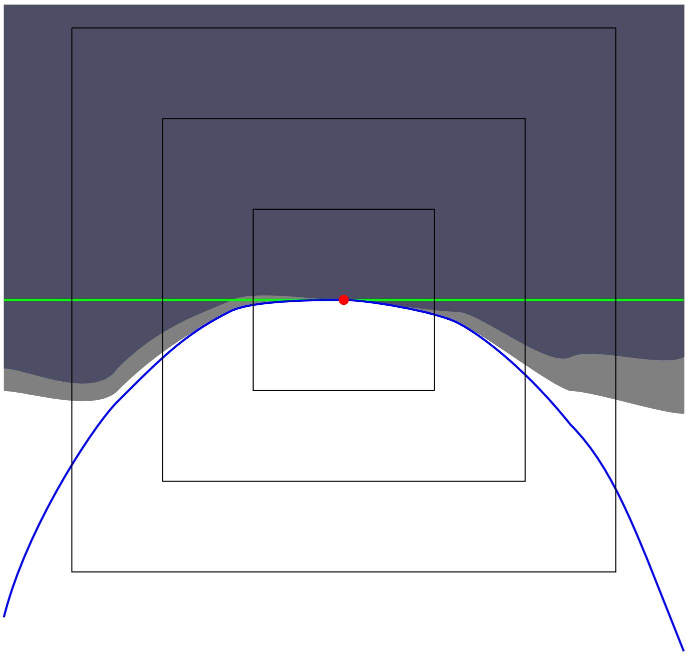


Fig. 1 A sketch of the situation of Lemma 7 and Lemma 8. The deep blue area corresponds to $\operatorname{supp} u_0$; the union of the grey and deep blue areas represents the set M. The boundary of the grey area corresponds to the graph of ξ.
The three boxes represent the sets Z_{3r}, Z_{2r}, Z_r. The red dot denotes the point 0. The green line marks the tangent plane H to ∂M in 0 (i.e. $\mathbb{R}^{d-1}$).
The blue curve corresponds to the graph of $\tilde{\xi}$. It is clearly visible that the graph of $\tilde{\xi}$ coincides with ∂M (i.e. the graph of ξ) in Z_r, but moves away from ∂M as one moves away from 0; in $Z_{3r} \setminus Z_{2r}$ the graph of $\tilde{\xi}$ lies at least Kr^2 below the set M.

Note that the reason for the appearance of the inhomogeneity is the usage of the cutoff ϕ.

Relying on the new almost monotonicity formula (7), in Lemma 8 below we deduce an upper bound on the waiting time in the neighbourhood Z_{3r} of x_0. At the level of this upper bound on waiting times, the inhomogeneity in the almost monotonicity formula translates to an additional condition on the initial data (more precisely, the initial mass near x_0 must not be too small in order for our estimate on the waiting time to hold). As we shall in the proof of our multidimensional theorem, this additional condition becomes irrelevant as we "zoom in" on the free boundary, at least for $n > 2$; for $n = 2$, the condition gives rise to a stronger condition on the initial data in the final statement of the theorem.

Lemma 7 *Let u be a strong energy solution of the thin-film equation on a domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with nonnegative initial data $u_0 \in H^1(\Omega)$ with bounded support and let $n \in (1,3)$, $\alpha \in (-1,0)$. Setting $b := n+\alpha$, assume that the conditions (H1), (H2), (H3), (H4) preceding Lemma 6 are satisfied. Given $\gamma \in [-20;-1]$, suppose furthermore that*

(H5) The condition

$$\left(2b-\frac{1}{2}n\right)\frac{\gamma-3}{(b+1)^2(\gamma-2)}-\frac{1}{b+1}\geq\tau$$

is satisfied for some $\tau\in(0,1)$.

Let M *be the closure of a* C^4 *domain and let* $x_0\in\partial M$*; w.l.o.g. we may assume that* $x_0=0$*. Denote the tangent plane to* ∂M *in* 0 *by* H*; w.l.o.g. (i.e. possibly after a rotation and reflection) we may assume that* $H=\{x\in\mathbb{R}^d: x_d=0\}$ *and that* $x_0+\mu\mathbf{e_d}\in M$ *for any* $\mu>0$ *small enough. Denote the projection onto* H *by* P*. Define*

$$Z_\rho:=\{x:|Px|<\rho,|x_d|<\rho\}\ . \tag{3}$$

Let $R>0$ *and let* $\xi:H\to\mathbb{R}$*,* $\xi\in C^4$*, be a function such that*

$$Z_R\cap M=Z_R\cap\{x\in\mathbb{R}^d:x_d\geq\xi(Px)\} \tag{4}$$

holds (for R *small enough such a function exists by the implicit function theorem). Note that* $\xi(0)=0$ *and that* $\nabla\xi(0)=0$ *as* H *is tangent to* ∂M *at* 0.

Assume that $Z_R\subset\subset\Omega$.

Take any $r\in\left(0,\frac{R}{3}\right)$ *and any* $K\in\mathbb{R}_0^+$ *such that*

(P1) $\operatorname{supp}u_0\cap Z_{3r}\subset M$*, i.e. locally near* x_0 *the support of* u_0 *is contained in* M.

(P2) $|D^2\xi(Px)|\leq K$*,* $|D^3\xi(Px)|\leq\frac{K}{r}$*, and* $|D^4\xi(Px)|\leq\frac{K}{r^2}$ *for any* $x\in\mathbb{R}^d$ *with* $|Px|\leq 3r$.

(P3) The inequality $Kr<\epsilon(d,n)\tau$ *holds for some small constant* $\epsilon(d,n)<\frac{1}{10}$ *which is to be determined in the course of the proof below.*

Let $\phi:\mathbb{R}^d\to\mathbb{R}$ *be a smooth cutoff with* $0\leq\phi\leq 1$*,* $\phi\equiv 1$ *on* Z_{2r}*,* $\operatorname{supp}\phi\subset Z_{3r}$*, and* $|\nabla\phi|\leq\frac{C(d)}{r}$*,* $|D^2\phi|\leq\frac{C(d)}{r^2}$*,* $|D^3\phi|\leq\frac{C(d)}{r^3}$*,* $|D^4\phi|\leq\frac{C(d)}{r^4}$*. Define* $\tilde{\xi}:H\to\mathbb{R}$ *by*

$$\tilde{\xi}(x):=\xi(x)-Kr^{-3}(|x|-r)_+^5\ . \tag{5}$$

Set

$$T:=\inf\{t>0:\operatorname{supp}u(.,t)\cap(\mathbb{R}^d\setminus M)\cap Z_{3r}\neq\emptyset\}\ . \tag{6}$$

Then we have for any $\delta\in(0,r)$

$$\begin{aligned}&\int_\Omega\frac{1}{1+\alpha}u^{1+\alpha}(.,t)|x_d-\tilde{\xi}(Px)+\delta|^\gamma\phi^2(x)\ dx\Bigg|_{t_1}^{t_2}\\ \geq&\tau\int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4}\\ &-C(d,n)(r^4(Kr^2)^{\gamma-4}+(Kr^2)^\gamma)\int_{t_1}^{t_2}\int_{Z_{3r}}|\nabla u^{\frac{b+1}{4}}|^4\end{aligned} \tag{7}$$

for a.e. $0<t_1<t_2<T$ *and a.e.* $0<t_2<T$ *in case* $t_1=0$.

Proof The proof is somewhat analogous to the first part of the proof of Lemma 4; however, we additionally make heavy use of cutoff arguments.

Set

$$\epsilon(d,n) := \min\left(\epsilon_0, \epsilon_1, \frac{1}{10}\right) \tag{8}$$

where ϵ_0 and ϵ_1 are to be chosen below depending only on d and n. From now on, to simplify notation we write ϵ instead of $\epsilon(d,n)$.

Note that we have $\tilde{\xi} \in C^4$. The function $\tilde{\xi}$ satisfies some estimates similar to (P2), namely:

(P2') We have $|D^2\tilde{\xi}(Px)| \leq C(d)K$, $|D^3\tilde{\xi}(Px)| \leq \frac{C(d)K}{r}$, and $|D^4\tilde{\xi}(Px)| \leq \frac{C(d)K}{r^2}$ for any $x \in \mathbb{R}^d$ with $|Px| \leq 3r$.

We abbreviate $\psi(x) := |x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)$. We obviously have $\psi \in C^4(M)$ (as the points at which the function has singularities do not belong to M). In Lemma 9 below, additional properties of this test function which we shall need in the sequel are proven.

By the assumptions of our Lemma, Lemma 6 is applicable. We use ψ as a test function in Lemma 6 (this is possible by the choice of T and the definition of ψ: for $t < T$ we have $\operatorname{supp} u(.,t) \cap Z_{3r} \setminus M = \emptyset$ and we have $\operatorname{supp} \psi \subset Z_{3r}$, i.e. the singularities of our weight are not touched by the support of the solution u). Making use of the estimates (14) and (15) from Lemma 9 below, we obtain for a.e. $t_1, t_2 \in [0,T)$ with $t_2 > t_1$ and a.e. $t_2 \in [0,T)$ in case $t_1 = 0$

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\;dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\;dx\\
\geq&\left(\frac{2}{3}b-\frac{1}{6}n\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi + \left(\frac{4}{3}b-\frac{1}{3}n\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u\\
&-\frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi\\
=&\frac{\frac{8}{3}b-\frac{4}{6}n}{(b+1)^2}\int_{t_1}^{t_2}\int_\Omega |\nabla u^{\frac{b+1}{2}}|^2\Delta\psi + \frac{\frac{16}{3}b-\frac{4}{3}n}{(b+1)^2}\int_{t_1}^{t_2}\int_\Omega \nabla u^{\frac{b+1}{2}}\cdot D^2\psi\cdot\nabla u^{\frac{b+1}{2}}\\
&-\frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi\\
\geq&\left(\gamma(\gamma-1)\frac{\frac{8}{3}b-\frac{4}{6}n}{(b+1)^2}-C(d,n)Kr\right) \qquad\qquad (9)\\
&\qquad\cdot\int_{t_1}^{t_2}\int_\Omega |\nabla u^{\frac{b+1}{2}}|^2\cdot\phi^2(x)|x_d-\tilde{\xi}(x)+\delta|^{\gamma-2}\\
&+\gamma(\gamma-1)\frac{\frac{16}{3}b-\frac{4}{3}n}{(b+1)^2}\int_{t_1}^{t_2}\int_\Omega |\partial_d u^{\frac{b+1}{2}}|^2\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-2}\\
&-\left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1}+C(d,n)Kr\right)\\
&\qquad\cdot\int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4}\\
&-C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}}|\nabla u^{\frac{b+1}{2}}|^2-C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}}u^{b+1}
\end{aligned}
$$

where we have used the fact that $\left|\frac{\frac{8}{3}b-\frac{4}{6}n}{(b+1)^2}\right| \leq C(d,n)$ by assumption (H1), the fact that $\left|\frac{1}{b+1}\right| \leq C(d,n)$ again by (H1), and the fact that $\operatorname{supp}\psi \subset Z_{3r}$.

By the assumption $Kr \leq \tau\epsilon$ (see (P3)), assumptions (H1) and (H3) and the condition $-20 \leq \gamma \leq -1$, we see that

$$
\begin{aligned}
&\left(\gamma(\gamma-1)\frac{\frac{8}{3}b-\frac{4}{6}n}{(b+1)^2}-C(d,n)Kr\right) \geq \left(\gamma(\gamma-1)\frac{\frac{8}{3}b-\frac{4}{6}n}{(b+1)^2}-C(d,n)\epsilon\tau\right)\\
\geq&\left(2\frac{\frac{6}{3}b-\frac{2}{3}}{9}-C(d,n)\epsilon\tau\right) \geq \left(\frac{8}{27}-C(d,n)\epsilon\tau\right)\ . \qquad\qquad (10)
\end{aligned}
$$

Thus, by $\tau < 1$ (see (H5)) we see that the prefactor of the first term on the right-hand side of (9) is nonnegative if we choose ϵ_1 small enough depending only on d and n. Thus we can estimate this term from below by dropping the derivatives in directions perpendicular to $\mathbf{e}_d$. Additionally taking into account our assumption $Kr \le \epsilon\tau$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
&\ge \left(\gamma(\gamma-1)\frac{8b-2n}{(b+1)^2} - C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega |\partial_d u^{\frac{b+1}{2}}|^2 \cdot \phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2} \\
&\quad - \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\ ,
\end{aligned}$$

where the prefactor of the first term on the right-hand side is still nonnegative (the term $\gamma(\gamma-1)\frac{\frac{16}{3}b-\frac{4}{3}n}{(b+1)^2}$, by which the prefactor has increased, is nonnegative as shown in (10)).

