BEHAVIOUR OF FREE BOUNDARIES IN THIN-FILM FLOW: THE REGIME OF STRONG SLIPPAGE AND THE REGIME OF VERY WEAK SLIPPAGE

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Abstract. We analyze the behaviour of free boundaries in thin-film flow in the regime of strong slippage $n \in [1,2)$ and in the regime of very weak slippage $n \in [\frac{32}{11},3)$ qualitatively and quantitatively. In the regime of strong slippage, we construct initial data which are bounded from above by the steady state but for which nevertheless instantaneous forward motion of the free boundary occurs. This shows that the initial behaviour of the free boundary is not determined just by the growth of the initial data at the free boundary. Note that this is a new phenomenon for degenerate parabolic equations which is specific for higher-order equations. Furthermore, this result resolves a controversy in the literature over optimality of sufficient conditions for the occurrence of a waiting time phenomenon. In contrast, in the regime of very weak slippage we derive lower bounds on free boundary propagation which are optimal in the sense that they coincide up to a constant factor with the known upper bounds. In particular, in this regime the growth of the initial data at the free boundary fully determines the initial behaviour of the interface.

1. Introduction

In this paper, we are concerned with the qualitative behaviour of free boundaries in solutions to the thin-film equation

$$\frac{d}{dt}u = -\operatorname{div}(u^n \nabla \Delta u), \qquad n \in \mathbb{R}^+,$$

in the case of strong slippage $n \in [1,2)$ and in the case of very weak slippage $n \in [\frac{32}{11},3)$. The thin-film equation describes the evolution of a thin viscous liquid film on a flat solid driven by surface tension. Different values of n correspond to different slip conditions on the fluid-solid interface: The case $n=3$ corresponds to a no-slip condition (see e.g. [1]), while the case $n=2$ (or more precisely, u^n replaced by u^2+u^3) corresponds to the Navier slip condition, the effective boundary condition for viscous flow on a rough surface [2]. For $n=1$ the thin-film equation arises as the lubrication approximation of the Hele-Shaw flow [3].

In order to prevent ill-posedness, one needs to prescribe an additional boundary condition at the free boundary. Typically one prescribes the contact angle of solutions. In this paper, we shall be concerned with the case of zero contact angle solutions only; i.e. we formally require $|\nabla u| = 0$ on $\partial \operatorname{supp} u(.,t)$. In the framework of weak solutions with zero contact angle, this condition is enforced by an additional regularity constraint on the solution. For existence of such weak solutions to the thin-film equation with zero contact angle, see the papers [4, 5, 6, 7, 8] (note that in the latter works, these solutions are called "strong solutions", as opposed to the weaker solutions of [6] without prescribed contact angle). For a stronger notion of solution (for which existence however is only guaranteed locally or for small initial data and which we shall not be concerned with in the sequel), see the recent works [9, 10, 11, 12, 13]. For solution concepts in case of nonzero contact angle and corresponding existence results, see [14, 15, 16, 17, 18].

The analysis of qualitative behaviour of solutions to the thin-film equation has a long history. Finite speed of support propagation of solutions has been shown in [5, 19, 20, 21, 22, 23]. If the initial data are "flat enough" at the free boundary, a waiting time phenomenon occurs [24]: The free boundary of the solution initially does not advance for some time before it starts moving forward. The waiting time has been estimated from below in [25]. However, all now-classical results on qualitative behaviour of weak solutions to the thin-film equation with zero contact angle have been concerned with proving upper bounds on free boundary propagation. Mainly due to the lack of a comparison principle and Harnack inequalities, no lower bounds on free boundary propagation have been available.

With no rigorous lower bounds on free boundary propagation available, there has been a controversy over the optimal condition for the occurrence of a waiting time phenomenon for $n < 2$. In [24], the authors have shown that an estimate of the form $u_0 \lesssim x_+^{\frac{4}{n}}$ is sufficient for a waiting time phenomenon to occur. The authors conjectured their condition to be optimal. In contrast, a formal analysis in [26] suggested the occurrence of a waiting time phenomenon also in case $u_0 \sim x_+^\beta$ with $\beta \geq 2$.

Only recently, the author of the present paper has developed a technique for the derivation of lower bounds on interface propagation. For the parameter range $n \in (1, \frac{32}{11})$, the author has shown that for large times the support of solutions spreads at roughly the same rate as the self-similar solution [27]. In the case of weak slippage $n \in (2, \frac{32}{11})$, sufficient conditions for instantaneous propagation of the free boundary in terms of the growth of initial data at the free boundary have been deduced [28]; with a grain of salt, these conditions are the converse of the sufficient conditions for the occurrence of a waiting time phenomenon in [24]. Thus, for $n \in (2, \frac{32}{11})$ the initial behaviour of the free boundary is entirely determined by the growth of the initial data at the free boundary. Nevertheless, the sharp conditions for the nonoccurrence of a waiting time being restricted to $n \geq 2$, the controversy regarding optimality of the sufficient conditions for a waiting time phenomenon in [24] for $n < 2$ has remained unresolved.

These recent results by the author are based on new monotonicity formulas for the thin-film equation of the form

$$\frac{d}{dt} \int u^{1+\alpha} |x - x_0|^\gamma \, dx \geq c \int u^{1+\alpha+n} |x - x_0|^{\gamma-4} \, dx \tag{1}$$

(for certain $\alpha \in (-1, 0)$ and $\gamma < 0$), which hold as long as the support of the solution does not touch the singularity of the weight. Combined with a differential inequality argument due to Chipot and Sideris [29], these formulas imply lower bounds on free boundary propagation. However, the procedure used for obtaining such formulas has been limited to the regime $n \in (1, \frac{32}{11})$. Moreover, in the regime $n \in (1, 2)$ the range of admissible values for γ has not been large enough to deduce conditions for instantaneous forward motion of the free boundary which are both necessary and sufficient at the same time.

It is well-known that the qualitative behaviour of solutions to the thin-film equation depends sensitively on the parameter n: for $n > 1.5$, no backward motion of the free boundary may happen, while for $n < 1.5$ the support of solutions may shrink. For $n \geq 3$, one expects the support of zero contact angle solutions to be constant in time. This sensitive dependence on the parameter is in contrast to the situation for the second-order analogue of the thin-film equation, the porous medium equation

$$u_t = \nabla \cdot (u^{m-1} \nabla u) \ ;$$

the qualitative behaviour of solutions to the porous medium equation is independent of the parameter $m > 1$.

Thus, it is of interest whether the limitations of the recent results by the author are caused by changes in qualitative behaviour of solutions to the thin-film equation or just by limitations of our technical tools.

In the present work, we show that the limitation of the monotonicity formulas to $n < \frac{32}{11}$ has been merely a technical issue of our estimates. Using an alternative strategy, we are able to prove monotonicity formulas of the form (1) also in the range $n \in [\frac{32}{11}, 3)$, thereby extending our results on asymptotic support propagation and waiting times to the full range $n \in (2,3)$. Due to the conjectured change in qualitative behaviour for $n \geq 3$, the upper bound of the interval $(2,3)$ is presumably optimal.

In contrast, the limitation of the sharp conditions for instantaneous propagation of the interface to $n \in (2,3)$ is caused by a change in qualitative behaviour: For $d = 1$, the solution with the profile x_+^2 is a steady state of the thin-film equation, although for $n < 2$ the profile x_+^2 violates the sufficient conditions for a waiting-time phenomenon of [24]. On the other hand, in the present work we construct for any $\beta \in (\frac{4}{n}, 2]$ some initial data u_0 satisfying $u_0(x) \leq x_+^\beta$ for which instantaneous propagation of the interface occurs.

Therefore for $n < 2$ the initial behaviour of the interface is not completely determined by the growth of the initial data at the free boundary. This is a yet unobserved phenomenon for degenerate parabolic PDEs. It is specific for higher-order equations as it entails a drastic violation of any comparison principle.

Furthermore, this result resolves the abovementioned controversy on optimality of the known sufficient conditions for a waiting time phenomenon for $n < 2$: the conditions in [24] are seen to be in general optimal also in this case. While we cannot exclude correctness of the predictions of the heuristics in [26] for “nice” initial data (i.e. data with $u_0(x) \sim x_+^\beta$ close to the free boundary at 0), our approach shows that the heuristics break down in case of oscillatory initial data (i.e. certain data for which only $u_0(x) \lesssim x_+^\beta$ is known). This result is interesting in that it is an example of the (heuristically known) peculiarities of fourth-order equations with respect to oscillatory initial data.

The idea of our construction of initial data u_0 with $u_0(x) \leq x_+^\beta$ (for $2 \leq \beta < \frac{4}{n}$) for which instantaneous interface propagation occurs is to consider an infinite sum of scaled droplets which accumulate at the initial left free boundary. After some time has passed, droplets spread at a rate comparable to the self-similar solution [27]. This enables us to show that every droplet has to spread beyond the initial left interface before it may merge with larger droplets.

To prove our monotonicity formulas for the range $n \in [\frac{32}{11}, 3)$, we apply a strategy suggested by a computer-based analysis of the problem. For $d > 1$, we additionally need to estimate the non-radial components of the derivatives of the solution carefully.

Throughout the paper we use standard notation for Sobolev spaces. We abbreviate $I := [0, \infty)$. By $L^p_{loc}(I; X)$ we denote the set consisting of all measurable mappings $u : I \to X$ which belong to $L^p([0,T]; X)$ for all $T > 0$.

2. Main Results

Let us recall the definition of weak solutions with zero contact angle for the thin-film equation.

Definition 1. *Let $1 \leq d \leq 3$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary of class $C^{1,1}$ which is piecewise smooth or let $\Omega = \mathbb{R}^d$. Let $u_0 \in H^1(\Omega)$ be nonnegative with bounded support and let $n \in (\frac{1}{8}, 2)$. A nonnegative function $u \in L^\infty(I; H^1(\Omega))$ is called a weak solution to the thin-film equation with zero contact angle if the following conditions are satisfied:*

a) *$u \in H^1_{loc}\left(I; \left[W^{1,p}(\Omega)\right]'\right)$ for all $p > \frac{4d}{2d+n(2-d)}$.*
b) *For any $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n)\backslash\{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I; L^2(\Omega))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I; L^4(\Omega))$.*
c) *For any $\xi \in C^\infty_c(\mathbb{R}^d \times I)$ we have*
$$\begin{aligned}\int_0^T \langle u_t, \xi\rangle \; dt = &\int_0^T \int_{\{u>0\}} u^n \nabla u \cdot \nabla\Delta\xi \; dx \; dt \\ &+ n \int_0^T \int_{\{u>0\}} u^{n-1} \nabla u \cdot D^2\xi \cdot \nabla u \; dx \; dt \\ &+ \frac{n}{2} \int_0^T \int_{\{u>0\}} u^{n-1} |\nabla u|^2 \Delta\xi \; dx \; dt \\ &+ \frac{n(n-1)}{2} \int_0^T \int_{\{u>0\}} u^{n-2} |\nabla u|^2 \nabla u \cdot \nabla\xi \; dx \; dt\end{aligned}$$
for all $T > 0$.
d) *u attains the initial data in the sense that $u(.,t) \to u_0$ in $L^1(\Omega)$ as $t \to 0$.*

Existence of such weak solutions with zero contact angle has been shown by Dal Passo, Garcke, and Grün [7] (note that these authors call these solutions “strong solutions”, as opposed to the weak solutions without prescribed contact angle of [6]).

