UNIQUENESS OF SOLUTIONS OF THE DERRIDA-LEBOWITZ-SPEER-SPOHN EQUATION AND QUANTUM DRIFT-DIFFUSION MODELS

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Abstract. We prove uniqueness of solutions of the DLSS equation in a class of sufficiently regular functions. The global weak solutions of the DLSS equation constructed by Jüngel and Matthes belong to this class of uniqueness. We also show uniqueness of solutions for the quantum drift-diffusion equation, which contains additional drift and second-order diffusion terms. The results hold in case of periodic or Dirichlet-Neumann boundary conditions. Our proof is based on a monotonicity property of the DLSS operator and sophisticated approximation arguments; we derive a PDE satisfied by the pointwise square root of the solution, which enables us to exploit the monotonicity property of the operator.

1. Introduction

In this paper, we are concerned with proving uniqueness of weak solutions of the DLSS equation. The DLSS equation has originally been derived by Derrida, Lebowitz, Speer and Spohn [13] while analyzing interfaces in the Toom model, a probabilistic cellular automaton describing the evolution of a spin lattice. It also arises as the zero temperature and vanishing electric field limit of the quantum drift-diffusion equation, a drift-diffusion model for charge transport in semiconductors which takes lowest-order quantum corrections into account; for a derivation of the quantum drift-diffusion equation see the article by Degond, Gallego, Mehats, and Ringhofer [12] and the references therein. The quantum drift diffusion equation reads

$$u_t = \nabla \cdot (\vartheta \nabla u + u \nabla V)$$

where

$$V = V_{el} - \frac{\epsilon^2 \Delta \sqrt{u}}{6\sqrt{u}}$$

is the sum of the electric potential and the so-called quantum Bohm potential and where ϑ denotes temperature. Neglecting second-order (thermal) diffusion and drift induced by the electric field, after rescaling we obtain the dimensionless multidimensional DLSS equation:

$$u_t = -\nabla \cdot \left(u \nabla \frac{\Delta \sqrt{u}}{\sqrt{u}} \right) \tag{1}$$

It can be rewritten as

$$u_t = -D^2 : \left(\sqrt{u} D^2 \sqrt{u} - \nabla \sqrt{u} \otimes \nabla \sqrt{u} \right)$$

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or equivalently

$$2\,\partial_t\sqrt{u} = -\Delta^2\sqrt{u} + \frac{(\Delta\sqrt{u})^2}{\sqrt{u}}\ . \tag{2}$$

Initially nonnegative solutions of many fourth-order parabolic equations have the tendency to become negative at some points of the boundary of their support, thereby violating any comparison principle. In fact, for a large class of degenerate fourth-order equations the convex hull of the support of the solution cannot increase as long as the solution stays nonnegative (see e.g. the paper by Bernis [1]). In contrast, the DLSS equation is one of the two prominent examples of parabolic partial differential equations of fourth order which admit nonnegative global solutions, the other example being the thin-film equation $u_t = -\operatorname{div}(u^n\nabla\Delta u)$, $n \in \mathbb{R}^+$. In the case of the thin-film equation, the diffusion term degenerates as $u \to 0$. In contrast, the DLSS equation is nondegenerate; nonnegativity of solutions is instead preserved due to the equation becoming singular as u approaches zero. However, as one expects for fourth-order equations, numerical examples [17] have shown that the DLSS equation still violates any comparison principle.

For initial data $u_0 \in H^1$ which is bounded away from zero, local in time existence of solutions has been established by Bleher, Lebowitz and Speer [4] by a semigroup approach. As long as the solution stays bounded and bounded away from zero, uniqueness of these mild solutions is guaranteed. In case of periodic or no-flux boundary conditions, mass is conserved.

Since no result on the preservation of strict positivity for solutions of the DLSS equation is known and the semigroup approach breaks down when the solution touches zero, Jüngel and Pinnau [19] constructed weak nonnegative solutions for nonnegative initial data with $u_0 - \log u_0 \in L^1$ in case $d = 1$ (i.e. one spatial dimension) for certain Dirichlet-Neumann boundary conditions using an exponential variable transform. These solutions were the first solutions of the DLSS equation known to be defined globally in time. Subsequently existence of nonnegative weak global solutions for $d \leq 3$ and more general initial data has been shown independently by Jüngel and Matthes [18] for periodic boundary conditions and by Gianazza, Savare and Toscani [14] for variational boundary conditions using different methods: Jüngel and Matthes employ a discretization in time, proving strict positivity of the solutions of a regularized version of the elliptic equation which arises in the discretization process and deriving entropy estimates which allow for the passage to the limit. On the other hand, Gianazza, Savare and Toscani apply ideas from the theory of optimal transport, viewing the DLSS equation as the gradient flow of the Fisher information

$$\int |\nabla\sqrt{u}|^2\ dx$$

with respect to the Wasserstein distance.

The existence results are strongly based on entropy estimates which are derived using repeated integration by parts. Jüngel and Matthes have cast this method into an algorithm [16]. They found the following entropies for the DLSS equation: Set

$$E_\gamma := \frac{1}{\gamma(\gamma-1)}\int u^\gamma(.,t)\ dx$$

in case $\gamma > 0$, $\gamma \neq 1$ and

$$E_1 := \int u(.,t)\log u(.,t)\ dx\ .$$

Then in case of periodic boundary conditions the entropy inequalities

$$\frac{d}{dt}E_\gamma + c(\gamma)\int |\Delta u^{\gamma/2}|^2 \, dx \le 0 \tag{3}$$

are satisfied by strictly positive smooth solutions u for some $c(\gamma) > 0$ and all $\gamma \in C_d$, where C_d is given by

$$C_d := \left(\frac{(\sqrt{d}-1)^2}{d+2}, \frac{(\sqrt{d}+1)^2}{d+2}\right)$$

in case $d \ge 2$ and by $C_1 := \left(0, \frac{3}{2}\right)$ in case $d = 1$. The constant $c(\gamma)$ remains bounded away from zero as long as γ remains in compact subsets of $C_d \setminus \{1\}$. A time-discrete version of these entropy inequalities which can be derived by the same methods is the key step to the proof of existence of weak solutions for $d \le 3$ by Jüngel and Matthes in [18].