We now put ϕ under the derivative in the first term on the right-hand side and (in the second inequality below) use the first assertion of Lemma 9 below, the fact that $\operatorname{supp} u(.,t) \cap Z_{3r} \subset M$ for $t < T$ (recall also $\operatorname{supp}\phi \subset Z_{3r}$), as well as the estimate $|\nabla\phi| \le \frac{C(d)}{r} \le \frac{C(d)}{Kr^2}$ (recall that $Kr \le \epsilon\tau \le 1$) and

Young's inequality to obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi \ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi \ dx\\
\geq &\left(\gamma(\gamma-1)\frac{8b-2n}{(b+1)^2} - C(d,n)\epsilon\tau\right)\int_{t_1}^{t_2}\int_\Omega |\partial_d(u^{\frac{b+1}{2}}\phi)|^2 \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2}\\
&- C(d,n)\int_{t_1}^{t_2}\int_\Omega |\partial_d u^{\frac{b+1}{2}}|\ u^{\frac{b+1}{2}}\ |\nabla\phi|\ \phi \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2}\\
&- C(d,n)\int_{t_1}^{t_2}\int_\Omega u^{b+1}|\nabla\phi|^2 \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2}\\
&- \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right)\\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4}\\
&- C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\\
\geq &\left(\gamma(\gamma-1)\frac{8b-2n}{(b+1)^2} - C(d,n)\epsilon\tau\right)\int_{t_1}^{t_2}\int_\Omega |\partial_d(u^{\frac{b+1}{2}}\phi)|^2 \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2}\\
&- \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right)\\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4}\\
&- C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2\\
&- C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\ .
\end{aligned}
$$

The prefactor of the first term on the right-hand side did not change and so is still nonnegative.

An application of Fubini's theorem and the one-dimensional Hardy inequality (see Lemma 3) to the first term on the right-hand side on all the lines $\{x : Px = y\}$, $y \in H$, with the weight $\psi = |x_d - \tilde{\xi}(y) + \delta|^{\gamma-2}$ (note that this function has its singularity at $x_d = \tilde{\xi}(y) - \delta$, so we must check that $(u^{\frac{b+1}{2}}\phi)(x_1, \ldots, x_{d-1}, \cdot, t)$ is zero on some neighbourhood of $\tilde{\xi}(y) - \delta$; this check is performed easily since $\operatorname{supp} u(., t) \subset M$ and since $x_d \geq \xi(Px) \geq \tilde{\xi}(Px)$ for $x \in M \cap Z_{3r}$), yields (recall that the prefactor of the first term on the right-hand side is nonnegative)

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
&\geq \left(\gamma(\gamma-1)(\gamma-3)^2 \frac{2b-\frac{1}{2}n}{(b+1)^2} - C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^\gamma \int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4 \\
&\quad - C(d,n)(Kr^2)^{\gamma-4} \int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}
\end{aligned}$$

where we have also used the fact that $-20 \leq \gamma \leq -1$ and applied Young's inequality to the penultimate term. Assumption (H5) now gives

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
&\geq (\gamma(\gamma-1)(\gamma-2)(\gamma-3)\tau - C(d,n)\epsilon\tau) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^\gamma \int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4 \\
&\quad - C(d,n)(Kr^2)^{\gamma-4} \int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\ .
\end{aligned} \tag{11}$$

Using $\gamma \leq -1$, we see that choosing ϵ_0 small enough depending only on n and d we can enforce that $\gamma(\gamma-1)(\gamma-2)(\gamma-3)\tau - C(d,n)\epsilon\tau > \tau$.

Recall that $\operatorname{supp} u(.,t) \cap Z_{3r} \cap \Omega \setminus M = \emptyset$ for any $0 \leq t < T$ (by (6)). Since $Kr \leq \epsilon\tau \leq \frac{1}{10}$ (by our choice of ϵ in (8) and by $\tau \leq 1$) and $|\xi(Px)| \leq 9Kr^2 < r$ in case $|Px| \leq 3r$ (due to $\xi(0)=0$, $D\xi(0)=0$, and $|D^2\xi| \leq K$) by (4) we see that $\operatorname{supp} u(.,t) \cap \{x : |Px| < 3r, x_d \in (-3r,-r)\} = \emptyset$ for any $t \in [0,T)$. Therefore we may apply Fubini's theorem and the one-dimensional Poincare inequality on the one-dimensional segments $\{x : Px = y, x_d \in (-3r,3r)\}$,

$y \in H$, to estimate

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\,dx\,dt = \int_{t_1}^{t_2}\int_{PZ_{3r}}\int_{\{x_d:|x_d|<3r\}} u^{b+1}\,dx_d\,d\overline{x}\,dt\\
\leq &\int_{t_1}^{t_2}\int_{PZ_{3r}} C(d)(4r)^4\int_{\{x_d:|x_d|<3r\}} |\nabla u^{\frac{b+1}{4}}|^4\,dx_d\,d\overline{x}\,dt\\
=&C(d)r^4\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4\,dx\,dt\ ,
\end{aligned}$$

where we have abbreviated $\overline{x} = (x_1, \dots, x_{d-1})$.

Putting these considerations together, (11) implies the statement of our lemma. □

Lemma 8 *We use the notation of Lemma 7; recall in particular that*

$$T := \inf\{t > 0 : \operatorname{supp} u(.,t) \cap (\mathbb{R}^d \setminus M) \cap Z_{3r} \neq \emptyset\}\ .$$

Suppose that the assumptions of Lemma 7 are satisfied.

Then there exist constants $C_0(d,n) > 0$, $C(d,n) > 0$ such that for any $\delta \in (0,r)$ the following statement holds: Assuming that the estimate

$$\begin{aligned}
&\int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha}\cdot|x_d - \xi(Px) + \delta|^\gamma\,dx \qquad (12)\\
\geq &C_0(d,n)r^4(Kr^2)^{\gamma-4}\int_0^{\tilde{T}}\int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4\,dx\,dt
\end{aligned}$$

is satisfied for some $\tilde{T} > 0$, we have

$$\begin{aligned}
T \leq &\frac{C(d,n,\alpha)}{\tau}\cdot\left(r^{d-1}\int_\delta^{C(d)r} z^{\gamma+4\frac{1+\alpha}{n}}\,dz\right)^{\frac{n}{1+\alpha}} \qquad (13)\\
&\cdot\left(\int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha}|x_d - \xi(Px) + \delta|^\gamma\,dx\right)^{-\frac{n}{1+\alpha}}
\end{aligned}$$

if the expression on the right-hand side does not exceed $\tilde{T}$.

Proof Hölder's inequality implies (since $\operatorname{supp} u(.,t) \cap \operatorname{supp}\phi \subset M$ for $t < T$)

$$\begin{aligned}
&\int_\Omega u^{1+\alpha}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^\gamma\,dx\\
\leq &\left(\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4}\,dx\right)^{\frac{1+\alpha}{b+1}}\\
&\cdot\left(\int_M \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right)^{\frac{n}{b+1}}\ .
\end{aligned}$$

Estimating the second integral on the right-hand side using Fubini's theorem, the local representation of M (see (4)), the definition of $\tilde{\xi}$ (see (5)), the fact that $\delta \le r$ (see the assumptions of the lemma), the estimate $\xi(Px) \ge \tilde{\xi}(Px)$, the fact that $\operatorname{supp}\phi \subset Z_{3r}$, and the estimate $|\tilde{\xi}(Px)| \le C(d)Kr^2 \le C(d)r$ for $x \in Z_{3r}$ (recall $\tilde{\xi}(0) = 0$, $D\tilde{\xi}(0) = 0$, and $|D^2\tilde{\xi}(Px)| \le C(d)K$ for $x \in Z_{3r}$), we get

$$\begin{aligned}
&\left(\int_\Omega u^{1+\alpha} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^\gamma \, dx\right)^{\frac{b+1}{1+\alpha}}\\
&\le \int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \, dx\\
&\quad \cdot \left(\int_{PZ_{3r}} \int_{\xi(y)}^{3r} |z - \tilde{\xi}(y) + \delta|^{\gamma+4\frac{1+\alpha}{n}} \, dz \, dy\right)^{\frac{n}{1+\alpha}}\\
&= \int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \, dx\\
&\quad \cdot \left(\int_{PZ_{3r}} \int_{\xi(y)-\tilde{\xi}(y)+\delta}^{3r-\tilde{\xi}(y)+\delta} |z|^{\gamma+4\frac{1+\alpha}{n}} \, dz \, dy\right)^{\frac{n}{1+\alpha}}\\
&\le \int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \, dx\\
&\quad \cdot \left(C(d)r^{d-1} \int_\delta^{C_3(d)r} |z|^{\gamma+4\frac{1+\alpha}{n}} \, dz\right)^{\frac{n}{1+\alpha}} .
\end{aligned}$$

Plugging this estimate into (7), multiplying by $1+\alpha$ and using $\delta \le r$ we see that

$$\begin{aligned}
&\int_\Omega u^{1+\alpha}(.,t_2)\psi \, dx - \int_\Omega u^{1+\alpha}(.,t_1)\psi \, dx\\
\ge& c_1(d,n,\alpha)\tau \left(r^{d-1} \int_\delta^{C_3(d)r} z^{\gamma+4\frac{1+\alpha}{n}} \, dz\right)^{-\frac{n}{1+\alpha}} \cdot \int_{t_1}^{t_2} \left(\int_\Omega u^{1+\alpha}\psi \, dx\right)^{\frac{b+1}{1+\alpha}} dt\\
&- C_2(d,n)(r^4(Kr^2)^{\gamma-4} + (Kr^2)^\gamma) \int_{t_1}^{t_2} \int_\Omega |\nabla u^{\frac{b+1}{4}}|^4 \, dx \, dt
\end{aligned}$$

holds for a.e. $t_1, t_2 \in [0,T)$ with $t_2 > t_1$ and a.e. $t_2 \in [0,T)$ in case $t_1 = 0$.

We therefore have derived a differential inequality for $\int_\Omega u^{1+\alpha}(.,t)\psi \, dx$. By the comparison principle, the solution of the corresponding differential equation yields a lower bound on $\int_\Omega u^{1+\alpha}(.,t)\psi \, dx$ (as the right-hand side of our differential inequality is locally Lipschitz with respect to the solution). The corresponding differential equation reads

$$\begin{aligned}
\frac{d}{dt} f =& c_1(d,n,\alpha)\tau \left(r^{d-1} \int_\delta^{C_3(d)r} z^{\gamma+4\frac{1+\alpha}{n}} \, dz\right)^{-\frac{n}{1+\alpha}} \cdot f^{\frac{b+1}{1+\alpha}}\\
&- C_2(d,n)(r^4(Kr^2)^{\gamma-4} + (Kr^2)^\gamma) \int_\Omega |\nabla u^{\frac{b+1}{4}}|^4 \, dx
\end{aligned}$$

and the initial condition is $f(0)=\int_\Omega u_0^{1+\alpha}\psi\,dx$.

Fixing some $\tilde T\in(0,T)$ we can show that the solution f is bounded from below at all $t\in[0,\tilde T]$ by the solution g of

$$\frac{d}{dt}g=c_1(d,n,\alpha)\tau\left(r^{d-1}\int_\delta^{C_3(d)r} z^{\gamma+4\frac{1+\alpha}{n}}\;dz\right)^{-\frac{n}{1+\alpha}}\cdot g^{\frac{b+1}{1+\alpha}}$$

with initial data

$$\begin{aligned}g(0):=&\int_\Omega u_0^{1+\alpha}\psi\;dx\\&-C_2(d,n)(r^4(Kr^2)^{\gamma-4}+(Kr^2)^\gamma)\int_0^{\tilde T}\int_\Omega|\nabla u^{\frac{b+1}{4}}|^4\;dx\;dt\;,\end{aligned}$$

provided that we have $g(0)>0$.