There is a stronger notion of weak solution with zero contact angle which is characterized by the additional requirement that the Dirichlet energy be dissipated. Existence of such energy-dissipating weak solutions (the author decided to use this name in order to distinguish this notion of solution from the weaker notion of weak solutions with zero contact angle defined above) for the thin-film equation has been shown by Bernis in the case of one spatial dimension [5]. In case $d = 2$ or $d = 3$, proving existence of these solutions is much more demanding. In this case the proof has been carried out by Grün [8].

Definition 2. *Let $1 \leq d \leq 3$. Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain with boundary of class $C^{1,1}$ or let $\Omega = \mathbb{R}^d$. Let $u_0 \in H^1(\Omega)$ be nonnegative with bounded support. We call $u \in L^\infty(I; H^1(\Omega))$ an energy-dissipating weak solution to the thin-film equation if it is nonnegative and if the following conditions are satisfied:*

a) *We have $\nabla u^{\frac{n+2}{6}} \in L^6(I; L^6(\Omega))$, $u^{\frac{n-2}{2}} \nabla u \otimes D^2 u \in L^2(I; L^2(\Omega))$, $u^{\frac{n}{2}} \nabla \Delta u \in L^2(I; L^2(\Omega))$.*
b) *For any $\alpha \in (\max\{-1, \frac{1}{2} - n\}, 2 - n) \setminus \{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I; L^2(\Omega))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I; L^4(\Omega))$.*
c) *It holds that $u \in H^1_{loc}\Big(I; \big(W^{1,p}(\Omega)\big)'\Big)$ for all $p > \frac{4d}{2d+n(d-2)}$.*
d) *For any $\xi \in C_c^\infty(\mathbb{R}^d \times I)$ it holds that*
$$\int_0^T \langle u_t, \xi \rangle \; dt = \int_0^T \int_{\{u>0\}} u^n \nabla \Delta u \cdot \nabla \xi \; dx \; dt \; .$$
e) *u attains its initial data u_0 in the sense that $\lim_{t \to 0} u(.,t) = u_0(.)$ in $L^1(\Omega)$.*

Dal Passo, Giacomelli, and Grün [24] have given sufficient conditions for the occurrence of a waiting time phenomenon for solutions to the thin-film equation. E.g. in case $d = 1$ and $n < 2$ they show that for initial data whose left free boundary is located at 0 and which satisfy for some $S > 0$ and a certain $p > 1$
$$\left(\fint_{[0,r]} |u_0|^p \; dx \right)^{\frac{1}{p}} \leq S \cdot r^{\frac{4}{n}}$$
for any $r > 0$, a waiting time phenomenon occurs.

We now give an example showing that for initial data growing steeper at the free boundary than the critical growth $x_+^{\frac{4}{n}}$, instantaneous support spreading may happen (for certain initial data):

Theorem 3. *Let $n \in [1, 2]$ and $1 \leq d \leq 3$. Let $g : (0, \infty) \to \mathbb{R}$ satisfy*
$$\lim_{s \to 0} \frac{g(s)}{s^{\frac{4}{n}}} = \infty \; .$$
Then there exist compactly supported continuous nonnegative initial data $u_0 \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$ with $\operatorname{supp} u_0 \subset \{x \in \mathbb{R}^d : x_1 \geq 0\}$ and $\operatorname{supp} u_0 \cap \{x : x_1 = 0\} \neq \emptyset$ as well as $u_0(x) \leq g(x_1)$ for any $x \in \mathbb{R}^d$ such that the following holds: For any (zero contact angle) weak solution to the thin-film equation u constructed as in [7] *(and for any energy-dissipating weak solution) with initial data u_0, we have*
$$\inf \Big\{ t > 0 : \operatorname{supp} u(.,t) \cap \{x : x_1 < 0\} \neq \emptyset \Big\} = 0 \; .$$

Thus, the sufficient condition of [24] for the occurrence of a waiting time phenomenon is in general optimal. However, in case $d = 1$ the function x_+^2 is a steady state for the thin-film equation; for $n < 2$, it locally grows steeper than $x_+^{\frac{4}{n}}$ at the free boundary. For $n \in [1,2)$ the behaviour of solutions to the thin-film equation is therefore not fully determined by the growth of the initial data at the free boundary.

We now turn to our results in the case of very weak slippage $n \in [\frac{32}{11}, 3)$.

The following result has already been proven for $n \in (1, \frac{32}{11})$ in [27]; in the present work we extend it to $n \in (1,3)$. Note that in the special case $n = 1$, $d = 1$ there is a much stronger result due to Carrillo and Toscani [30].

Theorem 4. *Let $u_0 \in H^1(\mathbb{R}^d)$ be nonnegative and compactly supported. Assume $d \leq 3$ and $n \in (1,3)$. Let*

- *$n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$ and let u be an energy-dissipating weak solution to the Cauchy problem for the thin-film equation, or*
- *let $n \in (1,2)$ and let u be a weak solution with zero contact angle of the Cauchy problem constructed as in [7].*

Let $x \in \mathbb{R}^d$ be a point.

Denote by T^* the infimum of all T satisfying $\inf_{t\in[0,T]} \operatorname{dist}(x, \operatorname{supp} u(.,t)) = 0$. Then there exists a constant C_{low} depending only on d and n such that the following estimate holds:

$$T^* \leq C_{low} \left[\operatorname{dist}(x, \operatorname{supp} u_0) + \operatorname{diam}(\operatorname{supp} u_0)\right]^{4+d\cdot n} ||u_0||_{L^1}^{-n}$$

Note that for $n > \frac{3}{2}$, the support of solutions as constructed in [8] is nondecreasing with respect to time. Thus we obtain the following corollary:

Corollary 5. *Let $u_0 \in H^1(\mathbb{R}^d)$ be nonnegative and compactly supported. Assume $1 \leq d \leq 3$ and $\frac{3}{2} < n < 3$. Let u be an energy-dissipating weak solution of the Cauchy problem for the thin-film equation.*

Suppose that $\operatorname{supp} u(.,t_1) \subset \operatorname{supp} u(.,t_2)$ holds for all $0 \leq t_1 \leq t_2$.

Let $x_s \in \operatorname{supp} u_0$ be some point. Then there exists a constant $c(d,n)$ depending only on n and d such that for any $t > 0$ with $R(t) > 0$ we have

$$B_{R(t)}(x_s) \subset \operatorname{supp} u(.,t) \ ,$$

where

$$R(t) := c(d,n)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} - \operatorname{diam}(\operatorname{supp} u_0) \ .$$

An analogous version of the following theorem has been proven in [28] for $n \in (2, \frac{32}{11})$; we now prove it for $n \in [\frac{32}{11}, 3)$.

Theorem 6. *Let $d = 1$ and $x_0 \in \mathbb{R}$. Let u be an energy-dissipating weak solution to the Cauchy problem for the thin-film equation with compactly supported nonnegative initial data $u_0 \in H^1(\Omega)$.*

a) *Suppose $n \in \left[\frac{32}{11}, 3\right)$. Assume $\operatorname{supp} u_0 \cap (-\infty, x_0) = \emptyset$. Then there exists a constant $C > 0$ which depends only on n such that the quantity $T := \inf\{t > 0 : (-\infty, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset\}$ is bounded by*

$$T \leq C(n) \inf_{\epsilon>0} \epsilon^{4-\frac{2n}{10(3-n)}} \left[\int_{\mathbb{R}} u_0^{\frac{3-n}{2}} |x - x_0 + \epsilon|^{-1.1} \, dx\right]^{-\frac{2n}{3-n}} \ .$$

b) *Suppose* $n \in \left[\frac{32}{11}, 3\right)$. *Assume* $\operatorname{supp} u_0 \cap (x_0-\delta, x_0) = \emptyset$ *for some* $\delta > 0$ *and* $x_0 \in \partial \operatorname{supp} u_0$. *Then there exists a constant* $C > 0$ *which depends only on* n *and such that the waiting time* T^* *at* x_0 *is bounded by*

$$T^* \leq C(n) \left[\limsup_{\epsilon\to 0} ⨍_{(x_0,x_0+\epsilon)} \left[\frac{1}{\epsilon^{\frac{4}{n}}} u_0\right]^{\frac{3-n}{2}} dx \right]^{-\frac{2n}{3-n}} .$$

This theorem easily implies the following corollary:

Corollary 7. *Suppose* $d = 1$. *Let* u *be an energy-dissipating weak solution to the Cauchy problem for the thin-film equation with compactly supported nonnegative initial data* $u_0 \in H^1(\Omega)$. *Let a point* $x_0 \in \partial \operatorname{supp} u_0$ *be given such that* $\operatorname{supp} u_0 \cap (x_0-\delta, x_0) = \emptyset$ *holds for some* $\delta > 0$.

a) *Let* $n \in (2,3)$. *If*

$$u_0(x) \geq \tilde{S} \cdot (x-x_0)_+^{\frac{4}{n}}$$

is satisfied for all $x \in B_\epsilon(x_0)$ *for some* $\epsilon > 0$, *then the waiting time* T^* *at* x_0 *is bounded from above by*

$$T^* \leq C(n) \tilde{S}^{-n} .$$

b) *Let* $n \in (2,3)$. *If*

$$\lim_{x \searrow x_0} \frac{u_0(x)}{(x-x_0)_+^{\frac{4}{n}}} = \infty$$

holds, then the interface at x_0 *starts moving forward instantaneously.*

c) *Let* $n \in (2,3)$. *If*

$$u_0(x) \geq \tilde{S} \cdot (x-x_0)_+^{\frac{4}{n}-\beta}$$

is satisfied for all $x \in B_\epsilon(x_0)$ *for some* $\epsilon > 0$ *and some* $\beta > 0$, *then we have for any* $\mu \in (0,\epsilon)$

$$\inf\{t \geq 0 : \operatorname{supp} u(.,t) \cap (-\infty, x_0-\mu) \neq \emptyset\} \leq C(n) \tilde{S}^{-n} \mu^{n\beta} .$$

In particular, in case $n\beta > 1$ *the free boundary (considered as a function of time) cannot have better regularity than* $C^{\frac{1}{\beta n}}([0,\infty))$.

In the multidimensional case, the following result has been proven in [28] for $n \in (2, \frac{32}{11})$; we extend it to $n \in (2,3)$.