Moreover, in case $d = 1$ the methods by Jüngel and Matthes yield first-order entropy dissipation inequalities for the DLSS equation. Let

$$E_\alpha := \int \left|\nabla u^{\alpha/2}\right|^2 \, dx \; .$$

For any fixed α with $\alpha \in A_1$, where $A_1 := \left(\frac{2}{52}(25-6\sqrt{10}), \frac{2}{52}(25+6\sqrt{10})\right)$, and any initial data u_0 with the appropriate regularity, Jüngel and Violet [21] have proven existence of a weak solution of the DLSS equation for periodic boundary conditions which satisfies the first-order dissipation inequality

$$\frac{d}{dt}\int \left|\left(u^{\alpha/2}\right)_x\right| \, dx + c(\alpha)\int \left|\left(u^{\alpha/2}\right)_{xxx}\right|^2 + \left|\left(u^{\alpha/6}\right)_x\right|^6 \, dx \le 0 \; . \tag{4}$$

However, the mutual relation of the solutions constructed for different values of α has remained unclear. As a consequence of our results, we are able to show that the solutions for $\alpha \in A_1 \cap (0, \frac{1}{2}]$ coincide with the solution constructed by the procedure of Jüngel and Matthes [18].

The existence theory for solutions of the DLSS equation is well-developed; one of the few remaining open problems is the question of existence of solutions for Dirichlet boundary conditions for both the solution u and the quantum Bohm potential $\frac{\Delta\sqrt{u}}{\sqrt{u}}$ in case $d > 1$.

As observed first in [19], the DLSS operator formally has a monotonicity property: more precisely, given two solutions u_1 and u_2 in case of periodic boundary conditions we obtain by formal calculations

$$\frac{d}{dt}\int (\sqrt{u_1} - \sqrt{u_2})^2 \, dx = -\int \left|\sqrt[4]{\frac{u_2}{u_1}}\Delta\sqrt{u_1} - \sqrt[4]{\frac{u_1}{u_2}}\Delta\sqrt{u_2}\right|^2 \, dx \; . \tag{5}$$

However, as shown by Jüngel and Matthes [18] we cannot expect uniqueness of weak solutions without imposing additional constraints on regularity: they have given an example of a stationary nontrivial weak solution of class C^∞ for the DLSS equation; on the other hand, they have shown that for any given nonnegative measurable initial data u_0 with $u_0 \log u_0 \in L^1$, a weak solution exists which converges to a constant function as $t \to \infty$. Thus, the formal manipulations leading to (5) cannot be justified for solutions with insufficient regularity. The counterexample is C^∞, but fails to have the regularity $\sqrt{u} \in H^2$.

To the best of the author's knowledge, up to now there is no rigorous argument available for proving uniqueness of nonnegative weak solutions in some class of sufficiently regular functions, where the class at the same time is large enough to ensure global existence of solutions. In this work, we close this gap in the theory of the DLSS equation and establish uniqueness of weak solutions of the DLSS equation in the class of functions u with regularity $u^{1/4} \in L^2(I;H^2(\Omega))$, $u^{1/2} \in L^2(I;H^2(\Omega))$. The solutions constructed by Jüngel and Matthes [18] belong to this class; this is not stated explicitly in their paper, but is an easy consequence of their methods; see below. The weak solutions constructed by Jüngel and Matthes are defined globally in time; they impose only the mild condition $\int u_0 \log u_0 \, dx < \infty$ on the initial data.

Our method for proving uniqueness works as follows:

- To exploit the monotonicity property of the operator, it is necessary to prove that weak solutions (as defined in Definition 1 below) also satisfy the equation (2) in a weak sense.
- Therefore we would like to test the weak formulation of the DLSS equation (6) with the test function $\frac{\psi}{\sqrt{u}}$. This attempt however faces two major obstacles:
  a) We only have $u \in W^{1,1}([0,T];H^{-2}(\Omega))$; therefore our test function must belong to $L^\infty([0,T];H^2(\Omega))$. However, we only know $\sqrt{u} \in L^2([0,T];H^2(\Omega))$.
  b) It is not known whether u is strictly positive. Thus the denominator of the test function may vanish somewhere.

  Therefore we use the regularized test function
  $$\rho_\delta * \left(\frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right) ,$$
  where ρ_δ denotes a mollifier *with respect to space*, and pass to the limits $\delta \to 0$ and $\epsilon \to 0$.
- Due to $u \in W^{1,1}([0,T];H^{-2}(\Omega))$, by the properties of mollification our regularized test function belongs to $L^\infty([0,T];H^2(\Omega))$. Moreover, the regularized test function allows the left-hand side of the PDE (i.e. the term involving the time derivative) to be rearranged; see Lemma 13 below.
- When letting $\delta \to 0$, convergence of the right-hand side (i.e. the terms associated with the stationary DLSS equation) cannot be proven directly due to a lack of integrability of the terms on the right-hand side. This is not unexpected in view of obstacle a): The mollification has been introduced to overcome the lack of integrability (with respect to time) of spatial derivatives of the test function; thus, when trying to remove the mollification, the issue surfaces again.

  However, making use of the special structure of our test function, we can nevertheless prove convergence: In Lemma 14 below, we e.g. show that terms of the form
  $$\frac{1}{\sqrt{\rho_\delta * u + \epsilon}} \left[\rho_\delta * (\sqrt{u}\, f)\right]$$
  (with $f \in L^2([0,T];L^2(\Omega))$) converge strongly in $L^2([0,T];L^2(\Omega))$ as $\delta \to 0$, even though $u \notin L^\infty([0,T];L^\infty(\Omega))$. Therefore, such terms have better convergence behaviour than deduced by just looking at the integrability of $\sqrt{u}\, f$.
- The next step consists of letting $\epsilon \to 0$. Here the additional regularity $u^{1/4} \in L^2([0,T];H^2(\Omega))$ is needed to prove convergence of the terms on the right-hand side (see especially Lemma 11). In particular, we show that for u with this additional regularity, the term $\frac{|\Delta\sqrt{u}|^2}{\sqrt{u}}$ is well-defined and

belongs to $L^1([0,T];L^1(\Omega))$. Note that formally

$$\frac{|\Delta\sqrt{u}|^2}{\sqrt{u}} = 4\left|\Delta u^{1/4} + 4|\nabla u^{1/8}|^2\right|^2 \ ,$$

which is the reason why the regularity $u^{1/4} \in L^2([0,T];H^2(\Omega))$ is required. We finally obtain the equation (11), which is the desired weak formulation of (2).