It suffices to prove $f\geq g_\mu$ in $[0,\tilde T]$ for all g_μ solving the same differential equation as g, but with initial data $g_\mu(0):=g(0)-\mu>0$: We know that $g_\mu(t)$ converges to $g(t)$ as $\mu\to 0$ for any fixed $t\geq 0$. To prove $f\geq g_\mu$ in $[0,\tilde T]$, we argue by contradiction and assume that $t_\mu:=\inf\{t\in[0,\tilde T]:g_\mu(t)>f(t)\}<\infty$. This gives

$$\begin{aligned}&f(t_\mu)-g_\mu(t_\mu)\\=&f(0)-g_\mu(0)-C_2(d,n)(r^4(Kr^2)^{\gamma-4}+(Kr^2)^\gamma)\int_0^{t_\mu}\int_\Omega|\nabla u^{\frac{b+1}{4}}|^4\;dx\;dt\\&+c_1(d,n,\alpha)\tau\left(r^{d-1}\int_\delta^{C_3(d)r}z^{\gamma+4\frac{1+\alpha}{n}}\;dz\right)^{-\frac{n}{1+\alpha}}\cdot\int_0^{t_\mu}f^{\frac{b+1}{1+\alpha}}(t)-g_\mu^{\frac{b+1}{1+\alpha}}(t)\;dt\\\geq&f(0)-g_\mu(0)-C_2(d,n)(r^4(Kr^2)^{\gamma-4}+(Kr^2)^\gamma)\int_0^{\tilde T}\int_\Omega|\nabla u^{\frac{b+1}{4}}|^4\;dx\;dt\\\geq&\mu\end{aligned}$$

where we have used the fact that $f(t)\geq g_\mu(t)$ for $t<t_\mu$ and the definition of $g_\mu(0)$ to obtain the desired contradiction (due to continuity of f and g_μ, the definition of t_μ would imply that $g_\mu(t_\mu)\geq f(t_\mu)$).

We now choose $C_0(d,n):=4C_2(d,n)$ in condition (12). Using the estimate $(Kr^2)^\gamma\leq r^4(Kr^2)^{\gamma-4}$ (which holds since $Kr\leq\tau\epsilon\leq 1$ by the conditions on τ and ϵ) as well as the fact that $\xi(Px)=\tilde\xi(Px)$ for $|Px|\leq r$ and the fact that $\phi\equiv 1$ on Z_{2r}, we see that (12) then implies

$$\begin{aligned}g(0)\geq&\int_{\{x:|Px|<r,|x_d|<2r\}}u_0^{1+\alpha}|x_d-\xi(Px)+\delta|^\gamma\;dx\\&-2C_2(d,n)r^4(Kr^2)^{\gamma-4}\int_0^{\tilde T}\int_\Omega|\nabla u^{\frac{b+1}{4}}|^4\;dx\;dt\\\geq&\frac{1}{2}\int_{\{x:|Px|<r,|x_d|<2r\}}u_0^{1+\alpha}|x_d-\xi(Px)+\delta|^\gamma\;dx\;.\end{aligned}$$

Since the equation for g can be solved explicitly (the solution of $\frac{d}{dt}g(t) = q \cdot [g(t)]^m$ is $g(t) = [g(0)^{1-m} - (m-1) \cdot q \cdot t]^{\frac{1}{1-m}}$), this implies that g and therefore f and therefore also $\int_\Omega u^{1+\alpha}(.,t)\psi \, dx$ needs to blow up before time

$$\frac{1+\alpha}{c_1(d,n,\alpha)\cdot n \cdot \tau} \cdot \left(r^{d-1} \int_\delta^{C_3(d)r} z^{\gamma+4\frac{1+\alpha}{n}} \, dz \right)^{\frac{n}{1+\alpha}}$$
$$\cdot \left(\frac{1}{2} \int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha} |x_d - \xi(Px) + \delta|^\gamma \, dx \right)^{-\frac{n}{1+\alpha}}$$

if this quantity does not exceed $\tilde{T}$.

This yields an upper bound on T: we know that $\phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^\gamma$ is compactly supported and bounded from above by δ^γ on M; moreover, for $t < T$ we have $\operatorname{supp} u(.,t) \cap \operatorname{supp} \phi \subset M$. As $\int_\Omega u(.,t) \, dx = \int_\Omega u_0 \, dx < \infty$ by conservation of mass, by Hölder's inequality $\int_\Omega u^{1+\alpha}(.,t)\psi \, dx$ must remain bounded for $t < T$. Thus, if this quantity blows up at some time T' we necessarily have $T' \geq T$.

This finishes the proof of the lemma. □

Lemma 9 *With ϕ and $\tilde{\xi}$ defined as in Lemma 7, for any $x \in M \cap \operatorname{supp} \nabla\phi$ we have $x_d - \tilde{\xi}(Px) \geq Kr^2$.*

Moreover, with ψ defined as in the proof of Lemma 7, the following estimate holds for the second derivative of ψ for any $x \in M$:

$$\begin{aligned} &\left| D^2\psi(x) - \gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) \cdot \mathbf{e}_d \otimes \mathbf{e}_d \right| \\ \leq &C(d)Kr|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) + C(d)[Kr^2]^{\gamma-2} \end{aligned} \tag{14}$$

For the fourth derivative, the following estimate is satisfied for any $x \in M$:

$$\begin{aligned} &\left| \Delta^2\psi(x) - \gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \cdot \phi^2(x) \right| \\ \leq &C(d)Kr|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \cdot \phi^2(x) + C(d)[Kr^2]^{\gamma-4} \end{aligned} \tag{15}$$

Proof For $x \in Z_{2r}$, we have $\phi(x) = 1$; additionally we have $\operatorname{supp} \phi \subset Z_{3r}$. Thus for $x \in M \cap \operatorname{supp} \nabla\phi$ we know $x \in Z_{3r}$; moreover, we either have $|x_d| \geq 2r$ or $|Px| \geq 2r$.

- In the latter case, by definition of $\tilde{\xi}$ (see (5)) we obtain $\xi(Px) - \tilde{\xi}(Px) \geq Kr^2$ which implies $x_d - \tilde{\xi}(Px) \geq Kr^2$ (since $x_d \geq \xi(Px)$ due to $x \in M \cap Z_{3r}$ and (4)).
- To deal with the former case, we observe that $|D\xi(Px)| \leq K \cdot 3r$ for $x \in Z_{3r}$ by our assumption (P2) since $|D^2\xi(Px)| \leq K$ for $x \in Z_{3r}$ and $D\xi(0) = 0$; using $\xi(0) = 0$ this implies $|\xi(Px)| \leq K(3r)^2 = 9Kr^2 \leq 9\epsilon(d,n)r \leq r$ for $x \in Z_{3r}$ by (P3), our choice of ϵ (see (8)), and $0 < \tau < 1$. Thus $|x_d| \geq 2r$ and $x \in M \cap Z_{3r}$ imply $x_d \geq 2r$, the case $x_d \leq -2r$ being impossible (as $x \in M \cap Z_{3r}$ implies $x_d \geq \xi(Px) \geq -r$). This gives $x_d - \xi(Px) \geq 2r - r = r \geq Kr^2$ by condition (P3), our choice of ϵ (8), and $0 < \tau < 1$. Since we have $\tilde{\xi}(Px) \leq \xi(Px)$ by (5), we deduce $x_d - \tilde{\xi}(Px) \geq Kr^2$.

This finishes the proof of the first assertion.

We calculate for $x \in M \cap Z_{3r}$ (which implies $x_d \geq \xi(Px) \geq \tilde{\xi}(Px)$)

$$\begin{aligned} &D^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma) \\ =&\gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot (\mathbf{e}_d - D\tilde{\xi}(Px)) \otimes (\mathbf{e}_d - D\tilde{\xi}(Px)) \\ &- \gamma|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \cdot D^2\tilde{\xi}(Px) \end{aligned} \tag{16}$$

(where we think of $D\tilde{\xi}$ as taking values in $\mathbb{R}^d$, $\mathbb{R}^d$ being a superspace of the tangent space of H; we also think of $D^2\tilde{\xi}$ as taking values in $\mathbb{R}^{d\times d}$) and using $-20 \leq \gamma \leq -1$ we obtain

$$\begin{aligned} &\left|\Delta^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma) - \gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4}\right| \\ \leq\ & C(d)|D\tilde{\xi}(Px)| \cdot (1 + |D\tilde{\xi}(Px)|)^3 \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \\ &+ C(d)|D^2\tilde{\xi}(Px)| \cdot (1 + |D\tilde{\xi}(Px)|)^2 \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3} \\ &+ C(d)\left(|D^3\tilde{\xi}(Px)| \cdot (1 + |D\tilde{\xi}(Px)|) + |D^2\tilde{\xi}(Px)|^2\right) \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\ &+ C(d)|D^4\tilde{\xi}(Px)| \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \ . \end{aligned} \tag{17}$$

From (16), for $x \in M$ it follows that

$$\begin{aligned} &\bigg|D^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)) \\ &\quad - \gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) \cdot \mathbf{e}_d \otimes \mathbf{e}_d\bigg| \\ &\leq C(d)(|D\tilde{\xi}(Px)| + |D\tilde{\xi}(Px)|^2)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2}\phi^2(x) \\ &\quad + C(d)|D^2\tilde{\xi}(Px)|\ |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}\phi^2(x) \\ &\quad + C(d)\sup_x |D\phi(x)| \sup_{x\in\operatorname{supp} D\phi\cap M} \left[|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}|\mathbf{e}_d - D\tilde{\xi}(Px)|\right] \\ &\quad + C(d)\sup_x(|D^2\phi(x)| + |D\phi(x)|^2) \sup_{x\in\operatorname{supp} D\phi\cap M} |x_d - \tilde{\xi}(Px) + \delta|^\gamma \\ &\leq C(d)(Kr + K^2r^2)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2}\phi^2(x) \\ &\quad + C(d)K|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}\phi^2(x) \\ &\quad + C(d)\sup_x |D\phi(x)| \sup_{x\in\operatorname{supp} D\phi\cap M} \left[|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}|\mathbf{e}_d - D\tilde{\xi}(Px)|\right] \\ &\quad + C(d)\sup_x(|D^2\phi(x)| + |D\phi(x)|^2) \sup_{x\in\operatorname{supp} D\phi\cap M} |x_d - \tilde{\xi}(Px) + \delta|^\gamma \end{aligned}$$

where we have used the fact that $|D\tilde{\xi}(Px)| \leq |D\tilde{\xi}(0)| + C(d)K|Px| \leq C(d)Kr$ for $x \in Z_{3r}$ (by (P2') and $D\tilde{\xi}(0) = 0$) and that $\operatorname{supp}\phi \subset Z_{3r}$; moreover, we have made use of the estimate $|D^2\tilde{\xi}| \leq C(d)K$ (by (P2')).

We have $|x_d - \tilde{\xi}(Px) + \delta| \leq C(d)r$ for any $x \in \operatorname{supp}\phi$: it holds that $\operatorname{supp}\phi \subset Z_{3r}$; moreover we have $0 < \delta < r$ and $|\tilde{\xi}(Px)| \leq C(d)Kr^2 \leq C(d)r$ (by (P2'), by $\tilde{\xi}(0) = 0$, $D\tilde{\xi}(0) = 0$ and since $Kr \leq \epsilon\tau \leq 1$) for any $x \in Z_{3r}$.