Theorem 8. *Let* u *be an energy-dissipating weak solution to the thin-film equation on a domain* $\Omega \subset \mathbb{R}^d$, $d \leq 3$, *with nonnegative initial data* $u_0 \in H^1(\Omega)$. *Assume that* $\operatorname{supp} u_0$ *is bounded. Let* $x_0 \in \partial \operatorname{supp} u_0 \cap \Omega$ *be some point with the property that in some neighbourhood of* x_0, $\operatorname{supp} u_0$ *is the closure of a* C^4 *domain.*

a) *Suppose* $n \in (2,3)$. *Provided that there exist constants* $r > 0$, $\tilde{S} > 0$, *such that for any* $x \in B_r(x_0) \cap \operatorname{supp} u_0$ *we have*

$$u_0(x) \geq \tilde{S} \cdot \operatorname{dist}(x, \partial \operatorname{supp} u_0)^{\frac{4}{n}} ,$$

the waiting time T^* *of* u *at* x_0 *is bounded from above by*

$$T^* \leq C(d,n) \tilde{S}^{-n} .$$

b) *Suppose that* $n \in (2,3)$. *Set* $A := (\operatorname{supp} u_0)^\circ$. *If we have*

$$\lim_{A \ni x \to x_0} \frac{u_0(x)}{\operatorname{dist}(x, \partial \operatorname{supp} u_0)^{\frac{4}{n}}} = \infty \ ,$$

then the interface at x_0 *starts moving forward instantaneously.*

3. Proof of the Main Results

3.1. **Optimality of Sufficient Conditions for the Existence of Waiting Times.** The following quantitative bound on support spreading due to Bernis [5], Hulshof and Shishkov [20], Bertsch, Dal Passo, Garcke, and Grün [21] and Grün [8] will be required for our proof:

Theorem 9. *Assume* $1 \leq d \leq 3$ *and* $\Omega = \mathbb{R}^d$. *Let* u *be an energy-dissipating weak solution to the thin-film equation and* $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$ *or let* u *be a (zero contact angle) weak solution to the thin-film equation constructed as in* [7] *and* $n \in (\frac{1}{8}, 2)$. *Assume that* $\operatorname{supp} u_0 \subset B_{R_0}(x)$ *for some* $R_0 > 0$ *and some* $x \in \mathbb{R}^d$. *Then for any* $t > 0$ *we have the estimate* $\operatorname{supp} u(.,t) \subset B_{R(t)}(x)$ *with*

$$R(t) := R_0 + C_{up} ||u_0||_{L^1}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} \ ,$$

where C_{up} *depends only on* d *and* n.

We now construct initial data growing steeper than $x_+^{\frac{4}{n}}$ such that in any corresponding solution to the thin-film equation the free boundary starts moving forward instantaneously.

Proof of Theorem 3. Let $y \in \mathbb{R}^d$ be given by $y = 4$ in case $d = 1$, $y = (4,0)^T$ in case $d = 2$, and $y = (4,0,0)^T$ in case $d = 3$. Take some nonnegative $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \varphi \subset B_1(y)$ and $\int_{\mathbb{R}^d} \varphi \, dx \geq 1$ as well as $\varphi \leq 1$.

Denote by (λ_k), (μ_k) two sequences of positive real numbers subject to the following conditions:

(S1) The sequence (λ_k) is decreasing with $\lambda_{k+1} \leq \frac{\lambda_k}{5}$, $\lambda_k \leq 1$, and $\lim_k \frac{\lambda_{k+1}}{\lambda_k} = 0$.
(S2) The sequence (μ_k) is increasing with $\lim_k \frac{\mu_{k+1}}{\mu_k} = \infty$.
(S3) We have $\mu_{k+1} \leq \mu_k \frac{\lambda_k}{\lambda_{k+1}}$.
(S4) The estimate $\mu_k \leq \lambda_k^{-1}$ is satisfied.
(S5) It holds that

$$\inf_{s \in (0, 5\lambda_k]} \frac{g(s)}{s^{\frac{4}{n}}} \geq \mu_k \ .$$

We shall show below that such sequences indeed exist.

Fix $K \in \mathbb{N}$ and define

$$u_0(x) := \sum_{k=K}^{\infty} \mu_k \lambda_k^{\frac{4}{n}} \varphi\left(\frac{1}{\lambda_k} x\right) \ .$$

It is immediate that $u_0 \in L^\infty(\mathbb{R}^d)$ since $\lim_k \mu_k \lambda_k^{\frac{4}{n}} = 0$ (recall that $n \leq 2$) and since the supports of the different $\varphi(\lambda_k^{-1} x)$ are disjoint (due to $\lambda_{k+1} \leq \frac{1}{4}\lambda_k$ and

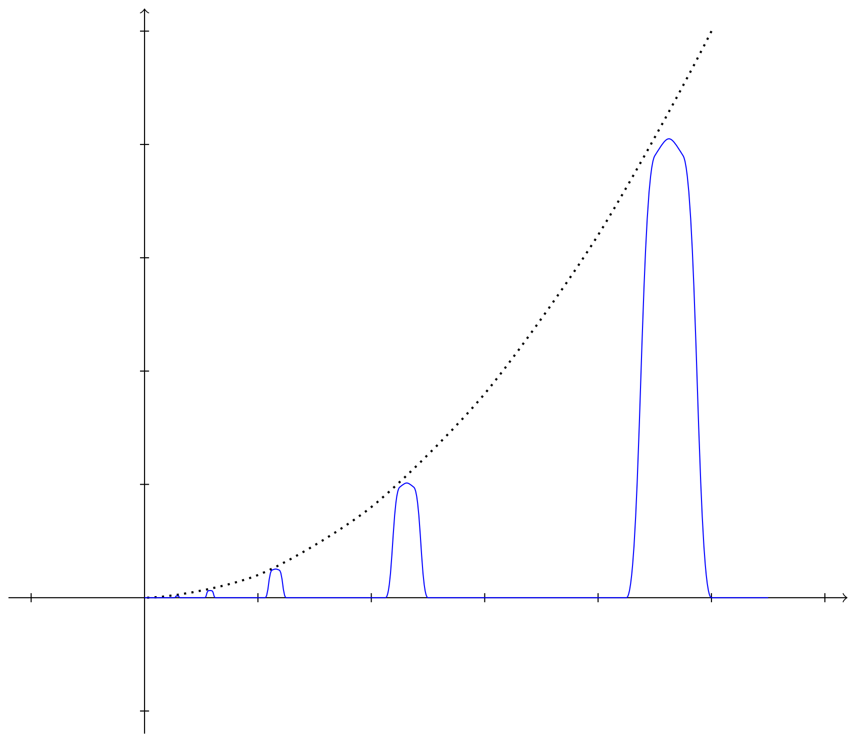

FIGURE 1. A sketch of the initial data u_0 (blue solid line) constructed as in the proof of Theorem 3. Note that it consists of an infinite sum of appropriately scaled droplets of the same kind. The function g (black line, dotted) dominates u_0.

$\operatorname{supp}\varphi \subset B_1(y)$). Moreover, we have $\operatorname{supp} u_0 \subset \{x \in \mathbb{R}^d : x_1 \geq 0\}$. Additionally we have the regularity $u_0 \in L^1(\mathbb{R}^d)$: we know that

$$\begin{aligned} ||u_0||_{L^1} \leq & \sum_{k=K}^{\infty} \mu_k \lambda_k^{\frac{4}{n}} \left|\left|\varphi\left(\frac{1}{\lambda_k} x\right)\right|\right|_{L^1} \\ \leq & \sum_{k=K}^{\infty} \mu_k \lambda_k^{\frac{4}{n}+d} ||\varphi||_{L^1} \\ \leq & ||\varphi||_{L^1} \sum_{k=K}^{\infty} \lambda_k^{\frac{4}{n}+d-1} \\ < & \infty \ , \end{aligned}$$

where in the penultimate step we have used (S4) and where the last estimate holds due to $\lambda_{k+1} \leq \frac{1}{4}\lambda_k$. Furthermore we observe that $u_0 \in H^1(\mathbb{R}^d)$ holds: we have

$$\begin{aligned}
||\nabla u_0||_{L^2(\mathbb{R}^d)} \leq & \sum_{k=K}^{\infty} \mu_k \lambda_k^{\frac{4}{n}} \left|\left|\nabla\left(\varphi\left(\frac{1}{\lambda_k}x\right)\right)\right|\right|_{L^2(\mathbb{R}^d)} \\
\leq & \sum_{k=K}^{\infty} \mu_k \lambda_k^{\frac{4}{n}-1} \left|\left|\nabla\varphi\left(\frac{1}{\lambda_k}x\right)\right|\right|_{L^2(\mathbb{R}^d)} \\
\leq & \sum_{k=K}^{\infty} \mu_k \lambda_k^{\frac{4}{n}-1+\frac{d}{2}} ||\nabla\varphi||_{L^2(\mathbb{R}^d)} \\
\leq & ||\nabla\varphi||_{L^2(\mathbb{R}^d)} \sum_{k=K}^{\infty} \lambda_k^{\frac{4}{n}-2+\frac{d}{2}} \\
< & \infty \ ,
\end{aligned}$$

where in the penultimate step we have used (S4) and where the last estimate follows since $n \leq 2$ and since $\lambda_{k+1} \leq \frac{1}{4}\lambda_k$. Finally we know that $u_0(x) \leq g(x_1)$ for all x: As the supports of the different droplets in the definition of u_0 are disjoint, it suffices to consider a single droplet. We have $\left|\mu_k \lambda_k^{\frac{4}{n}} \varphi\left(\frac{1}{\lambda_k}x\right)\right| \leq \mu_k \lambda_k^{\frac{4}{n}}$ and $\operatorname{supp}\varphi\left(\frac{1}{\lambda_k}\ \cdot\ \right) \subset B_{\lambda_k}(\lambda_k y)$ which implies $x_1 \in [3\lambda_k, 5\lambda_k]$ for any $x \in \operatorname{supp}\varphi\left(\frac{1}{\lambda_k}\ \cdot\ \right)$, i.e. (using (S5)) $g(x_1) \geq \mu_k x_1^{\frac{4}{n}} \geq \mu_k (3\lambda_k)^{\frac{4}{n}}$. Thus the estimate is established.