- Having derived the evolution equation for $\sqrt{u}$, we proceed to prove uniqueness using the monotonicity property

$$\begin{aligned}&\frac{d}{dt}\int |\sqrt{u_1}-\sqrt{u_2}|^2 \ dx\\ &= -\int \frac{\sqrt{u_2}}{\sqrt{u_1}}|\Delta\sqrt{u_1}|^2 + \frac{\sqrt{u_1}}{\sqrt{u_2}}|\Delta\sqrt{u_2}|^2 - 2\Delta\sqrt{u_1}\Delta\sqrt{u_2} \ dx \leq 0 \ .\end{aligned}$$

Note that a-priori the terms of the form $\frac{\sqrt{u_1}}{\sqrt{u_2}}|\Delta\sqrt{u_2}|^2$ are not known to belong to $L^1([0,T];L^1(\Omega))$. However, they have the "right" sign, which implies that we do not need to care about their integrability. At the level of the approximation argument, this is reflected in the usage of Fatou's lemma in the proof of Theorem 5 below (an approximation argument is necessary again since we cannot use $\sqrt{u_2}$ as a test function in (11), given that $\frac{|\Delta\sqrt{u_1}|^2}{\sqrt{u_1}}$ only belongs to $L^1([0,T];L^1(\Omega))$).

In the present paper we also construct solutions with weak initial trace, i.e. solutions for initial data which is a nonnegative Radon measure with finite mass. This is done by replacing the initial data by a mollified version $\rho_\delta * u_0$, allowing for the application of the existence result by Jüngel and Matthes in [18]. By proving entropy decay estimates we then show that the weak second derivative of a weak solution can be controlled in terms of the total mass of the initial data. These estimates then provide sufficient compactness for the passage to the limit $\delta \to 0$. For a result in the same direction for the thin-film equation, see the paper by Garcke and Dal Passo [11]. As a consequence of our uniqueness result, if two weak solutions with weak initial trace have the regularity inferred from the entropy inequalities and in addition satisfy

$$\lim_{t\to 0}\int \left|\sqrt{u_1(.,t)} - \sqrt{u_2(.,t)}\right|^2 \ dx = 0$$

they coincide globally. Unfortunately, we do not know whether the latter convergence assumption can be weakened or even dropped. Typical concepts used to obtain uniqueness of solutions with weak initial trace already experience difficulties in the case of nonlinear second-order equations (see e.g. [10]).

Additionally, we show that our proof of uniqueness extends to the case of quantum drift-diffusion models in which different species of charge carriers interact via the electric field. These models are given by

$$\begin{aligned}&\frac{d}{dt}u^i = \nabla\cdot\left(\vartheta_i\nabla u^i + \tilde{Q}_i u^i \nabla V_{el}\right) - \epsilon_i^2\nabla\cdot\left(u^i\nabla\frac{\Delta\sqrt{u^i}}{\sqrt{u^i}}\right), \qquad 1\leq i\leq N \ ,\\ &\Delta V_{el} = -\sum_{i=1}^N Q_i u^i \ ,\end{aligned}$$

where N denotes the number of species of charge-carriers and where $Q_i \in \mathbb{R}$, $\vartheta_i \in \mathbb{R}^+$, $\tilde{Q}_i \in \mathbb{R}$, $\epsilon_i \in \mathbb{R}^+$ are constants. Existence of solutions for such models has been shown e.g. by Chen, Chen and Jian [9], Chen and Chen [8], and Chen and Ju [7].

Finally, we show how uniqueness of solutions can be proven not only for periodic boundary conditions, but also for combined Dirichlet-Neumann boundary conditions.

Throughout the paper, we use standard notation for Sobolev spaces. By $C_c^\infty(\Omega)$ we denote the space of smooth compactly supported functions on Ω. We refer to the d-dimensional torus by the notation $[S^1]^d$. We shall use the abbreviation $I := [0,\infty)$. The notation $L^p_{loc}(I;X)$ shall be used to denote the space of all mappings $u : I \to X$ which belong to $L^p([0,T];X)$ for all $T > 0$. By ρ_δ we denote a standard mollifier with respect to space (i.e. defined on $\mathbb{R}^d$) with $\operatorname{supp}\rho_\delta \subset B_\delta(0)$. By $RM(\Omega)$ we denote the Banach space of all Radon measures on Ω with finite total variation.

2. Main results

Jüngel and Matthes [18] have introduced the following definition of weak solutions of the DLSS equation with periodic boundary conditions (recall that $I = [0,\infty)$):

Definition 1. *Suppose $\Omega = [S^1]^d$. Let $u_0 \in L^1(\Omega)$ be given with $u_0 \geq 0$. We call $u \in L^1(I;L^\infty(\Omega)) \cap W^{1,1}(I;H^{-2}(\Omega))$, $u \geq 0$, with $\sqrt{u} \in L^2(I;H^2(\Omega))$ a weak solution of the DLSS equation with initial data u_0 if for all $\psi \in L^\infty(I;H^2(\Omega))$ and all $T > 0$ we have*

$$\int_0^T \langle \partial_t u, \psi\rangle + \int_0^T \int_\Omega (\sqrt{u}D^2\sqrt{u} - \nabla\sqrt{u}\otimes\nabla\sqrt{u}) : D^2\psi \; dx \; dt = 0 \tag{6}$$

and if in addition the equality $u(.,0) = u_0(.)$ as elements of $H^{-2}(\Omega)$ holds.

In the same paper the following existence result has been established:

Theorem 2 (Jüngel and Matthes [18])**.** *Let $\Omega = [S^1]^d$ and $d \leq 3$. Let $u_0 \in L^1(\Omega)$ be given with $u_0 \log u_0 \in L^1(\Omega)$. Then there exists a weak solution of the DLSS equation with initial data u_0.*

The solutions constructed by Jüngel and Matthes have the additional regularity $u^{1/4} \in L^2(I;H^2(\Omega))$; see below.