Thus, the second term on the right-hand side in the previous inequality can be estimated from above by a constant times the first term on the right-hand side. Using the estimate on $D\tilde{\xi}$ and the bounds $|D\phi| \leq C(d)r^{-1}$ and $|D^2\phi| \leq C(d)r^{-2}$, we therefore obtain

$$\begin{aligned}
&\Big| D^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)) \\
&\quad - \gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) \cdot \mathbf{e}_d \otimes \mathbf{e}_d \Big| \\
&\leq C(d)(Kr + K^2r^2)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2}\phi^2(x) \\
&\quad + \frac{C(d)}{r}(1+Kr) \sup_{x \in \operatorname{supp} D\phi \cap M} |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \\
&\quad + \frac{C(d)}{r^2} \sup_{x \in \operatorname{supp} D\phi \cap M} |x_d - \tilde{\xi}(Px) + \delta|^{\gamma} \ .
\end{aligned}$$

By this estimate, the inequality $Kr \leq \epsilon \cdot \tau \leq 1$ (the latter inequality holds due to our conditions on τ and our choice of ϵ), and the first assertion of the present lemma, we obtain (14).

We now derive a similar estimate for the fourth derivative. Using the estimates on the derivatives of $\tilde{\xi}$ (see (P2')), the estimate $|D\tilde{\xi}(Px)| \leq C(d)Kr$ for $x \in Z_{3r}$ (see the proof of (14)), the fact that $|x_d - \tilde{\xi}(Px) + \delta| \leq C(d)r$ for any $x \in \operatorname{supp}\phi$ (see the proof of (14)), and the fact that $Kr \leq 1$, inequality (17) implies for any $x \in M$

$$\begin{aligned}
&\Big| \phi^2(x) \Delta^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma) \\
&\quad - \gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4}\phi^2(x) \Big| \\
&\leq C(d)Kr \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4}\phi^2(x) \ .
\end{aligned}$$

Thus, by the Leibniz formula and the estimates on the derivatives of ϕ, we obtain for $x \in M$

$$
\begin{aligned}
&\Big|\Delta^2(|x_d-\tilde\xi(Px)+\delta|^\gamma\phi^2(x))\\
&\qquad -\gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x)\Big|\\
&\leq C(d)Kr\cdot|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x)\\
&\quad+\sum_{j=0}^{3}C(d)r^{-4+j}\sup_{x\in\operatorname{supp}D\phi\cap M}\Big|D^j|x_d-\tilde\xi(Px)+\delta|^\gamma\Big|\\
&\leq C(d)Kr\cdot|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x)\\
&\quad+\sum_{j=0}^{3}C(d)r^{-4+j}\sup_{x\in\operatorname{supp}D\phi\cap M}C(d)|x_d-\tilde\xi(Px)+\delta|^{\gamma-j}\\
&\leq C(d)Kr\cdot|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x) \qquad (18)\\
&\quad+C(d)\sum_{j=0}^{3}r^{-4+j}(Kr^2)^{\gamma-j}\ ,
\end{aligned}
$$

where in the third step we have used the first assertion of the lemma and in the second step we have used the estimate

$$
\begin{aligned}
\Big|D|x_d-\tilde\xi(Px)+\delta|^\gamma\Big| &\leq C(d)(1+Kr)|x_d-\tilde\xi(Px)+\delta|^{\gamma-1}\\
&\leq C(d)|x_d-\tilde\xi(Px)+\delta|^{\gamma-1}
\end{aligned}
$$

which one easily verifies using $|D\tilde\xi(Px)|\leq C(d)Kr$ for $x\in Z_{3r}$ and $Kr\leq 1$, the estimate

$$
\begin{aligned}
&\Big|D^2|x_d-\tilde\xi(Px)+\delta|^\gamma\Big|\\
\leq& C(d)(1+Kr+K^2r^2)|x_d-\tilde\xi(Px)+\delta|^{\gamma-2}+C(d)K|x_d-\tilde\xi(Px)+\delta|^{\gamma-1}\\
\leq& C(d)(1+Kr+K^2r^2)|x_d-\tilde\xi(Px)+\delta|^{\gamma-2}\\
\leq& C(d)|x_d-\tilde\xi(Px)+\delta|^{\gamma-2}
\end{aligned}
$$

which follows from (16) in connection with the bound $|D\tilde\xi(Px)|\leq C(d)Kr$ and the bounds (P2') as well as the bound $|x_d-\tilde\xi(Px)+\delta|\leq C(d)r$ for

$x \in Z_{3r}$ (see above) and the inequality $Kr \leq 1$, and the estimate

$$\begin{aligned}
&\left|D^3|x_d - \tilde{\xi}(Px) + \delta|^\gamma\right| \\
\leq\, & C(d)|D^3\tilde{\xi}(Px)| \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \\
&+ C(d)|D^2\tilde{\xi}(Px)| \cdot |\mathbf{e}_d - D\tilde{\xi}(Px)| \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\
&+ C(d)|\mathbf{e}_d - D\tilde{\xi}(Px)|^3 |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3} \\
\leq\, & C(d)\frac{K}{r}|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \\
&+ C(d)K \cdot (1 + Kr) \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\
&+ C(d)(1 + Kr)^3 |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3} \\
\leq\, & C(d)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3}
\end{aligned}$$

which is obtained by differentiating (16) and using $|D\tilde{\xi}(Px)| \leq C(d)Kr$ for $x \in Z_{3r}$ as well as (P2') and $|x_d - \tilde{\xi}(Px) + \delta| \leq C(d)r$ for $x \in Z_{3r}$ and the inequality $Kr \leq 1$.

Applying $Kr \leq 1$ to (18) we obtain (15). □

We are now in position to prove our main theorem in the multidimensional case.

Proof (Theorem 2) Assertion a) is a consequence of Lemma 8: We set $b := \frac{9}{20}n + \frac{12}{20}$, $\gamma := -2$. For these choices, conditions (H1) to (H5) have already been checked in the proof of Theorem 1 (in case of (H5) for $\tau = \tau(n)$ sufficiently small).

W.l.o.g. we may assume that $x_0 = 0$, that $H = \{x \in \mathbb{R}^d : x_d = 0\}$, and that $x_0 + \mu\mathbf{e_d} \in M$ for any $\mu > 0$ small enough. Under the assumptions of Theorem 2, we can then find $R > 0$ such that in Z_R (as defined in (3)) our set M is the supergraph of a C^4 function $\xi : H \to \mathbb{R}$ with $D\xi(0) = 0$ and $\xi(0) = 0$; i.e. (4) holds. Set

$$K := \sup_{x \in Z_R} \max\left(|D^2\xi(Px)|, \frac{|D^3\xi(Px)|}{R}, \frac{|D^4\xi(Px)|}{R^2}\right) .$$

Then there exists $\tilde{R} \in (0, \frac{R}{3})$ such that $Z_{3\tilde{R}} \subset\subset \Omega$ and such that for any $r \in (0, \tilde{R}]$, the assumptions (P1), (P2) and (P3) of Lemma 8 (see Lemma 7) are fulfilled.

Possibly decreasing $\tilde{R}$, we may enforce

$$|x_d - \xi(Px)| \leq 2\,\mathrm{dist}(x, \partial M) \tag{19}$$

for any $x \in Z_{\tilde{R}}$: if $\tilde{R}$ is small enough, we know that for $x \in Z_{\tilde{R}}$ we have $\mathrm{dist}(x, \partial M \setminus Z_R) > \mathrm{dist}(x, \partial M)$. In this case, as $Z_R \cap \partial M$ is given by the graph of ξ over $Z_R \cap H$, for $x \in Z_{\tilde{R}}$ we obtain

$$\begin{aligned}
&\mathrm{dist}(x, \partial M) \\
=& \inf_{y \in H \cap Z_R} \sqrt{|Px - y|^2 + |x_d - \xi(y)|^2} \\
=& \inf_{y \in H \cap Z_R : |Px - y| \leq |x_d - \xi(Px)|} \sqrt{|Px - y|^2 + |x_d - \xi(y)|^2} \ ,
\end{aligned}$$

where in the second step we have used the fact $\operatorname{dist}(x,\partial M)\leq |x_d-\xi(Px)|$. By the triangle inequality, we obtain

$$
\begin{aligned}
&\operatorname{dist}(x,\partial M)\\
\geq & \inf_{y\in H\cap Z_R:|Px-y|\leq |x_d-\xi(Px)|}\left[\sqrt{|Px-y|^2+|x_d-\xi(Px)|^2}-|\xi(Px)-\xi(y)|\right]\\
\geq & |x_d-\xi(Px)|-\sup_{y\in H\cap Z_R:|Px-y|\leq |x_d-\xi(Px)|}|\xi(Px)-\xi(y)|\\
\geq & |x_d-\xi(Px)|-3K\tilde{R}|x_d-\xi(Px)|\ ,
\end{aligned}
$$

where in the last step we have used the fact that $|D\xi(Pz)|\leq 3K\tilde{R}$ for $z\in Z_{3\tilde{R}}$ (which follows from $D\xi(0)=0$ and $|D^2\xi(z)|\leq K$ for $z\in Z_{3\tilde{R}}$); note that $y\in Z_{3\tilde{R}}$ since otherwise $|Px-y|\leq |x_d-\xi(Px)|$ could not hold: we have $x\in Z_{\tilde{R}}$ which implies $|x_d|\leq \tilde{R}$, $|Px|<\tilde{R}$ as well as $|\xi(Px)|\leq K\tilde{R}^2\leq \tilde{R}$ (since $\xi(0)=0$, $D\xi(0)=0$, $|D^2\xi|\leq K$, $K\tilde{R}\leq 1$). Thus we obtain by (P2) and (P3) (recall that we have already checked (P2) and (P3) for any $r\in(0,\tilde{R}]$)

$$
\operatorname{dist}(x,\partial M)\geq (1-3\epsilon\tau)|x_d-\xi(Px)|
$$

which finishes the proof of (19) since $\tau<1$ and $\epsilon<\frac{1}{10}$.

From now on, let $r\in(0,\frac{\tilde{R}}{3})$.

It remains to check (12). Using $|\xi(Px)|\leq Kr^2\leq \frac{r}{2}$ for $x\in H\cap Z_r$ (which follows from (P2), the fact that $D\xi(0)=0$ and $\xi(0)=0$, and the fact that $Kr\leq \epsilon\tau\leq\frac{1}{10}$), in case $\delta<\frac{r}{4}$ we have

$$
\begin{aligned}
&\int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha}\cdot|x_d-\xi(Px)+\delta|^\gamma\,dx\\
\geq &\int_{\{x:|Px|<r,|x_d-\xi(Px)|<2\delta\}} u_0^{1+\alpha}\cdot|x_d-\xi(Px)+\delta|^{-2}\,dx\\
\geq & c(d)\delta^{\frac{4(1+\alpha)}{n}-1}r^{d-1}\fint_{\{x:|Px|<r,|x_d-\xi(Px)|<2\delta\}}\left|\frac{1}{\delta^{\frac{4}{n}}}u_0\right|^{1+\alpha}\,dx\\
\geq & c(d)\delta^{\frac{4(1+\alpha)}{n}-1}r^{d-1}\fint_{\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial M)<\delta\}}\left|\frac{1}{\delta^{\frac{4}{n}}}u_0\right|^{1+\alpha}\,dx
\end{aligned}
\tag{20}
$$

where in the third step we have used (19) and the estimate

$$
\begin{aligned}
&\mathcal{L}^d(\{x:|Px|<r,|x_d-\xi(Px)|<2\delta\})\\
\leq & 2\mathcal{L}^d(\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial M)<\delta\})
\end{aligned}
\tag{21}
$$

which holds since

$$
\{x:|Px|<r,|x_d-\xi(Px)|<\delta\}\subset\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial M)<\delta\}\ .
$$

We now fix $\tilde{T}>0$. Denote by $(r_i)_{i\in\mathbb{N}}$ a sequence for which the outer lim sup in the definition of W in Theorem 2 is approached. Denote by $(\delta_j^r)_j$

for fixed $r > 0$ a sequence for which the inner lim sup in the definition of W in Theorem 2 is approached.