Making use of the finite speed of propagation result Theorem 9, we can estimate the time it takes for droplet

$$D_K(x) := \mu_K \lambda_K^{\frac{4}{n}} \varphi\left(\frac{1}{\lambda_K}x\right)$$

to merge with the rest

$$D_K^R(x) := \sum_{k=K+1}^{\infty} \mu_k \lambda_k^{\frac{4}{n}} \varphi\left(\frac{1}{\lambda_k}x\right) \ .$$

Applying Theorem 9 with $x := \lambda_K y$ and $R_0 := \lambda_K$ to bound the support of the droplet D_K, we see that the evolved support of the (initial) droplet D_K is contained in $B_{2\lambda_K}(y)$ as long as we have $t \leq T_{K,1}$, where

$$\begin{aligned}
T_{K,1} =& C_{up}^{-4-d\cdot n} ||D_K||_{L^1}^{-n} \lambda_K^{4+d\cdot n} \\
=& C_{up}^{-4-d\cdot n} ||\varphi||_{L^1}^{-n} \mu_K^{-n} \lambda_K^{-4-d\cdot n} \lambda_K^{4+d\cdot n} \\
=& C_{up}^{-4-d\cdot n} ||\varphi||_{L^1}^{-n} \mu_K^{-n} \ .
\end{aligned}$$

We apply Theorem 9 once more with $x := 0$ and $R_0 := (4+1)\lambda_{K+1}$. We see that the evolved support of the (initial) rest D_K^R is contained in $B_{2\lambda_K}(0)$ (recall that

$\lambda_K \geq 5\lambda_{K+1}$) as long as $t \leq T_{K,2}$, where

$$\begin{aligned}
T_{K,2} =& C_{up}^{-4-n\cdot d} ||D_K^R||_{L^1}^{-n} (2\lambda_K - 5\lambda_{K+1})^{4+d\cdot n} \\
\geq & C_{up}^{-4-n\cdot d} ||D_K^R||_{L^1}^{-n} \lambda_K^{4+d\cdot n} \\
=& C_{up}^{-4-n\cdot d} \left(\sum_{k=K+1}^{\infty} \mu_k \lambda_k^{\frac{4}{n}+d} ||\varphi||_{L^1} \right)^{-n} \lambda_K^{4+d\cdot n} \\
=& C_{up}^{-4-n\cdot d} \left(\sum_{k=K+1}^{\infty} (\mu_k \lambda_k) \lambda_k^{\frac{4}{n}+d-1} ||\varphi||_{L^1} \right)^{-n} \lambda_K^{4+d\cdot n} \\
\geq & C_{up}^{-4-n\cdot d} \left((\mu_{K+1}\lambda_{K+1}) \lambda_{K+1}^{\frac{4}{n}+d-1} \sum_{k=K+1}^{\infty} \left(\frac{\lambda_k}{\lambda_{K+1}} \right)^{\frac{4}{n}+d-1} ||\varphi||_{L^1} \right)^{-n} \lambda_K^{4+d\cdot n} \\
\geq & C_{up}^{-4-n\cdot d} \left(\mu_{K+1} \lambda_{K+1}^{\frac{4}{n}+d} ||\varphi||_{L^1} \sum_{k=0}^{\infty} 5^{-k} \right)^{-n} \lambda_K^{4+d\cdot n} \\
=& C_{up}^{-4-n\cdot d} \left(\frac{5}{4} \right)^{-n} ||\varphi||_{L^1}^{-n} \mu_{K+1}^{-n} \lambda_{K+1}^{-4-d\cdot n} \lambda_K^{4+d\cdot n} \\
\geq & 2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_{K+1}^{-n} \lambda_{K+1}^{-n} \lambda_K^{n} \\
\geq & 2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_K^{-n} \ .
\end{aligned}$$

Here in the second inequality we have used (S3) and in the third inequality we have used (S1) as well as $n \leq 2$; in the penultimate inequality, we have used (S1) and, finally, in the last inequality we have used (S3). Summing up, the droplet D_K cannot merge with the rest D_K^R before

$$T_K = \min(T_{K,1}, T_{K,2}) \geq 2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_K^{-n} \ .$$

As long as the droplet D_K and the rest D_K^R remain bounded away from each other, we may consider them as separate solutions. Thus for $t < T_K$ we may consider the droplet

$$D_{K+1}(x) := \mu_{K+1} \lambda_{K+1}^{\frac{4}{n}} \varphi \left(\frac{1}{\lambda_{K+1}} x \right)$$

and the rest

$$D_{K+1}^R(x) := \sum_{k=K+1+1}^{\infty} \mu_k \lambda_k^{\frac{4}{n}} \varphi \left(\frac{1}{\lambda_k} x \right) \ .$$

In this case, by the same arguments as above we can show that the droplet D_{K+1} cannot merge with the rest D_{K+1}^R before T_{K+1}, where

$$\begin{aligned}
& T_{K+1} = \min(T_K, T_{K+1,1}, T_{K+1,2}) \\
& \geq \min(2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_K^{-n}, C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_{K+1}^{-n}, 2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_{K+1}^{-n}) \\
& \geq 2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_{K+1}^{-n}
\end{aligned}$$

(note that we need to include T_K in our minimum since our argument is based on the assumption that D_K^R has not yet merged with the larger droplet D_K). More generally, define

$$D_M(x) := \mu_M \lambda_M^{\frac{4}{n}} \varphi\left(\frac{1}{\lambda_M} x\right)$$

and

$$D_M^R(x) := \sum_{k=M+1}^{\infty} \mu_k \lambda_k^{\frac{4}{n}} \varphi\left(\frac{1}{\lambda_k} x\right) \ .$$

Repeating the previous argument, we see by induction that generally the rest D_M^R (where $M \geq K$) does not merge with any larger droplet (i.e. the droplets $D_K,\ldots,D_M$) before time

$$T_M = \min(T_K, \ldots, T_{M-1}, T_{M,1}, T_{M,2}) \geq 2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_M^{-n} \ .$$

The previous observation now enables us to use our lower bounds on asymptotic support propagation rates, as for $t < T_M$ we may treat D_M^R as a separate solution: Applying Theorem 4 to the rest D_M^R (instead of u_0) with $x := -\lambda_{M+1} y$ (to apply Theorem 4, we need the condition $n \geq 1$ of our theorem), we obtain

$$\begin{aligned} &\inf\{t > 0 : \operatorname{supp} u(.,t) \cap \{x : x_1 < 0\} \neq \emptyset\} \\ \leq & C_{low} \left[4\lambda_{M+1} + 5\lambda_{M+1}\right]^{4+d\cdot n} ||D_M^R||_{L^1}^{-n} \end{aligned}$$

if the right-hand side does not exceed T_M (the latter condition is necessary as our argument is based on the assumption that D_M^R has not yet merged with a larger droplet). This in particular implies

$$\begin{aligned} &\inf\{t > 0 : \operatorname{supp} u(.,t) \cap \{x : x_1 < 0\} \neq \emptyset\} \\ \leq & 9^{4+d\cdot n} C_{low} \lambda_{M+1}^{4+d\cdot n} \left|\left|\mu_{M+1} \lambda_{M+1}^{\frac{4}{n}} \varphi\left(\lambda_{M+1}^{-1} x\right)\right|\right|_{L^1}^{-n} \end{aligned}$$

if the right-hand side does not exceed T_M, which in turn implies

$$\begin{aligned} &\inf\{t > 0 : \operatorname{supp} u(.,t) \cap \{x : x_1 < 0\} \neq \emptyset\} \\ \leq & 9^{4+d\cdot n} C_{low} \lambda_{M+1}^{4+d\cdot n} \mu_{M+1}^{-n} \lambda_{M+1}^{-4-d\cdot n} ||\varphi||_{L^1}^{-n} \\ = & 9^{4+d\cdot n} C_{low} ||\varphi||_{L^1}^{-n} \mu_{M+1}^{-n} \end{aligned}$$

if the right-hand side does not exceed T_M. Since $T_M \geq 2^{-n} C_{up}^{-4-n\cdot d} ||\varphi||_{L^1}^{-n} \mu_M^{-n}$ and since $\lim_k \frac{\mu_{k+1}}{\mu_k} = \infty$, the right-hand side indeed does not exceed T_M if M is large enough.

Thus, we finally see that choosing $K \in \mathbb{N}$ large enough, we obtain some initial data u_0 which satisfies all properties stated in our theorem (since for K large, the condition in the previous paragraph is satisfied for any $M \geq K$ and thus we have $\inf\{t > 0 : \operatorname{supp} u \cap \{x : x_1 < 0\} \neq \emptyset\} \leq 9^{4+d\cdot n} C_{low} ||\varphi||_{L^1}^{-n} \mu_{M+1}^{-n}$ for any $M \geq K$, i.e. $\inf\{t > 0 : \operatorname{supp} u \cap \{x : x_1 < 0\} \neq \emptyset\} = 0$). □

Lemma 10. *Sequences (λ_k), (μ_k) subject to conditions (S1) to (S5) indeed exist.*

Proof. Set $\lambda_0 := 1$. We define

$$\mu_0^a := \inf_{s \in (0, 5\cdot\lambda_0]} \frac{g(s)}{s^{\frac{4}{n}}} \ .$$

We then choose $\lambda_1 \in (0, \frac{\lambda_0}{5+0})$ small enough such that

$$\mu_1^a := \inf_{s\in(0,5\cdot\lambda_1]} \frac{g(s)}{s^{\frac{4}{n}}}$$

satisfies $\mu_1^a \geq (2+0)\mu_0^a$; this is possible due to our assumptions on g.

Proceeding similarly, we construct sequences (λ_k), (μ_k^a) by induction: we choose $\lambda_{k+1} \in (0, \min(\frac{\lambda_k}{5+k}, \frac{\lambda_k^2}{\lambda_{k-1}}))$ small enough such that

$$\mu_{k+1}^a := \inf_{s\in(0,5\cdot\lambda_{k+1}]} \frac{g(s)}{s^{\frac{4}{n}}}$$

satisfies $\mu_{k+1}^a \geq (2+k)\mu_k^a$; this is possible due to our assumptions on g. Summing up, we see that the sequences (λ_k) and (μ_k^a) satisfy conditions (S1), (S2) and (S5); moreover, we have $\frac{\lambda_{k+1}}{\lambda_k} \leq \frac{\lambda_k}{\lambda_{k-1}}$ for $k \geq 1$.

We now define another sequence (μ_k): we define $\mu_0 := \min(\lambda_0^{-1}, \mu_0^a)$ and set (inductively)

$$\mu_{k+1} := \min\left(\lambda_{k+1}^{-1}, \frac{\lambda_k}{\lambda_{k+1}}\mu_k, \mu_{k+1}^a\right) .$$

Since property (S5) is preserved when decreasing μ_k, we see that the sequences (λ_k), (μ_k) also satisfy (S5). The conditions (S3) and (S4) are satisfied by construction. As we do not change (λ_k), the property (S1) also holds. It remains to check (S2). Since (λ_k) is decreasing, since $\frac{\lambda_{k+1}}{\lambda_k} \leq \frac{1}{5}$, and since (μ_k^a) is increasing, we conclude that (μ_k) is also increasing. Furthermore using our definition of μ_{k+1} and μ_k we see that

$$\frac{\mu_{k+1}}{\mu_k} = \min\left(\frac{1}{\lambda_{k+1}\mu_k}, \frac{\lambda_k}{\lambda_{k+1}}, \frac{\mu_{k+1}^a}{\mu_k}\right) \geq \min\left(\frac{\lambda_k}{\lambda_{k+1}}, \frac{\lambda_k}{\lambda_{k+1}}, \frac{\mu_{k+1}^a}{\mu_k^a}\right) .$$

As all terms in the minimum on the right-hand side tend to infinity as $k \to \infty$, we see that (S2) holds. □

3.2. Extension of the lower bounds on interface propagation to $n \in [\frac{32}{11}, 3)$.

In this section, we extend the upper bounds on waiting times and the lower bounds on asymptotic support propagation rates to the whole interval $n \in (2, 3)$.