For Dirichlet-Neumann boundary conditions, we define the following notion of weak solutions:

Definition 3. *Let $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, be a $C^{1,1}$ domain. Let $u_0 \in L^1(\Omega)$ be given with $u_0 \geq 0$. Let a nonnegative measureable function u_B be given with $\sqrt{u_B} \in L^2_{loc}(I;H^2(\Omega))$. We call $u \in L^\infty_{loc}(I;L^1(\Omega)) \cap W^{1,1}_{loc}(I;H^{-2}(\Omega))$, $u \geq 0$, with $\sqrt{u} \in L^2_{loc}(I;H^2(\Omega))$, a weak solution of the DLSS equation with initial data u_0 and boundary data u_B if for all $\psi \in L^\infty(I;H^2_0(\Omega))$ and all $T > 0$ we have*

$$\int_0^T \langle \partial_t u, \psi\rangle + \int_0^T \int_\Omega (\sqrt{u}D^2\sqrt{u} - \nabla\sqrt{u}\otimes\nabla\sqrt{u}) : D^2\psi \; dx \; dt = 0 \; , \tag{7}$$

if $\sqrt{u} - \sqrt{u_B} \in L^2_{loc}(I;H^2_0(\Omega))$ is satisfied, and if in addition the equality $u(.,0) = u_0(.)$ as elements of $H^{-2}(\Omega)$ holds.

The author is not aware of any proof of existence of such weak solutions; however, since by formal calculations one can derive energy estimates which would imply the stated regularity and even $u^{1/4} \in L^2_{loc}(I;H^2_{loc}(\Omega))$ (at least for boundary data which is regular enough and strictly positive; see the appendix for a sketch of the calculations), it seems likely that such solutions exist. Proving existence of such weak solutions may be the subject of future work.

For the quantum drift-diffusion model we introduce a similar definition:

Definition 4. *Let $\Omega = [S^1]^d$, $1 \leq d \leq 3$, or let $\Omega \subset\subset \mathbb{R}^d$, $1 \leq d \leq 3$, be a $C^{1,1}$ domain. Assume that we are given nonnegative functions $u_0^i \in L^1(\Omega)$. If $\Omega \neq [S^1]^d$, let additionally nonnegative measurable functions u_B^i with $\sqrt{u_B^i} \in L^2_{loc}(I; H^2(\Omega))$ be given.*

Let $u^i \in L^\infty_{loc}(I; L^1(\Omega))$, $1 \leq i \leq N$, be nonnegative. Assume that we have $\sqrt{u^i} \in L^2_{loc}(I; H^2(\Omega))$ and $(u^i)^{1/4} \in L^2_{loc}(I; H^2(\Omega))$ and that $u^i \log u^i \in L^\infty_{loc}(I; L^1(\Omega))$ is satisfied. Let $V_{el} \in L^\infty_{loc}(I; W^{1,1}(\Omega))$. We then call $(u^1, \ldots, u^N, V_{el})$ a weak solution of the quantum drift-diffusion model if for all $i \in \{1, \ldots, N\}$ and every $\psi \in L^\infty(I; C^2(\Omega))$ with $\operatorname{supp} \psi \subset\subset \Omega \times I$ we have

$$\begin{aligned} &- \int_\Omega u_0^i \psi \, dt - \int_0^\infty \int_\Omega \psi_t u^i \, dt \\ = &- \int_0^\infty \int_\Omega \sqrt{u^i} D^2 \sqrt{u^i} : D^2 \psi - (\nabla \sqrt{u^i} \otimes \nabla \sqrt{u^i}) : D^2 \psi \, dx \, dt \\ &- \int_0^\infty \int_\Omega \vartheta_i \nabla u^i \cdot \nabla \psi \, dx \, dt - \int_0^\infty \int_\Omega \tilde{Q}_i u^i \nabla V_{el} \cdot \nabla \psi \, dx \, dt \ , \end{aligned}$$

if for every $\psi \in C^\infty(\overline{\Omega})$ and a.e. $t > 0$ we have

$$\int_\Omega \nabla V_{el}(., t) \cdot \nabla \psi - \sum_{i=1}^N Q_i u^i(., t) \psi \, dx = 0 \ ,$$

and if (in case $\Omega \neq [S^1]^d$) we have $\sqrt{u^i} - \sqrt{u_B^i} \in L^2_{loc}(I; H_0^2(\Omega))$ for all i.

Note that existence of such solutions has not yet been established. In the case of periodic boundary conditions or in case of strictly positive and regular boundary data, formal calculations again yield energy estimates which would imply the stated regularity and even $u^{1/4} \in L^2_{loc}(I; H^2(\Omega))$.

Our main results read as follows:

Theorem 5. *Let $d \leq 3$. Suppose we are given two weak solutions u_1, u_2 of the DLSS equation on $\Omega = [S^1]^d$ with $u_i^{1/2} \in L^2(I; H^2(\Omega))$ and $u_i^{1/4} \in L^2(I; H^2(\Omega))$ and initial data u_{01}, u_{02}. We then have for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$*

$$\int_\Omega \left| \sqrt{u_1(., t_2)} - \sqrt{u_2(., t_2)} \right|^2 \, dx \leq \int_\Omega \left| \sqrt{u_1(., t_1)} - \sqrt{u_2(., t_1)} \right|^2 \, dx \ .$$

In particular, weak solutions with the stated regularity (and therefore the solutions constructed by Jüngel and Matthes [18]) are unique within this class of regularity. As a corollary we obtain:

Corollary 6. *Given $\beta \in \left(\frac{2}{52}(25 - 6\sqrt{10}), \frac{1}{2}\right]$ and some nonnegative $u_0 \in L^1(S^1)$ with $u_0^{\beta/2} \in H^1(S^1)$, there exists a weak solution of the DLSS equation with initial data u_0 in the sense of Definition 1 which satisfies the zeroth-order entropy estimates for all $\gamma \in C_1 = (0, \frac{3}{2})$ and the first-order entropy estimates for any $\alpha \in [\beta, \frac{1}{2}]$.*