Note that $\frac{4(1+\alpha)}{n} - 1 < 0$ since $\alpha < -\frac{1}{2}$ and $n > 2$. Using (20), we see that by our definition of W (note that $\mathrm{dist}_C(x, x_0) = \max(|Px|, |x_d|)$ since $x_0 = 0$ and $H = \{x \in \mathbb{R}^d : x_d = 0\}$) we have

$$\lim_{j\to\infty} \int_{\{x:|Px|<r_i,|x_d|<2r_i\}} u_0^{1+\alpha} \cdot |x_d - \xi(Px) + \delta_j^{r_i}|^\gamma \, dx = \infty$$

for any i for which the inner lim sup in the definition of W is nonzero for $r = r_i$, in particular for any i large enough. Thus for any i large enough there exists $j_0(i, \tilde{T})$ such that for any $j \geq j_0(i, \tilde{T})$ condition (12) is satisfied for our r_i, $\delta_j^{r_i}$ and our fixed $\tilde{T}$ (note that the regularity $u^{\frac{n+\alpha+1}{4}} \in L^4_{loc}(I; W^{1,4}(\Omega))$ is part of the definition of strong energy solutions).

Utilizing formula (20) to estimate the second integral on the right-hand side of (13) and estimating the first integral on the right-hand side of (13) (note that $-1 + 4\frac{1+\alpha}{n} < 0$ since $\alpha < -\frac{1}{2}$ and $n > 2$), we see that the waiting time T^* of u at x_0 is bounded from above by

$$\begin{aligned}
&T^* \leq \\
&\liminf_{r\to 0} \liminf_{\delta\to 0} \left[C(d,n) \cdot \left(r^{d-1} \frac{-1}{-1 + 4\frac{1+\alpha}{n}} \delta^{-1+4\frac{1+\alpha}{n}} \right)^{\frac{n}{1+\alpha}} \right. \\
&\qquad \left. \cdot \left(\delta^{\frac{4(1+\alpha)}{n} - 1} r^{d-1} \fint_{\{x:|Px|<r,|x_d|<r,\mathrm{dist}(x,\partial M)<\delta\}} \left[\frac{1}{\delta^{\frac{4}{n}}} u_0 \right]^{1+\alpha} dx \right)^{-\frac{n}{1+\alpha}} \right] \\
=&C(d,n) W^{-\frac{n}{1+\alpha}}
\end{aligned}$$

if the expression on the right-hand side does not exceed $\tilde{T}$. However, $\tilde{T} > 0$ was arbitrary and the expression does not depend on $\tilde{T}$. Choosing $\tilde{T}$ to be larger than this expression, this finishes the proof of assertion a).

Assertion b) is shown similarly. Again, w.l.o.g. we may assume that $x_0 = 0$, that $H = \{x \in \mathbb{R}^d : x_d = 0\}$, and that $x_0 + \mu \mathbf{e_d} \in M$ for any $\mu > 0$ small enough. Define ξ, R, K, $\tilde{R}$ as in the case of assertion a). Thus for $r \in (0, \frac{\tilde{R}}{3})$ conditions (P1) and (P2) are fulfilled and (19) holds.

However, to prove assertion b) we now let δ and r tend to zero simultaneously. Set $\alpha := -\frac{1}{2}$. Conditions (H1) to (H4) are readily verified. Condition (H5) is seen to be equivalent to

$$2\frac{\gamma - 3}{\gamma - 2} - \frac{5}{2} \geq \frac{25}{4}\tau$$

which in turn (due to $\gamma < 0$) is equivalent to

$$4(-\gamma + 3) - 5(-\gamma + 2) \geq \frac{25}{2}(-\gamma + 2)\tau$$

which in particular is satisfied if

$$\gamma \geq -2 + C\tau$$

holds for $C = 22 \cdot \frac{25}{2}$ (since we assume $\gamma \in [-20,-1]$).

We now set $r := \frac{1}{|\log\delta|}$ and $\tau := \frac{2K}{\epsilon|\log\delta|}$, i.e. $\gamma := -2 + \frac{CK}{\epsilon|\log\delta|}$ (with $\epsilon = \epsilon(d,n)$ from condition (P3)). By our choice of τ and r, the condition (P3) of Lemma 8 (see Lemma 7) is satisfied.

Let δ_i be a sequence converging to zero for which the lim sup in the assumptions of Theorem 2 b) is approached (with h replaced by δ).

It remains to check (12). We know $|\xi(Px)| \leq Kr^2 \leq \epsilon\tau r \leq \frac{r}{2}$ for $x \in Z_r$ (since $\xi(0) = 0$, $D\xi(0) = 0$, $|D^2\xi(Px)| \leq K$). For δ small enough, we have $r = |\log\delta|^{-1} > 4\delta$. Thus for δ small enough we see that condition (12) is satisfied for sure if

$$\begin{aligned}
&\int_{\{x:|Px|<|\log\delta|^{-1},|x_d-\xi(Px)|<2\delta\}} u_0^{1+\alpha} \cdot |x_d - \xi(Px) + \delta|^\gamma \, dx \\
\geq & C_0(d,n) K^{-6+\frac{CK}{\epsilon|\log\delta|}} |\log\delta|^{8-\frac{2CK}{\epsilon|\log\delta|}} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4 \, dx \, dt \ .
\end{aligned}$$

The previous inequality in turn is implied by the condition

$$\begin{aligned}
&\frac{c(d)|\log\delta|^{1-d}\delta}{3^{2-\frac{CK}{\epsilon|\log\delta|}}} \fint_{\{x:|Px|<\frac{1}{|\log\delta|},|x_d-\xi(Px)|<2\delta\}} u_0^{\frac{1}{2}} \delta^{-2+\frac{CK}{\epsilon|\log\delta|}} \, dx \\
\geq & C_0(d,n) K^{-6+\frac{CK}{\epsilon|\log\delta|}} |\log\delta|^{8-\frac{2CK}{\epsilon|\log\delta|}} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4 \, dx \, dt \ ,
\end{aligned}$$

which due to (19) and (21) in turn is implied by

$$\begin{aligned}
&c(d) e^{\frac{CK\log\delta}{\epsilon|\log\delta|}} \fint_{\{x:|Px|<\frac{1}{|\log\delta|},|x_d|<\frac{1}{|\log\delta|},\mathrm{dist}(x,\partial M)<\delta\}} \left[\frac{1}{\delta^2|\log\delta|^{14+2d}} u_0\right]^{\frac{1}{2}} \, dx \\
\geq & C_0(d,n) K^{-6+\frac{CK}{\epsilon|\log\delta|}} e^{-\frac{2CK\log|\log\delta|}{|\log\delta|\epsilon}} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4 \, dx \, dt \ .
\end{aligned}$$

Thus, evaluating at δ_i and passing to the limit $i \to \infty$, we see that the condition (12) is satisfied for any δ_i with i large enough if we have chosen $\tilde{T} > 0$ small enough (as the integral on the right-hand side of the present formula tends to zero as $\tilde{T} \to 0$).

Using $|\xi(Px)| \leq r$ for $x \in Z_{3r}$ and using (19), for δ so small that $r = |\log\delta|^{-1} > 4\delta$ the estimate (13) in connection with (19) and (21) yields

$$\begin{aligned}
&\inf\{t \geq 0 : \operatorname{supp} u(.,t) \cap Z_{\frac{3}{|\log\delta|}} \not\subset \operatorname{supp} u_0\}\\
\leq&\frac{C(d)\epsilon}{K}|\log\delta| \cdot \left(|\log\delta|^{-d+1}(C(d)|\log\delta|^{-1})^{\frac{CK}{\epsilon|\log\delta|}} \int_\delta^{\frac{C(d)}{|\log\delta|}} z^{-1}\ dz\right)^4\\
&\cdot\left(\int_{\{x:|Px|<|\log\delta|^{-1},|x_d-\xi(Px)|<2\delta\}} u_0^{\frac{1}{2}}|x_d-\xi(Px)+\delta|^\gamma\ dx\right)^{-4}\\
\leq&\frac{C(d)\epsilon}{K}|\log\delta| \cdot \left(|\log\delta|^{-d+1}(C(d)|\log\delta|^{-1})^{\frac{CK}{\epsilon|\log\delta|}} \int_\delta^{\frac{C(d)}{|\log\delta|}} z^{-1}\ dz\right)^4\\
&\cdot\left(\delta|\log\delta|^{-d+1}\fint_{\{x:|Px|<|\log\delta|^{-1},|x_d|<|\log\delta|^{-1},\operatorname{dist}(x,\partial M)<\delta\}} u_0^{\frac{1}{2}}\delta^{-2+\frac{CK}{\epsilon|\log\delta|}}\ dx\right)^{-4}
\end{aligned}$$

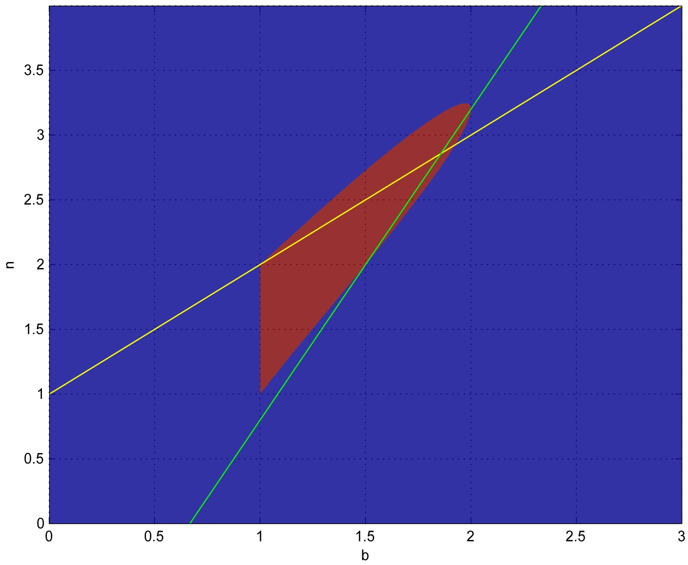
if the expression on the right-hand side does not exceed $\tilde{T}$. Rearranging, setting $\delta := \delta_i$, evaluating the first integral and letting $i \to \infty$, we obtain (since for i large enough we have $|\log|\log\delta_i|| + |\log C(d)| \leq |\log\delta_i|$)

$$\begin{aligned}
&T^* \leq\\
&\liminf_{i\to\infty}\Bigg[\frac{C(d)\epsilon}{K}|\log\delta_i|\\
&\quad\cdot\left(|\log\delta_i|^{-d+1}(C(d)|\log\delta_i|^{-1})^{\frac{CK}{\epsilon|\log\delta_i|}}\left(\log\frac{C(d)}{|\log\delta_i|}-\log\delta_i\right)\right)^4\\
&\quad\cdot\left(\delta_i|\log\delta_i|^{-d+1}\fint_{\{x:|Px|<|\log\delta_i|^{-1},|x_d|<|\log\delta_i|^{-1},\operatorname{dist}(x,\partial M)<\delta_i\}} u_0^{\frac{1}{2}}\delta_i^{-2+\frac{CK}{\epsilon|\log\delta_i|}}\ dx\right)^{-4}\Bigg]\\
&\leq\frac{C(d)\epsilon}{K}\lim_{i\to\infty}(C(d)|\log\delta_i|^{-1})^{\frac{4CK}{\epsilon|\log\delta_i|}}\\
&\quad\cdot\liminf_{i\to\infty}\left(|\log\delta_i|^{-\frac{5}{4}}\fint_{\{x:|Px|<|\log\delta_i|^{-1},|x_d|<|\log\delta_i|^{-1},\operatorname{dist}(x,\partial M)<\delta_i\}} u_0^{\frac{1}{2}}\delta_i^{-1}e^{-\frac{CK}{\epsilon}}\ dx\right)^{-4}
\end{aligned}$$

if the expression on the right-hand side is smaller than $\tilde{T}$. However, the first limit on the right-hand side is equal to 1, while the second limit on the right-hand side is zero by the assumptions in Theorem 2 b). This proves the second assertion of the theorem. □

6 Admissible Values for n and b and Limitations of our Approach

As a last point, we would like to discuss the results our approach yields for $n \in (1,2)$.



In the figure above, the red area marks the set of pairs (b, n) for which conditions (H1), (H2) and (H4) of Lemma 2 are satisfied. All pairs below the yellow line satisfy condition (H3). For all pairs below the green line, $\gamma = -2$ is an admissible value in condition (H5) of Lemma 4 and Lemma 7. The green line intersects the boundary of the red area at $n = 2$, $b = \frac{3}{2}$. The yellow line intersects the boundary of the red area at $n = \frac{2}{9}(10 + \sqrt{10}) \approx 2.92495$, $b = \frac{1}{9}(11 + 2\sqrt{10}) \approx 1.92495$.