We shall need the following version of Hardy's inequality.

Lemma 11 (Hardy's inequality)**.** *For $v \in H^1(\mathbb{R}^d)$ with* $\operatorname{supp} v \subset\subset \mathbb{R}^d \setminus \{0\}$ *and any $\psi \in C^\infty(\mathbb{R}^d \setminus \{0\})$ with $\Delta\psi > 0$ on $\mathbb{R}^d \setminus \{0\}$ the inequality*

$$\int v^2 \Delta\psi \, dx \leq 4 \int \left| \frac{\nabla\psi}{|\nabla\psi|} \cdot \nabla v \right|^2 \frac{|\nabla\psi|^2}{\Delta\psi} \, dx$$

holds.

A proof can be found e.g. in [28].

We now prove the following basic estimate which will allow for the derivation of the desired monotonicity formulas. Note that the strategy for integrating by parts which we use has been suggested by a computer-based analysis of the problem in [31] in case $d = 1$.

Lemma 12. *Let $n \in [\frac{32}{11}, 3)$ and let u be an energy-dissipating weak solution to the thin-film equation on a domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with initial data $u_0 \in H^1(\Omega)$. Assume that $\operatorname{supp} u_0$ is bounded. Let $\alpha \in (-1, 0)$. Set $b := n + \alpha$ and assume that the following conditions are satisfied:*

(H1) *Assume that $1 \leq b \leq 2$.*
(H2) *Suppose that $\frac{n}{2} \leq b \leq n$.*
(H3) *Assume that $n - 1 < b$.*

Let $x_0 \in \mathbb{R}^d$ and $T > 0$. Suppose that $\operatorname{dist}(\bigcup_{t \in [0,T]} \operatorname{supp} u(.,t), x_0) > 0$ holds.

Let $\psi \in C_c^\infty(\Omega)$ be nonnegative. Then for a.e. $t_1, t_2 \in [0, T)$ with $t_2 \geq t_1$ and for a.e. $t_2 \in [0, T)$ in case $t_1 = 0$ we have

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
\geq &\frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2 \Delta\psi + \left(b - \frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u \\
&+ \frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2 u|^2 \psi \\
&- \frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi \\
&+ \left(\frac{5}{3}n - \frac{8}{3}b\right)\int_\Omega u^{b-1}\nabla u \cdot D^2 u \cdot \nabla\psi
\end{aligned} \tag{2}
$$

as well as

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
\geq &\frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2 \Delta\psi + \left(b - \frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u \\
&- \frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi \\
&- \frac{(5n-8b)^2}{24(n-b)}\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2 \frac{|\nabla\psi|^2}{\psi}\ .
\end{aligned}
$$

Proof. In the proof of Lemma 6 in [28] it is shown that if the conditions of our lemma are satisfied, the following equation holds:

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
=&\left(b-\frac{1}{2}n+\frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi + \left(b-\frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u \\
&-\frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi \\
&+\frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2u|^2\psi + \frac{1}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\Delta u|^2\psi \\
&+\left(\frac{2}{3}+\frac{1}{3}\right)\left(b-\frac{n}{2}\right)(b-1)(2-b)\int_{t_1}^{t_2}\int_\Omega u^{b-3}|\nabla u|^4\psi \\
&+\left(\frac{5}{3}n-\frac{8}{3}b\right)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi \\
&+\left(\frac{5}{6}n-\frac{4}{3}b\right)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}|\nabla u|^2\Delta u\ \psi\ .
\end{aligned}
$$

Additionally we know that

$$
\begin{aligned}
0 =& 2\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi + \int_\Omega u^{b-2}|\nabla u|^2\Delta u\ \psi \\
&+(b-2)\int_\Omega u^{b-3}|\nabla u|^4\ \psi + \int_\Omega u^{b-2}|\nabla u|^2\nabla u\cdot\nabla\psi
\end{aligned}
\tag{3}
$$

holds for any nonnegative u with $u^{\frac{b+1}{2}} \in H^2(\Omega)$: The formula is obvious for smooth strictly positive u; for strictly positive u with $u^{\frac{b+1}{2}} \in H^2(\Omega)$ it follows by approximation. Considering $u^{\frac{b+1}{2}} + \delta$ and letting $\delta \to 0$, the formula is also seen to hold in case $u^{\frac{b+1}{2}} \in H^2(\Omega)$. Putting the previous equations together, we deduce

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
=&\left(b-\frac{1}{2}n+\frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi + \left(b-\frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u \\
&-\frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi \\
&+\frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2u|^2\psi + \frac{1}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\Delta u|^2\psi \\
&+\frac{1}{3}(n-b)(b-1)(2-b)\int_{t_1}^{t_2}\int_\Omega u^{b-3}|\nabla u|^4\psi \\
&-\left(\frac{5}{6}n-\frac{4}{3}b\right)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}|\nabla u|^2\nabla u\cdot\nabla\psi\ .
\end{aligned}
$$

We also know that

$$
\begin{aligned}
0 =&(b-1)\int_\Omega u^{b-2}|\nabla u|^2\nabla u\cdot\nabla\psi + 2\int_\Omega u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi\\
&+\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi
\end{aligned}
\tag{4}
$$

holds for any nonnegative u with $u^{\frac{b+1}{2}}\in H^2(\Omega)$. This implies

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)\psi(.)\;dx - \int_\Omega \frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)\psi(.)\;dx\\
=&\frac{2}{3}\,(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi + \left(b-\frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u\\
&-\frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi\\
&+\frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2u|^2\psi + \frac{1}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\Delta u|^2\psi\\
&+\frac{1}{3}\,(n-b)\,(b-1)(2-b)\int_{t_1}^{t_2}\int_\Omega u^{b-3}|\nabla u|^4\psi\\
&+\left(\frac{5}{3}n-\frac{8}{3}b\right)\int_\Omega u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi\ .
\end{aligned}
$$

Dropping the penultimate term and the second term in the third line of the right-hand side (note that they are both nonnegative), we obtain the first assertion. Applying Young's inequality to the last term, we get the second assertion. □

We now proceed to the proof of our estimates on waiting times in the case of very weak slippage $n\in[\frac{32}{11},3)$. We first deal with the one-dimensional case.

Proof of Theorem 6. We only need to establish an appropriate monotonicity formula; then the proof of Theorem 1 in [28] carries over verbatim. In order to avoid duplication of arguments, we only present the proof of the monotonicity formula.

Set $\alpha := \frac{1-n}{2}$. We apply Lemma 12 with $\psi := |x-x_0+\epsilon|^{-1.1}$ to obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0+\epsilon|^{-1.1}\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0+\epsilon|^{-1.1}\ dx \\
\geq & 2.31 \cdot \frac{n-1}{3} \int_{t_1}^{t_2} \int_\Omega u^{b-1}|u_x|^2 |x-x_0+\epsilon|^{-3.1} \\
&+ 2.31 \cdot \frac{n+2}{3} \int_{t_1}^{t_2} \int_\Omega u^{b-1}|u_x|^2 |x-x_0+\epsilon|^{-3.1} \\
&- 2.31 \cdot 3.1 \cdot 4.1 \cdot \frac{2}{n+3} \int_{t_1}^{t_2} \int_\Omega u^{b+1} |x-x_0+\epsilon|^{-5.1} \\
&- 1.21 \frac{(n-4)^2}{12(n-1)} \int_{t_1}^{t_2} \int_\Omega u^{b-1}|u_x|^2 |x-x_0+\epsilon|^{-3.1} \\
= & \left(2.31 \cdot \frac{2n+1}{3} - 1.21 \frac{(n-4)^2}{12(n-1)}\right) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|u_x|^2 |x-x_0+\epsilon|^{-3.1} \\
&- 2.31 \cdot 3.1 \cdot 4.1 \cdot \frac{2}{n+3} \int_{t_1}^{t_2} \int_\Omega u^{b+1} |x-x_0+\epsilon|^{-5.1} \ .
\end{aligned}$$

By Lemma 11 we infer that

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0+\epsilon|^{-1.1}\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0+\epsilon|^{-1.1}\ dx \\
\geq & \left(2.31 \cdot \frac{2n+1}{3} - 1.21 \cdot \frac{(n-4)^2}{12(n-1)}\right) \cdot \frac{16}{(n+3)^2} \int_{t_1}^{t_2} \int_\Omega |(u^{\frac{b+1}{2}})_x|^2 |x-x_0+\epsilon|^{-3.1} \\
&- 2.31 \cdot 3.1 \cdot 4.1 \cdot \frac{2}{n+3} \int_{t_1}^{t_2} \int_\Omega u^{b+1} |x-x_0+\epsilon|^{-5.1} \\
\geq & \left(2.31 \cdot \frac{2n+1}{3} - 1.21 \cdot \frac{(n-4)^2}{12(n-1)}\right) \cdot \frac{4}{(n+3)^2} \cdot 4.1^2 \int_{t_1}^{t_2} \int_\Omega u^{b+1} |x-x_0+\epsilon|^{-5.1} \\
&- 2.31 \cdot 3.1 \cdot 4.1 \cdot \frac{2}{n+3} \int_{t_1}^{t_2} \int_\Omega u^{b+1} |x-x_0+\epsilon|^{-5.1} \\
= & \frac{2}{n+3} \left(2 \cdot 4.1^2 \cdot 2.31 \cdot \frac{2n+1}{3(n+3)} - 4.1^2 \cdot 1.21 \cdot \frac{(n-4)^2}{6(n-1)(n+3)} - 2.31 \cdot 3.1 \cdot 4.1\right) \\
&\quad \cdot \int_{t_1}^{t_2} \int_\Omega u^{b+1} |x-x_0+\epsilon|^{-5.1} \ .
\end{aligned}$$

Factorizing the factor in front of the right-hand side, we see that it is nonnegative for $n \in [\frac{32}{11}, 3)$. We thus get

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0+\epsilon|^{-1.1}\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0+\epsilon|^{-1.1}\ dx \\
\geq & c(n) \int_{t_1}^{t_2} \int_\Omega u^{b+1} |x-x_0+\epsilon|^{-5.1} \ .
\end{aligned}$$

This estimate implies the theorem by an argument analogous to the proof of Theorem 1 in [28] since $\frac{4}{n} \cdot (1+\alpha) < 0.1 = -\gamma - 1$. □

To prove Theorem 8, we only need to establish the following lemma (since then again, the proofs of Lemma 8 and Theorem 2 in [28] carry over). This lemma is an analogue of Lemma 7 in [28].