We prove existence of solutions with weak initial trace which only near $t = 0$ fail to satisfy the regularity required for uniqueness:

Theorem 7. *Let $\Omega = [S^1]^d$, $1 \leq d \leq 3$. For any initial data $\mu \in RM(\Omega)$ with $\mu \geq 0$ and $\mu(\Omega) < \infty$ there exists a solution of the DLSS equation which satisfies $u \in L^\infty(I; L^1(\Omega))$, $u^{1/2} \in L^2_{loc}((0,\infty); H^2(\Omega))$, $u^{1/4} \in L^2(I; H^2(\Omega))$ and $u(.,t) \overset{*}{\rightharpoonup} \mu$ as $t \to 0$ in the sense of weak-$*$ convergence of measures. The function u satisfies the DLSS equation in the sense that (6) holds for any $\psi \in C_c^\infty(\Omega \times (0,\infty))$. Additionally, for any $\gamma \in C_d$ with $\gamma > 1$ the entropy decay estimate*

$$\int_\Omega u^\gamma(.,t)\ dx \leq C(d,\gamma)\,[\mu(\Omega)]^\gamma \left(t^{-d(\gamma-1)/4} + 1\right)$$

holds.

We obtain an analogous uniqueness result for the case of Dirichlet-Neumann boundary conditions:

Theorem 8. *Let $d \leq 3$. Given two weak solutions u_1, u_2 of the DLSS equation with common boundary data $u_B \in L^1_{loc}(I; L^1(\Omega))$ with $\sqrt{u_B} \in L^2_{loc}(I; H^2(\Omega))$ and with the additional regularity $u_i^{1/4} \in L^2_{loc}(I; H^2(\Omega))$, we have the stability estimate*

$$\int_\Omega \left|\sqrt{u_1(.,t_2)} - \sqrt{u_2(.,t_2)}\right|^2\ dx \leq \int_\Omega \left|\sqrt{u_1(.,t_1)} - \sqrt{u_2(.,t_1)}\right|^2\ dx$$

for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$.

A similar result can be proven for weak solutions of the quantum drift-diffusion equation:

Theorem 9. *Let $d \leq 3$ and let u_1, u_2 be two weak solutions of the quantum drift-diffusion model with $\sqrt{u_j^i} \in L^2_{loc}(I; H^2(\Omega))$, $(u_j^i)^{1/4} \in L^2_{loc}(I; H^2(\Omega))$ and initial data u_{10}^i, u_{20}^i. Then the following stability estimate holds for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$:*

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^N \left|\sqrt{u_1^i}(.,t_2) - \sqrt{u_2^i}(.,t_2)\right|^2\ dx \\
\leq &\exp\left(C\sum_{i=1}^N \int_{t_1}^{t_2} 1 + \left|\left|\sqrt{u_1^i}(.,t)\right|\right|^2_{H^2(\Omega)} + \left|\left|\sqrt{u_2^i}(.,t)\right|\right|^2_{H^2(\Omega)}\ dt\right) \\
&\cdot \int_\Omega \sum_{i=1}^N \left|\sqrt{u_1^i}(.,t_1) - \sqrt{u_2^i}(.,t_1)\right|^2\ dx
\end{aligned}$$

3. Uniqueness of sufficiently regular weak solutions

Since we want to show that the distance $\int_\Omega (\sqrt{u_1} - \sqrt{u_2})^2\ dx$ is nonincreasing for sufficiently regular weak solutions u_1 and u_2, we would like to derive an evolution equation for $\sqrt{u}$. Formally, the DLSS equation is equivalent to (2) as observed by Bleher, Lebowitz, and Speer [4]. Our challenge now is to prove that a weak form of (2) holds for weak solutions in the sense of Definition 1.

We need the following regularity lemma which in a slightly different form is due to Lions and Villani [22]:

Lemma 10. *Given $u \in H^2([S^1]^d)$ with $u \geq 0$, we have the estimate*

$$\int_{[S^1]^d} |\nabla u^{1/2}|^4\ dx \leq C(d) \int_{[S^1]^d} |\Delta u|^2\ dx\ .$$

This lemma is a special case of a family of similar inequalities; see Lemma 26 in [5]. We provide a proof of the lemma in the appendix, the proof of this special case being significantly shorter than the proof of the general case.

Additionally, we need the following convergence properties:

Lemma 11. *Given some nonnegative u with $\sqrt{u} \in H^2([S^1]^d)$, we have $\sqrt{u+\epsilon} \to \sqrt{u}$ in $H^2([S^1]^d)$ as $\epsilon \to 0$.*

Moreover, we have $\partial_i\partial_j\sqrt{u} \equiv 0$ a.e. on $\{u = 0\}$.

For nonnegative u with $u^{1/4} \in H^2([S^1]^d)$, we have $(u+\epsilon)^{1/4} \to u^{1/4}$ in $H^2([S^1]^d)$ as $\epsilon \to 0$.

Moreover, we have $\partial_i\partial_j u^{1/4} \equiv 0$ a.e. on $\{u = 0\}$.

The proofs of the previous lemma and the next two lemma are standard and can be found in the appendix.

Lemma 12. *Let ρ_δ denote a standard mollifier with respect to space. If $f_\delta \to f$ strongly in $L^p(\Omega)$ as $\delta \to 0$, then $\rho_\delta * f_\delta \to f$ strongly in $L^p(\Omega)$ as $\delta \to 0$.*

Lemma 13. *Let ρ_δ denote a standard mollifier with respect to space. Let $u \in W^{1,1}_{loc}(I;[H^2(\Omega)]')$. Define $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$ (note that $\Omega_\delta = \Omega$ in case $\Omega = [S^1]^d$). Then we have $\rho_\delta * u \in W^{1,1}_{loc}(I; C^2(\Omega_\delta))$ and for any test function $\xi \in L^2_{loc}(I; L^2(\Omega))$ satisfying $\bigcup_{t\in I} \operatorname{supp}\xi(.,t) \subset\subset \Omega_\delta$ and any $T > 0$ the representation*

$$\int_0^T \langle(\rho_\delta * u)_t, \xi\rangle \, dt = \int_0^T \langle u_t, \rho_\delta * \xi\rangle \, dt$$

holds.