Starting at $n = 2$, $b = \frac{3}{2}$ and tracking the boundary of the red area as n decreases, we see that for $n < 2$ the minimal values of γ which are admissible become larger until for $n = 1$ only values in $(-1, 0)$ are admissible. At the same time, $\alpha = b - n$ also increases until for $n = 1$ we have $\alpha = 0$. In particular, for $n < 2$ we have $1 + \gamma + 4\frac{1+\alpha}{n} > 0$.

Considering the case $d = 1$, let $x_0 \in \partial \operatorname{supp} u_0$ be a point with $\operatorname{supp} u_0 \cap (-\infty, x_0) = \emptyset$. Applying Lemma 4 with $x_1 := x_0 - \epsilon$, we see that the estimate on T^* provided by the lemma converges to zero as $\epsilon \to 0$ if near the free boundary the growth condition $u_0(x) \geq \tilde{S} \cdot (x - x_0)_+^{\frac{-\gamma-1}{1+\alpha}}$ is satisfied for some $\tilde{S} > 0$. Thus for $n < 2$ we only obtain immediate support spreading if u_0 grows steeper than $(x - x_0)_+^{\beta}$ at the free boundary for some $\beta = \beta(n) < 2$; this β tends to zero as n tends to 1. Note that for $\beta \leq \frac{1}{2}$ the condition $u_0 \in H^1(\mathbb{R})$ can no longer be satisfied; thus we have to work with the notion of solutions with weak initial trace.

Proof (Proof of Theorem 3) Dal Passo and Garcke [17] approximate u_0 by mollified versions $u_{0\delta} := \rho_\delta * u_0$ and consider the strong energy solution u_δ of the thin-film equation with initial data $u_{0\delta}$; then they pass to the limit $\delta \to 0$ and construct the solution u to be the limit of an appropriate subsequence.

First, observe that we have strong convergence of $u_\delta^{1+\alpha+n}$ towards $u^{1+\alpha+n}$ in $L^1_{loc}(\mathbb{R}\times[0,\infty))$ as $\delta\to 0$ if $n\in[1,2)$ and $\alpha\in(-\frac{1}{2},0]$: we know strong convergence of u_δ to u in $L^1_{loc}(\mathbb{R}\times(0,\infty))$ (see [17]); in connection with the estimate (see also [17])

$$\begin{aligned}||u_\delta(.,t)||_{L^\infty} \leq& C||\nabla u_\delta(.,t)||_{L^2}+C||u_\delta(.,t)||_{L^1}\\ \leq& C(n)||u_0||_{L^1}^{\frac{8-n}{8+2n}}t^{-\frac{3}{8+2n}}+C||u_0||_{L^1}\end{aligned}$$

which implies that u_δ^3 is bounded uniformly in $L^1([0,T);L^1)$ for every $T>0$, we obtain strong convergence of $u_\delta^{1+\alpha+n}$ to $u^{1+\alpha+n}$ in $L^1_{loc}(\mathbb{R}\times[0,\infty))$ since $1+\alpha+n<3$.

We now notice that the inequality from Lemma 2 survives the approximation procedure: For the solutions u_δ, Lemma 2 applies. Assume that $\operatorname{supp} u_0\subset B_{R_0}(0)$; by the finite speed of propagation result due to Bernis [5], we obtain $\operatorname{supp} u_\delta\subset B_{R_\delta(t)}(0)$, where $R_\delta(t):=R_0+\delta+C(n,||u_{0\delta}||_{L^1})t^{\frac{1}{4+n}}$. Note that $||u_{0\delta}||_{L^1}=||u_0||_{L^1}$. By the lower semicontinuity of weighted L^2 norms with continuous weights with respect to weak convergence, using our convergence $u_\delta^{1+\alpha+n}\to u^{1+\alpha+n}$ we see that the inequality from Lemma 2 also holds for the limit u if we have $t_1,t_2\leq T$ and if the test function ψ satisfies $\psi\in C_c^4(\mathbb{R})$ and $\psi_{xx}\geq 0$ on $B_{R_1(T)}(0)$.

Let $x_1<x_0$. Since the proof of Lemma 4 only used the inequality from Lemma 2 for the test function $|x-x_1|^\gamma$ and since for every $T>0$ this test function coincides on $(x_1+\epsilon,\infty)$ with a test function $\psi\in C_c^4(\mathbb{R})$ with $\psi_{xx}\geq 0$ on $B_{R_1(T)}(0)$, we see that Lemma 4 also applies to our limit u.

Setting $x_1:=x_0-\epsilon$ in Lemma 4, for $T:=\inf\{t>0:\operatorname{supp} u(.,t)\cap(-\infty,x_0)\neq\emptyset\}$ we obtain the estimate

$$T\leq\frac{1}{n\tau(1+\alpha)}\left[\int_{\bigcup_{t\in[0,1]}\operatorname{supp} u(.,t)}|x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{\frac{n}{1+\alpha}}\left[\int u_0^{1+\alpha}|x-x_1|^\gamma\,dx\right]^{-\frac{n}{1+\alpha}}$$

if the expression on the right-hand side does not exceed 1. Thus, we get

$$T\leq C(n,\alpha,\gamma)\left[\int_{\bigcup_{t\in[0,1]}\operatorname{supp} u(.,t)}|x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{\frac{n}{1+\alpha}}\left[⨍_{(x_0,x_0+\epsilon)}u_0^{1+\alpha}\epsilon^{\gamma+1}\,dx\right]^{-\frac{n}{1+\alpha}}$$

if the expression on the right-hand side does not exceed 1. Since $\gamma+4\frac{1+\alpha}{n}>-1$ for $n\in[1,2)$, by the finite speed of propagation estimate the first integral on the right-hand side converges to some constant value as $\epsilon\to 0$. Let γ_{inf} denote the infimum of the admissible values for γ for our fixed (b,n). The second integral tends to infinity as $\epsilon\to 0$ if the growth condition from our theorem is satisfied for $\beta:=\frac{-\gamma_{inf}-1}{1+\alpha}\in[0,2)$ and if γ has been chosen close enough to γ_{inf} (depending on the τ from our growth condition). Note that $\alpha\in(-\frac{1}{2},0)$ and $\gamma_{inf}\in(-2,-1)$ for $n\in(1,2)$ and that $\alpha=0$ and $\gamma_{inf}=-1$ for $n=1$. Thus, the first assertion of the theorem is established.

The assertions on the behaviour of β and α as $n\to 1$ or $n\to 2$ follow from the considerations regarding the admissible values for α and γ preceding the proof. □

Summing up, we have obtained sharp bounds on waiting times for solutions of the thin-film equation for $n \in (2, \frac{32}{11})$; in the regime of weak slippage $n \in (2, \frac{32}{11})$ the thin-film equation is seen to induce support spreading of solutions exactly as predicted by the order of degeneracy of the operator. The critical growth of initial data at the free boundary for the occurrence of a waiting time is $x_+^{\frac{4}{n}}$.

However, for $n \leq 2$ the situation changes drastically: for $n = 2$ we can only prove nonexistence of waiting times for initial data with growth steeper than $x_+^2 |\log x|^{\frac{1}{2}}$, whereas the existence of a waiting time has only be shown for growth like x_+^2 or slower. This gap becomes significantly larger when $n < 2$; both the minimal growth exponent $\frac{4}{n}$ known to be sufficient for the existence of a waiting time and the maximal exponent known to be sufficient for the nonexistence of a waiting time move away from 2 in opposite directions.

Note that the formal considerations by Blowey, King, and Langdon [13] suggest that in case $n \in (0, 2)$, at least for initial data with growth steeper than x_+^2 one might have instantaneous front propagation. However, the derivation of such an improved estimate remains an open problem.

A Derivation of the Equation for the Weighted Entropy

Proof (Lemma 1) Denoting a standard mollifier with respect to space by ρ_δ, we notice that $(\rho_\delta * u) \in H^1_{loc}(I; C^2(\Omega))$: for any $\xi \in C_c^\infty(\Omega \times (0, \infty))$ we have for $\delta > 0$ small enough (such that $\operatorname{supp}(\rho_\delta * \xi) \subset \Omega \times (0, \infty)$)

$$
\begin{aligned}
&\int_0^\infty \int_\Omega (\rho_\delta * u(.,t))(x) \frac{d}{dt}\xi(x,t)\ dx\ dt \\
&= \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_\delta(x-y) u(y,t) \frac{d}{dt}\xi(x,t)\ dy\ dx\ dt \\
&= \int_0^\infty \int_\Omega u(x,t) (\rho_\delta * \frac{d}{dt}\xi(.,t))(x)\ dx\ dt \\
&= \int_0^\infty \int_\Omega u(x,t) \frac{d}{dt}(\rho_\delta * \xi)(x,t)\ dx\ dt \\
&= -\int_0^\infty \left\langle \frac{d}{dt}u(x,t), \rho_\delta * \xi \right\rangle\ dt = -\int_0^\infty \left\langle \rho_\delta * \frac{d}{dt}u(x,t), \xi \right\rangle\ dt\ ,
\end{aligned}
$$

where we have used the symmetry of ρ_δ twice. This shows that the weak derivative of $\rho_\delta * u$ with respect to time exists and belongs to $L^2_{loc}(I; C^2(\Omega))$ (since we have $u \in H^1_{loc}(I; (W^{1,p}(\Omega))')$ and since the mollification of a distribution is smooth); moreover, we have the representation $\int_{t_1}^{t_2} \langle (\rho_\delta * u)_t, \xi \rangle\ dt = \int_{t_1}^{t_2} \langle u_t, \rho_\delta * \xi \rangle\ dt$ which holds for any smooth compactly supported ξ and for any $\xi \in L^2_{loc}(I; L^2(\Omega))$ by approximation.

Thus the function $\rho_\delta * [(\rho_\delta * u + \epsilon)^\alpha \psi]$ is an admissible test function in the weak formulation of the thin-film equation (see Definition 1). Taking $\psi \in C_c^\infty(\Omega)$, we can therefore compute

$$
\begin{aligned}
&\frac{1}{1+\alpha}\int (\rho_\delta * u+\epsilon)^{1+\alpha}\psi \ dx\Big|_{t_1}^{t_2}\\
=&\int_{t_1}^{t_2} \langle (\rho_\delta * u)_t, (\rho_\delta * u+\epsilon)^{\alpha}\psi\rangle \ dt\\
=&\int_{t_1}^{t_2} \langle u_t, \rho_\delta * [(\rho_\delta * u+\epsilon)^{\alpha}\psi]\rangle \ dt\\
=&\int_{t_1}^{t_2}\int u^n\nabla\Delta u\cdot\nabla\left(\rho_\delta * [(\rho_\delta * u+\epsilon)^{\alpha}\psi]\right) \ dx \ dt \ .
\end{aligned}
$$

We now pass to the limit $\delta\to 0$. Convergence of the left-hand side for a.e. t_1, t_2 and a.e. t_2 in case $t_1=0$ is immediate.

Recall that by our definition of strong energy solutions (Definition 1) we have $\nabla u^{\frac{n+2}{6}} \in L^6(I;L^6)$ and $u^{\frac{n}{2}}\nabla\Delta u \in L^2(I;L^2)$. By the Sobolev embedding and conservation of mass, we have $u^{\frac{n+2}{6}}\in L^6_{loc}(I;L^6(K))$ for any compact set $K\subset\subset \Omega$. Note that therefore $\nabla u = \frac{6}{n+2}u^{\frac{4-n}{6}}\nabla u^{\frac{n+2}{6}} \in L^{n+2}_{loc}(I;L^{n+2}(K))$ since $\frac{n+2}{6}+(4-n)\cdot\frac{1}{6}=1$. Moreover, $u^{\frac{n}{2}}=\left(u^{\frac{n+2}{6}}\right)^{\frac{3n}{n+2}}\in L_{loc}^{\frac{2(n+2)}{n}}(I;L^{\frac{2(n+2)}{n}}(K))$.