Lemma 13. *Let u be an energy-dissipating weak solution of the thin-film equation on a domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with nonnegative initial data $u_0 \in H^1(\Omega)$ with bounded support and let $n \in (\frac{29}{10}, 3)$. Set $\alpha := \frac{1-n}{2}$ and $b := n + \alpha$. Set $\gamma := -1.1$.*

Let M be the closure of a C^4 domain and let $x_0 \in \partial M$; w.l.o.g. we may assume that $x_0 = 0$. Denote the tangent plane to ∂M in 0 by H; w.l.o.g. (i.e. possibly after a rotation and reflection) we may assume that $H = \{x \in \mathbb{R}^d : x_d = 0\}$ and that $x_0 + \mu \vec{e_d} \in M$ for any $\mu > 0$ small enough. Denote the projection onto H by P. Define

$$Z_\rho := \{x : |Px| < \rho, |x_d| < \rho\} \ . \tag{5}$$

Let $R > 0$ and let $\xi : H \to \mathbb{R}$, $\xi \in C^4$, be a function such that

$$Z_R \cap M = Z_R \cap \{x \in \mathbb{R}^d : x_d \geq \xi(Px)\} \tag{6}$$

holds (for R small enough such a function exists by the implicit function theorem). Note that $\xi(0) = 0$ and that $\nabla\xi(0) = 0$ as H is tangent to ∂M at 0.

Assume that $Z_R \subset\subset \Omega$.

Take any $r \in \left(0, \frac{R}{3}\right)$ and any $K \in \mathbb{R}_0^+$ such that

- (P1) $\operatorname{supp} u_0 \cap Z_{3r} \subset M$, *i.e. locally near x_0 the support of u_0 is contained in M.*
- (P2) $|D^2\xi(Px)| \leq K$, $|D^3\xi(Px)| \leq \frac{K}{r}$, *and* $|D^4\xi(Px)| \leq \frac{K}{r^2}$ *for any $x \in \mathbb{R}^d$ with $|Px| \leq 3r$.*
- (P3) *The inequality $Kr < \epsilon(d,n)$ holds for some small constant $\epsilon(d,n) < \frac{1}{10}$ which is to be determined in the course of the proof below.*

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth cutoff with $0 \leq \phi \leq 1$, $\phi \equiv 1$ on Z_{2r}, $\operatorname{supp}\phi \subset Z_{3r}$, and $|\nabla\phi| \leq \frac{C(d)}{r}$, $|D^2\phi| \leq \frac{C(d)}{r^2}$, $|D^3\phi| \leq \frac{C(d)}{r^3}$, $|D^4\phi| \leq \frac{C(d)}{r^4}$. Define $\tilde{\xi} : H \to \mathbb{R}$ by

$$\tilde{\xi}(x) := \xi(x) - Kr^{-3}(|x| - r)_+^5 \ . \tag{7}$$

Set

$$T := \inf\{t > 0 : \operatorname{supp} u(.,t) \cap (\mathbb{R}^d \setminus M) \cap Z_{3r} \neq \emptyset\} \ . \tag{8}$$

Then we have for any $\delta \in (0, r)$

$$\begin{aligned} &\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t)|x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)\ dx\Bigg|_{t_1}^{t_2} \\ \geq &c(n) \int_{t_1}^{t_2} \int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \\ &- C(d,n)(r^4(Kr^2)^{\gamma-4} + (Kr^2)^\gamma) \int_{t_1}^{t_2} \int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4 \end{aligned} \tag{9}$$

for a.e. $0 < t_1 < t_2 < T$ and a.e. $0 < t_2 < T$ in case $t_1 = 0$.

Proof. Set

$$\epsilon(d,n) := \min\left(\epsilon_0, \epsilon_1, \frac{1}{10}\right) \tag{10}$$

where ϵ_0 and ϵ_1 are to be chosen below depending only on d and n. From now on, to simplify notation we write ϵ instead of $\epsilon(d,n)$.

Note that $\tilde{\xi} \in C^4$. The function $\tilde{\xi}$ satisfies some estimates similar to (P2), namely:

(P2') We have $|D^2\tilde{\xi}(Px)| \leq C(d)K$, $|D^3\tilde{\xi}(Px)| \leq \frac{C(d)K}{r}$, and $|D^4\tilde{\xi}(Px)| \leq \frac{C(d)K}{r^2}$ for any $x \in \mathbb{R}^d$ with $|Px| \leq 3r$.

We abbreviate $\psi(x) := |x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)$. This function obviously satisfies $\psi \in C^4(M)$ (as the points at which the function has singularities do not belong to M). We now use ψ as a weight in Lemma 12 and the estimates (11) and (12) from the lemma below to obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\, dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\, dx \\
\geq& \frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2 \Delta\psi + \left(b - \frac{n-b}{3}\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u \\
&- \frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi \\
&- \frac{(5n-8b)^2}{24(n-b)}\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2 \frac{|\nabla\psi|^2}{\psi} \\
\geq& \left(\gamma(\gamma-1)\frac{8}{3(b+1)^2}(n-b) - \gamma^2\frac{4(5n-8b)^2}{24(n-b)(b+1)^2} - C(d,n)Kr\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega |\nabla u^{\frac{b+1}{2}}|^2 \cdot \phi^2(x)|x_d - \tilde{\xi}(x) + \delta|^{\gamma-2} \\
&+ \gamma(\gamma-1)\frac{4(4b-n)}{3(b+1)^2}\int_{t_1}^{t_2}\int_\Omega |\partial_d u^{\frac{b+1}{2}}|^2 \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\
&- \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)Kr\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \\
&- C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\ .
\end{aligned}$$

Choosing $\epsilon_0 > 0$ small enough depending only on d and n, we see that the factor in front of the first term on the right-hand side is nonnegative (note that $n - b \geq \frac{9}{10}$, that $\frac{19}{10} \leq b \leq 2$, and that $|5n - 8b| \leq \frac{3}{2}$). We may thus drop the derivatives in directions perpendicular to $\vec{e}_d$. Rearranging, we obtain (note that

$|\nabla\phi| \leq C(d)r^{-1} \leq C(d)(Kr^2)^{-1})$

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\;dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\;dx \\
&\geq \left(\gamma(\gamma-1)\frac{8(n-b)+4(4b-n)}{3(b+1)^2} - \gamma^2\frac{4(5n-8b)^2}{24(n-b)(b+1)^2} - C(d,n)Kr\right) \\
&\qquad\qquad \cdot \int_{t_1}^{t_2}\int_\Omega |\partial_d(\phi\cdot u^{\frac{b+1}{2}})|^2 \cdot |x_d - \tilde\xi(Px)+\delta|^{\gamma-2} \\
&\quad - \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)Kr\right) \\
&\qquad\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde\xi(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\;.
\end{aligned}$$

Hardy's inequality (Lemma 11) implies

$$\begin{aligned}
&\frac{(\gamma-3)^2}{4}\int \phi^2 u^{b+1}|x_d - \tilde\xi(Px)+\delta|^{\gamma-4}dx_d \\
&\leq \int |\partial_d(\phi u^{\frac{b+1}{2}})|^2|x_d - \tilde\xi(Px)+\delta|^{\gamma-2}\;dx_d
\end{aligned}$$

since by assumption u is zero on some neighbourhood of $x_d = \tilde\xi(Px)-\delta$. Integrating this inequality with respect to $(x_1,\ldots,x_{d-1})$, we deduce that

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\;dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\;dx \\
&\geq \Bigg[\gamma(\gamma-1)(\gamma-3)^2\frac{2b+n}{3(b+1)^2} - \gamma^2(\gamma-3)^2\frac{(5n-8b)^2}{24(n-b)(b+1)^2} \\
&\qquad - \frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} - C(d,n)Kr\Bigg] \\
&\qquad\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde\xi(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{\frac{b+1}{2}}|\nabla u^{\frac{b+1}{4}}|^2 - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\;.
\end{aligned}$$

Note that $|\xi(Px)| \leq Kr^2 \leq r$ since $D\xi(0) = 0$ and $\xi(0) = 0$. Recalling that $u \equiv 0$ in $Z_{3r} \cap \{x : x_d < \xi(Px)\}$, we see that the Poincare inequality implies that $\int_{(-3r,3r)} u^{b+1}\;dx_d \leq Cr^{-4}\int_{(-3r,3r)} |\partial_d u^{\frac{b+1}{4}}|^4\;dx_d$. We therefore obtain by applying Young's inequality to the penultimate term in the previous estimate and using the Poincare inequality to estimate the last term on the right-hand side (note that

$r^{-1} \leq (Kr^2)^{-1}$)

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
\geq &\frac{\gamma(\gamma-1)(3-\gamma)}{n+3} \\
&\quad \cdot \left[\frac{41}{10}\cdot\frac{4(2n+1)}{3(n+3)} - \frac{11}{10}\cdot\frac{41}{10}\cdot\frac{10}{21}\cdot\frac{4(n-4)^2}{12(n-1)(n+3)} - \frac{31}{10}\cdot 2 - C(d,n)Kr\right] \\
&\quad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&- C(d,n)(Kr^2)^\gamma \int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4\ ,
\end{aligned}$$

where we have used our choices $\gamma = -1.1$ and $\alpha = \frac{1-n}{2}$. We see using e.g. a computer algebra program that for $n \in (2.9, 3)$ the factor in front of the first term on the right-hand side is strictly positive if $C(d,n)Kr$ is small enough; thus, if ϵ_1 is chosen small enough depending only on d and n, we get

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
\geq &c(n,d)\int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&- C(d,n)(Kr^2)^\gamma \int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4\ .
\end{aligned}$$

□

Proof of Theorem 8. Proceeding as in the proof of Theorem 2 in [28] (note that this proof relies on the estimate from Lemma 13), we obtain Theorem 8. It is essential for this proof to work that for the choices of γ and α in Lemma 13, we have $\gamma + \frac{4}{n}\cdot(1+\alpha)+1 < 0$. □

The following lemma has been proven in [28] (there it is called Lemma 9); we do not repeat its proof here.

Lemma 14. *With ϕ defined as at the beginning of the proof of the previous lemma, for any $x \in M \cap \operatorname{supp}\nabla\phi$ we have $x_d - \tilde{\xi}(Px) \geq Kr^2$.*

Moreover, the following estimate holds for the second derivative of ψ for any $x \in M$:

$$\begin{aligned}
&\left|D^2\psi(x) - \gamma(\gamma-1)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2}\cdot\phi^2(x)\cdot\vec{e}_d\otimes\vec{e}_d\right| \\
\leq &C(d)Kr|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2}\cdot\phi^2(x) + C(d)[Kr^2]^{\gamma-2}
\end{aligned} \tag{11}$$

For the fourth derivative, the following estimate is satisfied for any $x \in M$:

$$\begin{aligned}
&\left|\Delta^2\psi(x) - \gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4}\cdot\phi^2(x)\right| \\
\leq &C(d)Kr|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4}\cdot\phi^2(x) + C(d)[Kr^2]^{\gamma-4}
\end{aligned} \tag{12}$$

Our next aim is to extend the lower bounds on asymptotic support propagation rates to the range $n \in [\frac{32}{11}, 3)$.