The following convergence properties are central for our result:

Lemma 14. *Let ρ_δ denote a standard mollifier with respect to space. Suppose that we are given a measureable function u on $\Omega := [S^1]^d$, $u \geq 0$, with $\sqrt{u} \in L^2(I; H^2(\Omega))$ and $u^{1/4} \in L^4(I; W^{1,4}(\Omega))$. Then the following convergence properties hold:*

a) *We have $\nabla\left[(\rho_\delta * u + \epsilon)^{1/4}\right] \to \nabla\left[(u+\epsilon)^{1/4}\right]$ strongly in $L^4(I; L^4(\Omega))$ as $\delta \to 0$.*
b) *It holds that $D^2\left[(\rho_\delta * u + \epsilon)^{1/2}\right] \to D^2\left[(u+\epsilon)^{1/2}\right]$ strongly in $L^2(I; L^2(\Omega))$ as $\delta \to 0$.*
c) *We deduce that $\frac{1}{\sqrt{\rho_\delta * u + \epsilon}}\left(\rho_\delta * (\sqrt{u}\partial_i\partial_i\sqrt{u})\right) \to \frac{1}{\sqrt{u+\epsilon}}\left(\sqrt{u}\partial_i\partial_i\sqrt{u}\right)$ strongly in the space $L^2(I; L^2(\Omega))$ as $\delta \to 0$.*
d) *We obtain the strong convergence $\frac{1}{(\rho_\delta * u + \epsilon)^{1/4}}\left(\rho_\delta * \left(u^{1/4}\partial_i\partial_i\sqrt{u}\partial_j u^{1/4}\right)\right) \to \frac{1}{(u+\epsilon)^{1/4}} u^{1/4}\partial_i\partial_i\sqrt{u}\partial_j u^{1/4}$ in $L^{\frac{4}{3}}(I; L^{\frac{4}{3}}(\Omega))$ as $\delta \to 0$.*

Proof. We only prove the first assertion; a sketch of the proofs of the remaining assertions (which are mainly analogous) can be found in the appendix. We have

$$\nabla\left[(\rho_\delta * u + \epsilon)^{1/4}\right] = \frac{1}{4}(\rho_\delta * u + \epsilon)^{-3/4}\nabla(\rho_\delta * u + \epsilon) \, .$$

Pointwise convergence a.e. to the desired limit is immediate. Denote by $S^\delta_\tau(t)$ the set on which the difference between $\nabla\left[(\rho_\delta * u + \epsilon)^{1/4}\right](.,t)$ and $\nabla\left[(u+\epsilon)^{1/4}\right](.,t)$

exceeds τ. We can estimate by Hölder's inequality

$$\begin{aligned}\frac{1}{4}\left|\nabla(\rho_\delta * u+\epsilon)\right|(x,t) &= \frac{1}{4}\left|\int \rho_\delta(x-y)\nabla u(y,t)\;dy\right|\\ \leq \left(\int \rho_\delta(x-y)|\nabla u^{1/4}|^4(y,t)\;dy\right)^{1/4} &\cdot \left(\int \rho_\delta(x-y)u(y,t)\;dy\right)^{3/4} .\end{aligned}$$

This implies that

$$\begin{aligned}&\int \chi_{S_\tau^\delta(t)}(x)\left|\frac{1}{4}(\rho_\delta * u+\epsilon)^{-3/4}\nabla(\rho_\delta * u+\epsilon)\right|^4\;dx\\ \leq &\int \chi_{S_\tau^\delta(t)}(x)\int \rho_\delta(x-y)|\nabla u^{1/4}|^4(y,t)\;dy\;dx \qquad (8)\\ = &\int |\nabla u^{1/4}|^4(y,t)\int \chi_{S_\tau^\delta(t)}(x)\rho_\delta(x-y)\;dx\;dy\;.\end{aligned}$$

For a.e. $t\in I$ and any fixed $\tau>0$ we see that by the definition of $S_\tau^\delta(t)$ and by pointwise convergence a.e. of $\nabla\left[(\rho_\delta * u+\epsilon)^{1/4}\right](.,t)$, the Lebesgue measure of $S_\tau^\delta(t)$ tends to zero as $\delta\to 0$. Thus we obtain $\chi_{S_\tau^\delta(t)}\to 0$ in L^1 as $\delta\to 0$ for a.e. t and any $\tau>0$. By Young's inequality for convolutions we deduce that

$$\int \chi_{S_\tau^\delta(t)}(x)\rho_\delta(x-y)\;dx\to 0$$

in L^1 as $\delta\to 0$. Since it is immediate that $|\int \chi_{S_\tau^\delta(t)}(x)\rho_\delta(x-y)\;dx|\leq 1$, using dominated convergence we see that for any fixed $\tau>0$ and a.e. $t\in I$

$$\lim_{\delta\to 0}\int |\nabla u^{1/4}|^4(y,t)\int \chi_{S_\tau^\delta(t)}(x)\rho_\delta(x-y)\;dx\;dy=0$$

and therefore (by (8))

$$\lim_{\delta\to 0}\int \chi_{S_\tau^\delta(t)}(x)\left|\frac{1}{4}(\rho_\delta * u+\epsilon)^{-3/4}\nabla(\rho_\delta * u+\epsilon)\right|^4\;dx=0\;. \qquad (9)$$

By dominated convergence, we have

$$\lim_{\delta\to 0}\int \chi_{S_\tau^\delta(t)}(x)\left|\nabla(u+\epsilon)^{1/4}\right|^4\;dx=0 \qquad (10)$$

for any $\tau>0$. Recalling the definition of $S_\tau^\delta(t)$, using (9) and (10), and finally letting $\tau\to 0$, for a.e. $t\in I$ we obtain $\nabla\left[(\rho_\delta * u+\epsilon)^{1/4}\right](.,t)\to\nabla\left[(u+\epsilon)^{1/4}\right](.,t)$ strongly in $L^4(\Omega)$ as $\delta\to 0$.