We calculate $\nabla(\rho_\delta * u+\epsilon)^{\alpha} = \alpha(\rho_\delta * u+\epsilon)^{\alpha-1}\cdot(\rho_\delta * \nabla u)$ and notice that $(\rho_\delta * u+\epsilon)^{-1+\alpha}\leq \epsilon^{-1+\alpha}$ since $\alpha\leq 0$. Putting these results together and rewriting the term on the right-hand side as

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int u^{\frac{n}{2}}\cdot u^{\frac{n}{2}}\nabla\Delta u\cdot\left(\rho_\delta * [\alpha(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot\psi\cdot(\rho_\delta * \nabla u)]\right) \ dx \ dt\\
&+\int_{t_1}^{t_2}\int u^{\frac{n}{2}}\cdot u^{\frac{n}{2}}\nabla\Delta u\cdot\left(\rho_\delta * [(\rho_\delta * u+\epsilon)^{\alpha}\cdot\nabla\psi]\right) \ dx \ dt \ ,
\end{aligned}
$$

we obtain convergence of the right-hand side since $\frac{n}{2(n+2)}+\frac{1}{2}+\frac{1}{n+2}=1$, since the convergence

$$
\begin{aligned}
&\left|\left|(\rho_\delta * [(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot(\rho_\delta * \nabla u)]) - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
\leq&\left|\left|\rho_\delta * \Big[(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot(\rho_\delta * \nabla u) - (u+\epsilon)^{-1+\alpha}\nabla u\Big]\right|\right|_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
&+\left|\left|\rho_\delta * \Big[(u+\epsilon)^{-1+\alpha}\nabla u\Big] - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
\leq&\left|\left|(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot(\rho_\delta * \nabla u) - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
&+\left|\left|\rho_\delta * \Big[(u+\epsilon)^{-1+\alpha}\nabla u\Big] - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
\underset{\delta\to 0}{\to}&\, 0
\end{aligned}
$$

holds for every $T>0$ (here in the second step we have used the fact that mollification does not increase the L^p norms), and since the convergence

$$\begin{aligned}
&||\rho_\delta * [(\rho_\delta * u+\epsilon)^\alpha \cdot \nabla\psi] - (u+\epsilon)^\alpha \cdot \nabla\psi||_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
\leq &||\rho_\delta * [(\rho_\delta * u+\epsilon)^\alpha \cdot \nabla\psi - (u+\epsilon)^\alpha \cdot \nabla\psi]||_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
&+||\rho_\delta * [(u+\epsilon)^\alpha \cdot \nabla\psi] - (u+\epsilon)^\alpha \cdot \nabla\psi||_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
\leq &||(\rho_\delta * u+\epsilon)^\alpha \cdot \nabla\psi - (u+\epsilon)^\alpha \cdot \nabla\psi||_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
&+||\rho_\delta * [(u+\epsilon)^\alpha \cdot \nabla\psi] - (u+\epsilon)^\alpha \cdot \nabla\psi||_{L^{n+2}([0,T];L^{n+2}(\Omega))}\\
\underset{\delta\to 0}{\to} &0
\end{aligned}$$

holds for every $T>0$ (again using the fact that mollification does not increase L^p norms).

Therefore the formula

$$\frac{1}{1+\alpha}\int (u+\epsilon)^{1+\alpha}\ dx\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int u^n \nabla\Delta u \cdot \nabla\left[(u+\epsilon)^\alpha \psi\right]\ dx\ dt \tag{22}$$

is valid for a.e. $t_1, t_2 \in I$ and a.e. $t_2 \in I$ in case $t_1 = 0$.

By expressions like $[(v+\epsilon)^\alpha]'$ we denote the derivative with respect to v of the function in brackets evaluated at v, i.e. in the this case $\alpha(v+\epsilon)^{\alpha-1}$. Given an arbitrary smooth strictly positive function v and a smooth compactly supported function ψ, we compute using repeated integrations by parts

$$
\begin{aligned}
&\int v^n \nabla \Delta v \cdot \nabla \left((v+\epsilon)^\alpha \psi\right) \\
=&\int (v+\epsilon)^\alpha v^n \nabla \Delta v \cdot \nabla \psi + \alpha \int (v+\epsilon)^{\alpha-1} v^n \nabla \Delta v \cdot \nabla v \ \psi \\
=&-\int [(v+\epsilon)^\alpha v^n]' \nabla v \cdot D^2 v \cdot \nabla \psi - \int (v+\epsilon)^\alpha v^n D^2 v : D^2 \psi \\
&-\alpha \int (v+\epsilon)^{\alpha-1} v^n |D^2 v|^2 \psi - \alpha \int [(v+\epsilon)^{\alpha-1} v^n]' \nabla v \cdot D^2 v \cdot \nabla v \ \psi \\
&-\alpha \int (v+\epsilon)^{\alpha-1} v^n \nabla v \cdot D^2 v \cdot \nabla \psi \\
=&\frac{1}{2} \int [(v+\epsilon)^\alpha v^n]'' |\nabla v|^2 \nabla v \cdot \nabla \psi + \frac{1}{2} \int [(v+\epsilon)^\alpha v^n]' |\nabla v|^2 \Delta \psi \\
&+\int [(v+\epsilon)^\alpha v^n]' \nabla v \cdot D^2 \psi \cdot \nabla v + \int (v+\epsilon)^\alpha v^n \nabla v \cdot \nabla \Delta \psi \\
&-\alpha \int (v+\epsilon)^{\alpha-1} v^n |D^2 v|^2 \psi - \alpha \int [(v+\epsilon)^{\alpha-1} v^n]' \nabla v \cdot D^2 v \cdot \nabla v \ \psi \\
&+\frac{\alpha}{2} \int (v+\epsilon)^{\alpha-1} v^n |\nabla v|^2 \Delta \psi + \frac{\alpha}{2} \int [(v+\epsilon)^{\alpha-1} v^n]' |\nabla v|^2 \nabla v \cdot \nabla \psi \\
=&-\frac{1}{2} \int \Big[[(v+\epsilon)^\alpha v^n]''' + \alpha [(v+\epsilon)^{\alpha-1} v^n]'' \Big] |\nabla v|^4 \psi \\
&-\frac{1}{2} \int \Big[[(v+\epsilon)^\alpha v^n]'' + \alpha [(v+\epsilon)^{\alpha-1} v^n]' \Big] |\nabla v|^2 \Delta v \ \psi \\
&-\int \Big[[(v+\epsilon)^\alpha v^n]'' + 2\alpha [(v+\epsilon)^{\alpha-1} v^n]' \Big] \nabla v \cdot D^2 v \cdot \nabla v \ \psi \\
&-\alpha \int (v+\epsilon)^{\alpha-1} v^n |D^2 v|^2 \psi \\
&+\frac{1}{2} \int \Big[[(v+\epsilon)^\alpha v^n]' + \alpha (v+\epsilon)^{\alpha-1} v^n \Big] |\nabla v|^2 \Delta \psi \\
&+\int [(v+\epsilon)^\alpha v^n]' \nabla v \cdot D^2 \psi \cdot \nabla v - \int \int_0^v (s+\epsilon)^\alpha s^n \ ds \ \Delta^2 \psi \ .
\end{aligned}
$$

Considering $\rho_\delta * v$ and passing to the limit $\delta \to 0$, one can prove that for any $v \in H^3_{loc}(\Omega)$ with $\inf_\Omega v > 0$ we have

$$\begin{aligned}
&\int v^n \nabla\Delta v \cdot \nabla\left((v+\epsilon)^\alpha \psi\right)\\
=&-\frac{1}{2}\int \Big[[(v+\epsilon)^\alpha v^n]''' + \alpha[(v+\epsilon)^{\alpha-1}v^n]''\Big]\,|\nabla v|^4\psi\\
&-\frac{1}{2}\int \Big[[(v+\epsilon)^\alpha v^n]'' + \alpha[(v+\epsilon)^{\alpha-1}v^n]'\Big]\,|\nabla v|^2\Delta v\ \psi\\
&-\int \Big[[(v+\epsilon)^\alpha v^n]'' + 2\alpha[(v+\epsilon)^{\alpha-1}v^n]'\Big]\,\nabla v\cdot D^2 v\cdot\nabla v\ \psi \qquad (23)\\
&-\alpha\int (v+\epsilon)^{\alpha-1}v^n|D^2 v|^2\psi\\
&+\frac{1}{2}\int \Big[[(v+\epsilon)^\alpha v^n]' + \alpha(v+\epsilon)^{\alpha-1}v^n\Big]\,|\nabla v|^2\Delta\psi\\
&+\int [(v+\epsilon)^\alpha v^n]'\nabla v\cdot D^2\psi\cdot\nabla v - \int\int_0^v (s+\epsilon)^\alpha s^n\ ds\ \Delta^2\psi\\
=:&\ I+II+III+IV+V+VI+VII\ .
\end{aligned}$$

Suppose now that $v \geq 0$ satisfies $\nabla v^{\frac{n+2}{6}} \in L^6(\Omega)$ and $v^{\frac{n}{2}}\nabla\Delta v \in L^2(\Omega)$ as well as $v^{\frac{n-2}{2}}\nabla v \otimes D^2 v \in L^2(\Omega)$; moreover, assume that $\nabla v^{\frac{1+n+\alpha}{4}} \in L^4(\Omega)$, $D^2 v^{\frac{1+n+\alpha}{2}} \in L^2(\Omega)$.

In this case, due to $d \leq 3$ and the Sobolev embedding we see that $v^{\frac{n+2}{6}}$ (and therefore also v) is continuous, so the sets $A_\delta := \{x : v(x) > \frac{1}{2}\delta\}$ are open and we have $\nabla\Delta v \in L^2(A_\delta \cap K)$, $\nabla v \otimes D^2 v \in L^2(A_\delta \cap K)$, $\nabla v \in L^6(A_\delta \cap K)$ for any $\delta > 0$ and any domain $K \subset\subset \Omega$. Take a smooth monotonous function ν with $0 \leq \nu \leq 1$, $\nu \equiv 0$ for $x < 0$ and $\nu \equiv 1$ for $x > 1$. Let

$$f_\delta(v) := \int_0^v \nu\left(\frac{s-\delta}{\delta}\right)\ ds + \delta\ . \qquad (24)$$

Using the fact that $f_\delta(v(.)) \equiv \delta$ in some neighbourhood of $\Omega \setminus A_\delta$, by $\nabla\Delta v \in L^2(A_\delta \cap K)$ and $\nabla v \otimes D^2 v \in L^2(A_\delta \cap K)$ and $\nabla v \in L^6(A_\delta \cap K)$ we infer that $\nabla\Delta f_\delta(v) = 0$ in some neighbourhood of $\Omega \setminus A_\delta$ and that

$$\begin{aligned}
&\nabla\Delta f_\delta(v)\\
=&f_\delta'(v)\nabla\Delta v + f_\delta''(v)(\nabla v\Delta v + 2D^2 v\cdot\nabla v) + f_\delta'''(v)|\nabla v|^2\nabla v
\end{aligned}$$

in A_δ; thus, recalling that all derivatives of f_δ are bounded we see that $\nabla\Delta f_\delta(v) \in L^2(K)$ for any domain $K \subset\subset \Omega$ and therefore $f_\delta(v) \in H^3_{loc}(\Omega)$. As $f_\delta(v) \in H^3_{loc}(\Omega)$ and $f_\delta(v) \geq \delta$, formula (23) applies to $f_\delta(v)$. We now pass to the limit $\delta \to 0$.

It is easy to check that $(f_\delta(v) - v) \to 0$ in L^∞: we have $|f_\delta(v) - v| \leq \delta + \int_0^v \left|\nu\left(\frac{s-\delta}{\delta}\right) - 1\right|\ ds \leq 3\delta$. Moreover, we obtain by dominated convergence

$$\nabla f_\delta(v) = f_\delta'(v)\nabla v = \frac{6}{n+2}f_\delta'(v)v^{\frac{4-n}{6}}\nabla v^{\frac{n+2}{6}} \to \frac{6}{n+2}v^{\frac{4-n}{6}}\nabla v^{\frac{n+2}{6}} = \nabla v$$

strongly in L^{n+2} as since $f_\delta'(v)$ is bounded uniformly and converges pointwise to 1 a.e. on $\{\nabla v \neq 0\}$. Since $n \geq 1$, this establishes convergence of the terms V, VI, and VII.