Proof of Theorem 4. Again, it is sufficient to establish appropriate monotonicity formulas, since then the arguments of [27] carry over.

Using $|x-x_0|^\gamma$ as a weight function in inequality (2) and using the abbreviation $P := Id - \frac{x-x_0}{|x-x_0|} \otimes \frac{x-x_0}{|x-x_0|}$, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \,dx - \int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \,dx \\
\geq & \frac{2\gamma(\gamma-2+d)}{3}(n-b)\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b-1}|\nabla u|^2|x-x_0|^{\gamma-2} \\
&+\gamma(\gamma-1)\frac{4b-n}{3}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b-1}\left|\nabla u\cdot\frac{x-x_0}{|x-x_0|}\right|^2|x-x_0|^{\gamma-2} \\
&+\gamma\frac{4b-n}{3}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b-1}|P\nabla u|^2|x-x_0|^{\gamma-2} \\
&+\frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b-1}|D^2u|^2|x-x_0|^{\gamma} \\
&-\frac{\gamma(\gamma+d-2)(\gamma-2)(\gamma+d-4)}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b+1}|x-x_0|^{\gamma-4} \\
&+\gamma\left(\frac{5}{3}n-\frac{8}{3}b\right)\int_{\mathbb{R}^d} u^{b-1}\nabla u\cdot D^2u\cdot\frac{x-x_0}{|x-x_0|}|x-x_0|^{\gamma-2}\ .
\end{aligned}$$

Applying the estimate

$$\begin{aligned}
&\left|\nabla u\cdot D^2u\cdot\frac{x-x_0}{|x-x_0|}\right| \\
\leq & |P\nabla u|\cdot\left|PD^2u\cdot\frac{x-x_0}{|x-x_0|}\right| + \left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|\cdot\left|\frac{x-x_0}{|x-x_0|}\cdot D^2u\cdot\frac{x-x_0}{|x-x_0|}\right|
\end{aligned}$$

as well as Young's inequality to the last term in the previous inequality (note that mixed derivatives occur twice in $|D^2u|^2$, i.e. $|D^2u| = |\partial_1\partial_1 u|^2 + 2|\partial_1\partial_2 u|^2 + \ldots$), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \,dx - \int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \,dx \\
\geq & \left(\frac{2\gamma(\gamma-2+d)}{3}(n-b)+\gamma(\gamma-1)\frac{4b-n}{3}-\gamma^2\frac{(5n-8b)^2}{24(n-b)}\right) \\
&\quad\cdot\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b-1}\left|\nabla u\cdot\frac{x-x_0}{|x-x_0|}\right|^2|x-x_0|^{\gamma-2} \\
&+\left(\frac{2\gamma(\gamma-2+d)}{3}(n-b)+\gamma\frac{4b-n}{3}-\gamma^2\frac{(5n-8b)^2}{48(n-b)}\right) \\
&\quad\cdot\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b-1}|P\nabla u|^2|x-x_0|^{\gamma-2} \\
&+\frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b-1}|P\ D^2u\ P|^2|x-x_0|^{\gamma} \\
&-\frac{\gamma(\gamma+d-2)(\gamma-2)(\gamma+d-4)}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{b+1}|x-x_0|^{\gamma-4}\ .
\end{aligned} \tag{13}$$

We now subsequently treat the cases $d = 2$, $d = 1$, and $d = 3$, as the calculations differ to some amount. Suppose that $d = 2$. We then have

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^{b-1} |P\nabla u|^2 |x - x_0|^{\gamma-2} \\
=&\int_{t_1}^{t_2} \int_{(0,\infty)} \int_{(0,2\pi)} u^{b-1} |r^{-1}\partial_\phi u|^2 r^{\gamma-2} \cdot r\ d\phi\ dr\ dt \\
=&-b^{-1} \int_{t_1}^{t_2} \int_{(0,\infty)} \int_{(0,2\pi)} u^b r^{-2} \partial_\phi \partial_\phi u\ r^{\gamma-2} \cdot r\ d\phi\ dr\ dt \\
=&-b^{-1} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^b \operatorname{tr}(P\ D^2u\ P) |x - x_0|^{\gamma-2} .
\end{aligned}$$

We therefore obtain from (13), taking into account that $d = 2$

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2) |x - x_0|^\gamma\ dx - \int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1) |x - x_0|^\gamma\ dx \\
\geq &\left(\frac{2\gamma^2}{3}(n-b) + \gamma(\gamma-1)\frac{4b-n}{3} - \gamma^2 \frac{(5n-8b)^2}{24(n-b)} \right) \\
&\quad \cdot \frac{4}{(b+1)^2} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \left| \nabla u^{\frac{b+1}{2}} \cdot \frac{x - x_0}{|x - x_0|} \right|^2 |x - x_0|^{\gamma-2} \\
&- b^{-1} \left(\frac{2\gamma^2}{3}(n-b) + \gamma \frac{4b-n}{3} - \gamma^2 \frac{(5n-8b)^2}{48(n-b)} \right) \\
&\quad \cdot \int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^b \operatorname{tr}(P\ D^2u\ P) |x - x_0|^{\gamma-2} \\
&+ \frac{2}{3}(n-b) \int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^{b-1} |P\ D^2u\ P|^2 |x - x_0|^\gamma \\
&- \frac{\gamma^2(\gamma-2)^2}{b+1} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^{b+1} |x - x_0|^{\gamma-4} .
\end{aligned}$$

Using Lemma 11 to bound the first term on the right-hand side from below and applying Young's inequality to the second term on the right-hand side, we infer the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2) |x - x_0|^\gamma\ dx - \int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1) |x - x_0|^\gamma\ dx \\
\geq &\Bigg[\left(\frac{2\gamma^2}{3}(n-b) + \gamma(\gamma-1)\frac{4b-n}{3} - \gamma^2 \frac{(5n-8b)^2}{24(n-b)} \right) \cdot \frac{(\gamma-2)^2}{(b+1)^2} \\
&\quad - \frac{3}{8b^2(n-b)} \left(\frac{2\gamma^2}{3}(n-b) + \gamma \frac{4b-n}{3} - \gamma^2 \frac{(5n-8b)^2}{48(n-b)} \right)^2 \\
&\quad - \frac{\gamma^2(\gamma-2)^2}{b+1} \Bigg] \\
&\quad \cdot \int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^{b+1} |x - x_0|^{\gamma-4} .
\end{aligned}$$

Setting $\gamma := -2$ and $b := 2$, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \; dx - \int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \; dx \\ \geq&\Bigg[\left(\frac{8}{3}(n-2) + 2(8-n) - \frac{(5n-16)^2}{6(n-2)}\right) \cdot \frac{16}{9} \\ &- \frac{3}{32(n-2)} \left(\frac{8}{3}(n-2) - 2\frac{8-n}{3} - \frac{(5n-16)^2}{12(n-2)}\right)^2 - \frac{64}{3}\Bigg] \\ &\cdot \int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^{b+1}|x-x_0|^{\gamma-4} \; .\end{aligned}$$

Using e.g. a computer algebra program, we can show that the factor on the right-hand side is strictly positive for $n \in [\frac{32}{11}, 3)$. This yields

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^{-2} \; dx - \int_{\mathbb{R}^2} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^{-2} \; dx \\ \geq& c(n) \int_{t_1}^{t_2} \int_{\mathbb{R}^2} u^{b+1}|x-x_0|^{-6} \; .\end{aligned}$$

From this point on, we can follow the lines of the proof of the lower bounds on asymptotic support propagation rates from [27] to prove our theorem in case $d = 2$ (more precisely, we may repeat the steps below inequality (4) in [27] to prove an analogue of Lemma 11 from [27]. Having established such an analogue, the proof in Section 3.4 in [27] applies).

For $d = 1$, an analogous estimate follows directly from the second inequality in Lemma 12: we set $b := 2$ and use $|x-x_0|^{-1}$ as a weight to obtain

$$\begin{aligned}&\int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^{-1} \; dx - \int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^{-1} \; dx \\ \geq& 2\frac{2}{3}(n-2) \int_{t_1}^{t_2} \int_{\mathbb{R}} u^{b-1}|u_x|^2|x-x_0|^{-3} + 2 \cdot \frac{8-n}{3} \int_{t_1}^{t_2} \int_{\mathbb{R}} u^{b-1}|u_x|^2|x-x_0|^{-3} \\ &- \frac{2\cdot 3\cdot 4}{3} \int_{t_1}^{t_2} \int_{\mathbb{R}} u^{b+1}|x-x_0|^{-5} \\ &- \frac{(5n-16)^2}{24(n-2)} \int_{t_1}^{t_2} \int_{\mathbb{R}} u^{b-1}|u_x|^2|x-x_0|^{-3}\end{aligned}$$

which yields

$$\begin{aligned}&\int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^{-1} \; dx - \int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^{-1} \; dx \\ \geq& \left(\frac{2n+8}{3} - \frac{(5n-16)^2}{24(n-2)}\right) \int_{t_1}^{t_2} \int_{\mathbb{R}} u^{b-1}|u_x|^2|x-x_0|^{-3} \\ &- \frac{2\cdot 3\cdot 4}{3} \int_{t_1}^{t_2} \int_{\mathbb{R}} u^{b+1}|x-x_0|^{-5} \; .\end{aligned}$$

An application of Hardy's inequality (Lemma 11) (note that the factor in front of the first term is nonnegative since $n \in [\frac{32}{11}, 3))$ gives

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^{-1}\ dx - \int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^{-1}\ dx \\
\geq& \left[\left(\frac{2n+8}{3} - \frac{(5n-16)^2}{24(n-2)}\right)\cdot\frac{16}{9} - \frac{24}{3}\right] \int_{t_1}^{t_2}\int_{\mathbb{R}} u^{b+1}|x-x_0|^{-5}\ .
\end{aligned}
$$

Writing the factor on the right-hand side as a product, we see that for $n \in [\frac{32}{11}, 3)$ it is strictly positive, yielding

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^{-1}\ dx - \int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^{-1}\ dx \\
\geq& c(n) \int_{t_1}^{t_2}\int_{\mathbb{R}} u^{b+1}|x-x_0|^{-5}\ .
\end{aligned}
$$

This monotonicity formula again enables us to follow the lines of the proof of the lower bounds on asymptotic support propagation rates from [27] to prove our theorem in case $d=1$ (again, this formula provides a substitute for the estimate just below inequality (4) in [27]).