Inequality (8) implies (for $\tau=-1$)

$$\int\left|\frac{1}{4}(\rho_\delta * u+\epsilon)^{-3/4}\nabla(\rho_\delta * u+\epsilon)\right|^4\;dx\leq\int |\nabla u^{1/4}|^4(y,t)\;dy\;.$$

Arguing by dominated convergence (note that we have just proven convergence in $L^4(\Omega)$ for a.e. $t>0$), we see that $\nabla\left[(\rho_\delta * u+\epsilon)^{1/4}\right]\to\nabla\left[(u+\epsilon)^{1/4}\right]$ in $L^4(I;L^4(\Omega))$ as $\delta\to 0$. This establishes the first assertion. □

We now derive the evolution equation for $\sqrt{u}$.

Lemma 15. *Given any weak solution u of the DLSS equation on $\Omega=[S^1]^d$ with $u^{1/2}\in L^2(I;H^2(\Omega))$ and $u^{1/4}\in L^2(I;H^2(\Omega))$, we have for any $T>0$ and any*

$\psi \in L^\infty(I; W^{2,\infty}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ *with* $\psi(.,T) \equiv 0$

$$
\begin{aligned}
(11) \qquad & -2\int_0^T \int_\Omega \sqrt{u}\ \psi_t\ dx\ dt - 2\int_\Omega \sqrt{u_0}\ \psi(.,0)\ dx \\
&= \int_0^T \int_\Omega \frac{\psi}{\sqrt{u}}|\Delta\sqrt{u}|^2\ dx\ dt - \int_0^T \int_\Omega \Delta\sqrt{u}\Delta\psi\ dx\ dt\ .
\end{aligned}
$$

Since $\partial_i\partial_j\sqrt{u} = 0$ a.e. on $\{u = 0\}$ by Lemma 11, the term $\frac{1}{\sqrt{u}}|\Delta\sqrt{u}|^2$ is well-defined a.e.. As shown below, the required regularity is sufficient to deduce that this term belongs to $L^1(I; L^1(\Omega))$. Note that the formula in the lemma may also be used as a definition to yield another notion of weak solution of the DLSS equation.

Proof. The first basic problem which we have to tackle is the low regularity of the solution: we only know $u_t \in W^{1,1}(I; H^{-2}(\Omega))$ and $u \in L^2(I; H^2(\Omega))$ which is not enough for inserting functions of u as test function in the weak formulation. We overcome this problem by a regularization via mollifications.

Suppose that $\psi \in C_c^\infty(\Omega \times [0,T))$. Let ρ_δ denote a standard mollifier with respect to space. We start with a smooth strictly positive function u and calculate using repeated integrations by parts (for details see the formula B1 in the appendix)

$$
\begin{aligned}
&\int_\Omega \left(\sqrt{u}D^2\sqrt{u} - \nabla\sqrt{u}\otimes\nabla\sqrt{u}\right) : D^2\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\ dx \\
&= -\int_\Omega \Delta\sqrt{u}\Delta\sqrt{u}\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\ dx \\
&\quad + \int_\Omega \Delta\sqrt{u}\Delta\sqrt{u}\ \left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\ dx \\
&\quad + 2\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\left(\rho_\delta * \frac{\nabla\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\ dx \\
&\quad + \int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \frac{\Delta\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\ dx \\
&\quad - 2\int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \left(\frac{\nabla\psi}{\rho_\delta * u + \epsilon}\cdot\nabla\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx \\
&\quad - 2\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\left(\rho_\delta * \left(\frac{\psi}{\rho_\delta * u + \epsilon}\nabla\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx \\
&\quad - \int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \left(\frac{\psi}{\rho_\delta * u + \epsilon}\Delta\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx \\
&\quad + 2\int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \left(\frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}^3}\nabla\sqrt{\rho_\delta * u + \epsilon}\cdot\nabla\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx\ .
\end{aligned}
$$

By approximation (e.g. mollification of $\sqrt{u}$), the equation holds for all u with $\sqrt{u} \in H^2(\Omega)$ and $\inf u > 0$. Letting $\tilde{u} := u + \delta$ and $\delta \to 0$, by Lemma 11 and the regularizing effect of mollification the equation holds for all nonnegative u with $\sqrt{u} \in H^2(\Omega)$.

We now plug in the test function $\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}$ into the weak formulation of the DLSS equation (see Definition 1). Note that the function $\frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}$ belongs to $W^{1,1}_{loc}(I; C^2(\Omega))$ (this is a consequence of $\rho_\delta * u \in W^{1,1}_{loc}(I; C^2(\Omega))$ by Lemma 13); thus, it especially belongs to $C^0_{loc}(I; C^2(\Omega))$. Therefore it follows using Lemma 13 and the previous calculation (for details see formula B2 in the appendix)

$$
\begin{aligned}
&2\int_0^T\int_\Omega \psi_t\sqrt{\rho_\delta * u+\epsilon}\;dx\;dt\;+2\int_\Omega \sqrt{\rho_\delta * u_0+\epsilon}\;\psi(.,0)\;dx\\
=&2\int_0^T\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\left(\rho_\delta * \frac{\nabla\psi}{\sqrt{\rho_\delta * u+\epsilon}}\right)\;dx\;dt\\
&+\int_0^T\int_\Omega \Delta\sqrt{u}\sqrt{u}\;\left(\rho_\delta * \frac{\Delta\psi}{\sqrt{\rho_\delta * u+\epsilon}}\right)\;dx\;dt\\
&-2\int_0^T\int_\Omega \left(\rho_\delta * (\sqrt{u}\Delta\sqrt{u})\right)\;\left(\frac{\nabla\psi}{\rho_\delta * u+\epsilon}\cdot\nabla\sqrt{\rho_\delta * u+\epsilon}\right)\;dx\;dt\\
&-2\int_0^T\int_\Omega \left(\rho_\delta * (\nabla\sqrt{u}\Delta\sqrt{u})\right)\cdot\left(\frac{\psi}{\rho_\delta * u+\epsilon}\nabla\sqrt{\rho_\delta * u+\epsilon}\right)\;dx\;dt\\
&-\int_0^T\int_\Omega \left(\rho_\delta * (\sqrt{u}\Delta\sqrt{u})\right)\;\left(\frac{\psi}{\rho_\delta * u+\epsilon}\Delta\sqrt{\rho_\delta * u+\epsilon}\right)\;dx\;dt\\
&+2\int_0^T\int_\Omega \left(\rho_\delta * (\sqrt{u}\Delta\sqrt{u})\right)\;\left(\frac{\psi}{\sqrt{\rho_\delta * u+\epsilon}^3}\nabla\sqrt{\rho_\delta * u+\epsilon}\cdot\nabla\sqrt{\rho_\delta * u+\epsilon}\right)\;dx\;dt\;.
\end{aligned}
$$