Convergence of term I is shown similarly: we see that

$$I = \int \frac{[[(f_\delta(v)+\epsilon)^\alpha f_\delta(v)^n]''' + \alpha[(f_\delta(v)+\epsilon)^{\alpha-1} f_\delta(v)^n]'']}{f_\delta(v)^{n+\alpha-3}} \cdot f_\delta(v)^{n+\alpha-3} |\nabla f_\delta(v)|^4 \psi \, dx \ . \tag{25}$$

Note that an estimate of the form

$$\frac{\left|[(v+\epsilon)^\alpha v^n]''' + \alpha[(v+\epsilon)^{\alpha-1} v^n]''\right|}{v^{\alpha+n-3}} \leq C(\alpha,n) \tag{26}$$

can be shown to hold: recalling that $\alpha \leq 0$, by the Leibniz formula we have

$$\begin{aligned} &\left|[(v+\epsilon)^\alpha v^n]''' + \alpha[(v+\epsilon)^{\alpha-1} v^n]''\right| \\ =&\left|\sum_{i=0}^{3} c_i(\alpha,n)(v+\epsilon)^{\alpha-i} v^{n+i-3}\right| \leq C(\alpha,n) v^{\alpha+n-3} \ . \end{aligned}$$

By dominated convergence we therefore get

$$\begin{aligned} &[f_\delta(v)]^{\frac{n+\alpha-3}{4}} \nabla f_\delta(v) = [f_\delta(v)]^{\frac{n+\alpha-3}{4}} f_\delta'(v) \frac{4}{n+1+\alpha} v^{\frac{3-n-\alpha}{4}} \nabla v^{\frac{n+1+\alpha}{4}} \\ &\to \frac{4}{n+1+\alpha} \nabla v^{\frac{n+1+\alpha}{4}} = v^{\frac{n+\alpha-3}{4}} \nabla v \end{aligned} \tag{27}$$

strongly in L^4; to obtain the dominating function we have made use of the estimate $\frac{v}{f_\delta(v)} \leq 2$ which holds since $f_\delta(v) \geq \delta + (v-2\delta)_+$ and of the fact that $\alpha \leq 0$, $n < 3$. Combining this convergence property with formula (25) and estimate (26) as well as convergence of $f_\delta(v)$ in L^∞, we deduce convergence of term I.

We now turn our attention to the terms II, III and IV which involve second derivatives. We calculate

$$[f_\delta(v)]^{\frac{\alpha+n-1}{2}} D^2 f_\delta(v) = [f_\delta(v)]^{\frac{\alpha+n-1}{2}} f_\delta'(v) D^2 v + [f_\delta(v)]^{\frac{\alpha+n-1}{2}} f_\delta''(v) \nabla v \otimes \nabla v \ . \tag{28}$$

The second term on the right-hand side equals

$$\frac{16}{(n+1+\alpha)^2} f_\delta''(v) \cdot v \cdot \nabla v^{\frac{n+1+\alpha}{4}} \otimes \nabla v^{\frac{n+1+\alpha}{4}} \ .$$

By the definition of $f_\delta(v)$, it holds that $f_\delta''(v) = 0$ for any $v < \delta$ and any $v > 2\delta$; moreover $|f_\delta''(v)| \leq C\delta^{-1}$. We thus see that $f_\delta''(v) \cdot v \leq C\delta^{-1} \cdot 2\delta \leq C$. As the second term on the right-hand side in (28) converges to zero pointwise a.e., by dominated convergence we therefore see that it converges to zero strongly in L^2 as $\delta \to 0$.

Convergence of the first term on the right-hand side of (28) to $\chi_{v\neq 0} v^{\frac{n+\alpha-1}{2}} D^2 v$ is immediate by dominated convergence: we have $v^{\frac{n+\alpha-1}{2}} D^2 v \in L^2$ and

$$f_\delta'(v) \frac{[f_\delta(v)]^{\frac{n+\alpha-1}{2}}}{v^{\frac{n+\alpha-1}{2}}} \leq C(n,\alpha,d)$$

since $f_\delta'(v) = 0$ for $v < \delta$, $|f_\delta'(v)| \leq 1$ for any v, and $\frac{1}{2} v \leq f_\delta(v) \leq \delta + v$ for any v.

In case $n + \alpha > 1$, we have $\chi_{v\neq 0} v^{\frac{n+\alpha-1}{2}} D^2 v = v^{\frac{n+\alpha-1}{2}} D^2 v$. In case $n + \alpha = 1$ we also see that $\chi_{v\neq 0} v^{\frac{n+\alpha-1}{2}} D^2 v = \chi_{v\neq 0} D^2 v = D^2 v = v^{\frac{n+\alpha-1}{2}} D^2 v$ a.e. as otherwise we would obtain the inequality $\lim_{\delta\to 0} ||D^2 f_\delta(v)||_{L^2}^2 = \int \chi_{v\neq 0} |D^2 v|^2 \, dx <$

$\int D^2 v \; dx = ||D^2 v||_{L^2}$ which clearly contradicts the lower semicontinuity of the L^2 norm with respect to convergence in the sense of distributions. Thus we have proven

$$[f_\delta(v)]^{\frac{\alpha+n-1}{2}} D^2 f_\delta(v) \to v^{\frac{\alpha+n-1}{2}} D^2 v \tag{29}$$

strongly in L^2 as $\delta \to 0$.

Using the strong convergence (29) in connection with the convergence result regarding the first derivative (27), the convergence of $(f_\delta(v) - v)$ in L^∞, and the estimates

$$\frac{\left|[(v+\epsilon)^\alpha v^n]'' + \alpha[(v+\epsilon)^{\alpha-1} v^n]'\right|}{v^{\alpha+n-2}} \leq C(n,\alpha) \tag{30}$$

and

$$\frac{\left|[(v+\epsilon)^\alpha v^n]'' + 2\alpha[(v+\epsilon)^{\alpha-1} v^n]'\right|}{v^{\alpha+n-2}} \leq C(n,\alpha) \tag{31}$$

as well as

$$\frac{\left|(v+\epsilon)^{\alpha-1} v^n\right|}{v^{n+\alpha-1}} \leq C(n,\alpha) \ , \tag{32}$$

we establish convergence of the terms II, III, IV by rewriting these terms analogous to the rearrangement (25) of term I.

It remains to prove convergence of the left-hand side in (23). It is easily seen that $(f_\delta(v) + \epsilon)^\alpha \to (v+\epsilon)^\alpha$ in $W^{1,6}$. We calculate

$$\begin{aligned} &[f_\delta(v)]^{\frac{n}{2}} \nabla \Delta f_\delta(v) = [f_\delta(v)]^{\frac{n}{2}} \Delta [f'_\delta(v) \nabla v] \\ =&[f_\delta(v)]^{\frac{n}{2}} \nabla \cdot [f''_\delta(v) \nabla v \otimes \nabla v + f'_\delta(v) D^2 v] \\ =&[f_\delta(v)]^{\frac{n}{2}} [f'''_\delta(v) |\nabla v|^2 \nabla v + 2 f''_\delta(v) D^2 v \cdot \nabla v + f''_\delta(v) \Delta v \nabla v + f'_\delta(v) \nabla \Delta v] \ . \end{aligned} \tag{33}$$

Using the regularity $\nabla v^{\frac{n+2}{6}} \in L^6$, $v^{\frac{n-2}{2}} \nabla v \otimes D^2 v \in L^2$, $v^{\frac{n}{2}} \nabla \Delta v \in L^2$, the fact that $\operatorname{supp} f''_\delta \subset [\delta, 2\delta]$, $\operatorname{supp} f'''_\delta \subset [\delta, 2\delta]$ and the estimates $|f''_\delta| \leq C(d,n)\delta^{-1}$, $|f'''_\delta| \leq C(d,n)\delta^{-2}$, $|f'-1| \leq 1$, we see that $[f_\delta(v)]^{\frac{n}{2}} \nabla \Delta f_\delta(v) \to v^{\frac{n}{2}} \nabla \Delta u$ strongly in L^2 by the dominated convergence theorem: estimating the first term on the right-hand side of (33), we get

$$\begin{aligned} &[f_\delta(v)]^{\frac{n}{2}} \cdot |f'''_\delta(v)| \cdot |\nabla v|^3 \leq [f_\delta(v)]^{\frac{n}{2}} v^{\frac{4-n}{2}} \cdot f'''_\delta(v) |\nabla v^{\frac{n+2}{6}}|^3 \\ \leq& C(d,n) \chi_{v \in (\delta, 2\delta)} \delta^{\frac{n}{2}} \delta^{\frac{4-n}{2}} \delta^{-2} |\nabla v^{\frac{n+2}{6}}|^3 = C(d,n) \chi_{v \in (\delta, 2\delta)} |\nabla v^{\frac{n+2}{6}}|^3 \end{aligned}$$

which implies pointwise convergence to 0 a.e. and yields the dominating function $C(d,n)|\nabla v^{\frac{n+2}{6}}|^3$. The second term on the right-hand side of (33) can be treated similarly. Regarding the third term, we immediately obtain convergence a.e. to the desired limit; moreover, we notice that the third term is bounded from above by $C(d,n) v^{\frac{n}{2}} |\nabla \Delta v|$ since $f_\delta(v) \leq 2v$ for any v.

This finishes the proof of (23) under the weakened regularity assumptions.

Now assume that u is a strong solution of the thin-film equation. We may then rewrite (22) using (23): for a.e. $t \in I$ we have $\nabla u^{\frac{n+2}{6}} \in L^6$, $u^{\frac{n}{2}} \nabla \Delta u \in L^2$, $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4$, $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2$; thus, for a.e. $t \in I$ formula (23) can be applied with $v := u(.,t)$. We get

$$\begin{aligned}
&\frac{1}{1+\alpha}\int (u+\epsilon)^{1+\alpha}\psi\ dx\Big|_{t_1}^{t_2}\\
=&-\frac{1}{2}\int\Big[[(u+\epsilon)^\alpha u^n]''' + \alpha[(u+\epsilon)^{\alpha-1}u^n]''\Big]\,|\nabla u|^4\psi\\
&-\frac{1}{2}\int\Big[[(u+\epsilon)^\alpha u^n]'' + \alpha[(u+\epsilon)^{\alpha-1}u^n]'\Big]\,|\nabla u|^2\Delta u\ \psi\\
&-\int\Big[[(u+\epsilon)^\alpha u^n]'' + 2\alpha[(u+\epsilon)^{\alpha-1}u^n]'\Big]\,\nabla u\cdot D^2u\cdot\nabla u\ \psi\\
&-\alpha\int (u+\epsilon)^{\alpha-1}u^n|D^2u|^2\psi\\
&+\frac{1}{2}\int\Big[[(u+\epsilon)^\alpha u^n]' + \alpha(u+\epsilon)^{\alpha-1}u^n\Big]\,|\nabla u|^2\Delta\psi\\
&+\int [(u+\epsilon)^\alpha u^n]'\nabla u\cdot D^2\psi\cdot\nabla u - \int\int_0^u (v+\epsilon)^\alpha v^n\ dv\ \Delta^2\psi\\
=&:I+II+III+IV+V+VI+VII+VIII\ .
\end{aligned}$$

Passing to the limit $\epsilon \to 0$, by dominated convergence the desired result is obtained: we just need to use the inequalities (26), (30), (31), (32) in connection with pointwise convergence a.e. to deal with the first four terms on the right-hand side and the inequalities

$$\frac{\big|[(v+\epsilon)^\alpha v^n]' + \alpha(v+\epsilon)^{\alpha-1}v^n\big|}{v^{\alpha+n-1}} \le C(\alpha,n)$$

and

$$\frac{|[(v+\epsilon)^\alpha v^n]'|}{v^{\alpha+n-1}} \le C(\alpha,n)$$

to deal with the fifth and sixth term. The last term is immediately seen to converge to the desired limit. □

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