In case $d=3$, denoting by P_ϕ the projection onto the space spanned by $\vec{e}_\phi$ we compute

$$
\begin{aligned}
&\int_{\mathbb{R}^3} u^{b-1}|P_\phi \nabla u|^2 |x-x_0|^{\gamma-2} \\
=& \int_{(0,\infty)}\int_{(0,\pi)}\int_{(0,2\pi)} u^{b-1}|[r\sin(\theta)]^{-1}\partial_\phi u|^2 r^{\gamma-2}\cdot r\sin(\theta)\ d\phi\ d\theta\ dr \\
=& -b^{-1}\int_{(0,\infty)}\int_{(0,\pi)}\int_{(0,2\pi)} u^b [r\sin(\theta)]^{-2}\partial_\phi\partial_\phi u\ r^{\gamma-2}\cdot r\sin(\theta)\ d\phi\ d\theta\ dr \\
=& -b^{-1}\int_{\mathbb{R}^3} u^b\,\mathrm{tr}(P_\phi\ D^2u\ P_\phi)|x-x_0|^{\gamma-2}\ .
\end{aligned}
$$

Denoting by P_θ the projection onto the space spanned by $\vec{e}_\theta$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} u^{b-1}|P_\theta \nabla u|^2 |x-x_0|^{\gamma-2} \\
=&\int_{(0,\infty)}\int_{(0,2\pi)}\int_{(0,\pi)} u^{b-1}|r^{-1}\partial_\theta u|^2 r^{\gamma-2}\cdot r\sin(\theta)\ d\theta\ d\phi\ dr \\
=&-b^{-1}\int_{(0,\infty)}\int_{(0,2\pi)}\int_{(0,\pi)} u^b r^{-2}\partial_\theta\partial_\theta u\ r^{\gamma-2}\cdot r\sin(\theta)\ d\theta\ d\phi\ dr \\
&-b^{-1}\int_{(0,\infty)}\int_{(0,2\pi)}\int_{(0,\pi)} u^b r^{-2}\partial_\theta u\ r^{\gamma-2}\cdot r\cos(\theta)\ d\theta\ d\phi\ dr \\
=&-b^{-1}\int_{(0,\infty)}\int_{(0,2\pi)}\int_{(0,\pi)} u^b r^{-2}\partial_\theta\partial_\theta u\ r^{\gamma-2}\cdot r\sin(\theta)\ d\theta\ d\phi\ dr \\
&-\frac{1}{b(b+1)}\int_{(0,\infty)}\int_{(0,2\pi)}\int_{(0,\pi)} u^{b+1} r^{-2}\ r^{\gamma-2}\cdot r\sin(\theta)\ d\theta\ d\phi\ dr \\
&-\frac{1}{b(b+1)}\int_{(0,\infty)}\int_{(0,2\pi)} \left[u^{b+1} r^{-2}\ r^{\gamma-2}\cdot r\cos(\theta)\right]_{\theta=0}^{\theta=\pi}\ d\phi\ dr \\
=&-b^{-1}\int_{\mathbb{R}^3} u^b\,\mathrm{tr}(P_\theta D^2 u P_\theta)\ |x-x_0|^{\gamma-2} \\
&-\frac{1}{b(b+1)}\int_{\mathbb{R}^3} u^{b+1}|x-x_0|^{\gamma-4} \\
&+\frac{2\pi}{b(b+1)}\int_{(0,\infty)} [u^{b+1}(r\vec{e}_z)+u^{b+1}(-r\vec{e}_z)]\ r^{\gamma-4}\cdot r\ dr\ .
\end{aligned}$$

These two formulas enable us to use arguments analogous to the twodimensional situation: inserting these formulas into equation (13) and noting that $\mathrm{tr}(P_\theta A P_\theta)+$

$\mathrm{tr}(P_\phi A P_\phi) = \mathrm{tr}(PAP)$ holds for any matrix A, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\geq &\left(\frac{2\gamma(\gamma-2+d)}{3}(n-b) + \gamma(\gamma-1)\frac{4b-n}{3} - \gamma^2\frac{(5n-8b)^2}{24(n-b)}\right) \\
&\quad \cdot \int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b-1}\left|\nabla u \cdot \frac{x-x_0}{|x-x_0|}\right|^2 |x-x_0|^{\gamma-2} \\
&- b^{-1}\left(\frac{2\gamma(\gamma-2+d)}{3}(n-b) + \gamma\frac{4b-n}{3} - \gamma^2\frac{(5n-8b)^2}{48(n-b)}\right) \\
&\quad \cdot \int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b-1}\,\mathrm{tr}(PD^2uP)|x-x_0|^{\gamma-2} \\
&- \frac{1}{b(b+1)}\left(\frac{2\gamma(\gamma-2+d)}{3}(n-b) + \gamma\frac{4b-n}{3} - \gamma^2\frac{(5n-8b)^2}{48(n-b)}\right) \\
&\quad \cdot \left(\int u^{b+1}|x-x_0|^{\gamma-4} - 2\pi\int_{(0,\infty)} [u^{b+1}(r\vec{e_z}) + u^{b+1}(-r\vec{e_z})]\; r^{\gamma-4}\cdot r \; dr\right) \\
&+ \frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b-1}|P\; D^2u\; P|^2|x-x_0|^\gamma \\
&- \frac{\gamma(\gamma+d-2)(\gamma-2)(\gamma+d-4)}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b+1}|x-x_0|^{\gamma-4}\,.
\end{aligned}$$

After symmetrization (letting $\vec{e_z}$ take all possible values in S^2 and taking the average integral) the third term on the right-hand side is seen to vanish (note that all the other terms do not depend on the orientation of $\vec{e_z}$). Taking into account that $d=3$, this yields

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\geq &\left(\frac{2\gamma(\gamma+1)}{3}(n-b) + \gamma(\gamma-1)\frac{4b-n}{3} - \gamma^2\frac{(5n-8b)^2}{24(n-b)}\right) \\
&\quad \cdot \int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b-1}\left|\nabla u \cdot \frac{x-x_0}{|x-x_0|}\right|^2 |x-x_0|^{\gamma-2} \\
&- b^{-1}\left(\frac{2\gamma(\gamma+1)}{3}(n-b) + \gamma\frac{4b-n}{3} - \gamma^2\frac{(5n-8b)^2}{48(n-b)}\right) \\
&\quad \cdot \int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b-1}\,\mathrm{tr}(PD^2uP)|x-x_0|^{\gamma-2} \\
&+ \frac{2}{3}(n-b)\int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b-1}|P\; D^2u\; P|^2|x-x_0|^\gamma \\
&- \frac{\gamma(\gamma+1)(\gamma-2)(\gamma-1)}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} u^{b+1}|x-x_0|^{\gamma-4}\,.
\end{aligned}$$

An application of Young's inequality to the penultimate term and an application of Hardy's inequality (Lemma 11) to the first term on the right-hand side gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \ dx - \int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \ dx \\
\geq &\Bigg[\left(\frac{2\gamma(\gamma+1)}{3}(n-b) + \gamma(\gamma-1)\frac{4b-n}{3} - \gamma^2 \frac{(5n-8b)^2}{24(n-b)} \right) \cdot \frac{(\gamma-1)^2}{(b+1)^2} \\
&\quad - 2 \cdot \frac{3}{8b^2(n-b)} \left(\frac{2\gamma(\gamma+1)}{3}(n-b) + \gamma \frac{4b-n}{3} - \gamma^2 \frac{(5n-8b)^2}{48(n-b)} \right)^2 \\
&\quad - \frac{\gamma(\gamma+1)(\gamma-2)(\gamma-1)}{b+1} \Bigg] \\
&\quad \cdot \int_{t_1}^{t_2} \int_{\mathbb{R}^3} u^{b+1} |x-x_0|^{\gamma-4} \ .
\end{aligned}$$

Setting $b := 2$ and $\gamma := -3$, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \ dx - \int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \ dx \\
\geq &\Bigg[\left(4(n-2) + 4(8-n) - 9\frac{(5n-16)^2}{24(n-2)} \right) \cdot \frac{16}{9} \\
&\quad - 2 \cdot \frac{3}{32(n-2)} \left(4(n-2) - (8-n) - 9\frac{(5n-16)^2}{48(n-2)} \right)^2 - \frac{120}{3} \Bigg] \\
&\quad \cdot \int_{t_1}^{t_2} \int_{\mathbb{R}^3} u^{b+1} |x-x_0|^{\gamma-4} \ .
\end{aligned}$$

Using e.g. a computer algebra program, one can show that the factor on the right-hand side is strictly positive for $n \in [\frac{32}{11}, 3)$. This gives

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^{-3} \ dx - \int_{\mathbb{R}^3} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^{-3} \ dx \\
\geq &c(n) \int_{t_1}^{t_2} \int_{\mathbb{R}^3} u^{b+1} |x-x_0|^{-7}
\end{aligned}$$

which again may be used as a starting point for the arguments in [27] (we may use our formula as a substitute for the inequality just below estimate (4) in [27]). This finishes the proof of our theorem. □

4. Conclusion

To conclude, we would like to provide an overview of the known results on initial behaviour of free boundaries in thin-film flow; for simplicity, we restrict ourselves to the one-dimensional case.

- In general (for $n \in (0,3)$): If (with a grain of salt) the initial data u_0 grow at most like $S \cdot x_+^{\frac{4}{n}}$ at the free boundary, a waiting time phenomenon occurs.
- Case $n \in (0,1)$: Apart from the result just mentioned, nothing is known about the initial behaviour of free boundaries. Essentially this is due to the

fact that for $n<1$ no backward entropies exist. Whether this is merely a technical limitation or whether this fact points to a fundamental change in behaviour of solutions is also an open question.

- Case $n \in (1,2)$: There exist initial data growing just a bit steeper than $x_+^{\frac{4}{n}}$ at the free boundary for which immediate support spreading takes place. However, there also exist initial data growing steeper than $x_+^{\frac{4}{n}}$ for which the interface remains stationary. Thus, the initial behaviour of the free boundary is not determined by the growth of the initial data at the free boundary only.
- Case $n=2$: If the initial data u_0 grow steeper than $x_+^2 |\log x|^{\frac{1}{2}}$ at the free boundary, instantaneous forward motion of the interface occurs. The gap between the sufficient conditions for instantaneous support spreading and the sufficient conditions for the occurrence of a waiting time phenomenon is very small.
- Case $n \in (2,3)$: If the initial data u_0 grow steeper than $x_+^{\frac{4}{n}}$ at the free boundary, immediate support spreading takes place. In particular, the initial behaviour of the free boundary is entirely determined by the growth of the initial data at the free boundary.

One might conjecture that for $n \in (0,2)$ and initial data which grows steeper than x_+^2, immediate support spreading must happen. However, a proof of this assertion is presently out of reach.

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