We now let $\delta\to 0$. For this step, we need the regularity $u^{1/2}\in L^2(I;H^2(\Omega))$ which implies $u^{1/4}\in L^4(I;W^{1,4}(\Omega))$ by Lemma 10. By this regularity and $\psi\in L^\infty(I;W^{2,\infty}(\Omega))$, from Lemma 12 we immediately obtain convergence of the first two terms on the right-hand side since $\frac{1}{\sqrt{\rho_\delta * u+\epsilon}}$ is uniformly bounded and converges pointwise a.e.. In order to show that

$$
\begin{aligned}
&2\int_0^T\int_\Omega \psi_t\sqrt{u+\epsilon}\;dx\;dt+2\int_\Omega \sqrt{u_0+\epsilon}\psi(.,0)\;dx\\
=&2\int_0^T\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\frac{\nabla\psi}{\sqrt{u+\epsilon}}\;dx\;dt\\
&+\int_0^T\int_\Omega \Delta\sqrt{u}\;\sqrt{u}\;\frac{\Delta\psi}{\sqrt{u+\epsilon}}\;dx\;dt\\
&-2\int_0^T\int_\Omega \sqrt{u}\;\Delta\sqrt{u}\;\frac{\nabla\psi}{u+\epsilon}\cdot\nabla\sqrt{u+\epsilon}\;dx\;dt\\
&-2\int_0^T\int_\Omega \Delta\sqrt{u}\;\nabla\sqrt{u}\cdot\frac{\psi}{u+\epsilon}\nabla\sqrt{u+\epsilon}\;dx\;dt\\
&-\int_0^T\int_\Omega \sqrt{u}\;\Delta\sqrt{u}\;\frac{\psi}{u+\epsilon}\Delta\sqrt{u+\epsilon}\;dx\;dt\\
&+2\int_0^T\int_\Omega \sqrt{u}\;\Delta\sqrt{u}\;\frac{\psi}{\sqrt{u+\epsilon}^3}\nabla\sqrt{u+\epsilon}\cdot\nabla\sqrt{u+\epsilon}\;dx\;dt
\end{aligned}
\tag{12}
$$

holds, we still need to ensure convergence of the remaining four terms on the right-hand side. To this end, let us rewrite

$$\begin{aligned}
&-2\int_0^T\int_\Omega \left(\rho_\delta*(\sqrt{u}\Delta\sqrt{u})\right)\left(\frac{\nabla\psi}{\rho_\delta*u+\epsilon}\cdot\nabla\sqrt{\rho_\delta*u+\epsilon}\right)\,dx\,dt\\
&-2\int_0^T\int_\Omega \left(\rho_\delta*(\nabla\sqrt{u}\Delta\sqrt{u})\right)\cdot\left(\frac{\psi}{\rho_\delta*u+\epsilon}\nabla\sqrt{\rho_\delta*u+\epsilon}\right)\,dx\,dt\\
&-\int_0^T\int_\Omega \left(\rho_\delta*(\sqrt{u}\Delta\sqrt{u})\right)\left(\frac{\psi}{\rho_\delta*u+\epsilon}\Delta\sqrt{\rho_\delta*u+\epsilon}\right)\,dx\,dt\\
&+2\int_0^T\int_\Omega \left(\rho_\delta*(\sqrt{u}\Delta\sqrt{u})\right)\left(\frac{\psi}{\sqrt{\rho_\delta*u+\epsilon}^3}\nabla\sqrt{\rho_\delta*u+\epsilon}\cdot\nabla\sqrt{\rho_\delta*u+\epsilon}\right)\,dx\,dt\\
=&-4\int_0^T\int_\Omega \frac{1}{\sqrt{\rho_\delta*u+\epsilon}}\left(\rho_\delta*(\sqrt{u}\Delta\sqrt{u})\right)\frac{\nabla\psi}{(\rho_\delta*u+\epsilon)^{1/4}}\cdot\nabla(\rho_\delta*u+\epsilon)^{1/4}\,dx\,dt\\
&-8\int_0^T\int_\Omega \left(\rho_\delta*\left(u^{1/4}\Delta\sqrt{u}\nabla u^{1/4}\right)\right)\cdot\frac{\psi}{(\rho_\delta*u+\epsilon)^{3/4}}\nabla\left(\rho_\delta*u+\epsilon\right)^{1/4}\,dx\,dt\\
&-\int_0^T\int_\Omega \frac{1}{\sqrt{\rho_\delta*u+\epsilon}}\left(\rho_\delta*\left(\sqrt{u}\Delta\sqrt{u}\right)\right)\frac{\psi}{\sqrt{\rho_\delta*u+\epsilon}}\Delta\sqrt{\rho_\delta*u+\epsilon}\,dx\,dt\\
&+8\int_0^T\int_\Omega \left(\rho_\delta*\left(\sqrt{u}\Delta\sqrt{u}\right)\right)\frac{\psi}{\sqrt{\rho_\delta*u+\epsilon}^2}\nabla\left(\rho_\delta*u+\epsilon\right)^{1/4}\cdot\nabla\left(\rho_\delta*u+\epsilon\right)^{1/4}\,dx\,dt\,.
\end{aligned}$$

Convergence of these terms is now immediate from the assertions of Lemma 14.

We now intend to let $\epsilon\to 0$ in formula (12); here the additional regularity $u^{1/4}\in L^2(I;H^2(\Omega))$ which implies $u^{1/8}\in L^4(I;W^{1,4}(\Omega))$ by Lemma 10 will be required. We rearrange (12) to get

$$\begin{aligned}
&2\int_0^T\int_\Omega \psi_t\sqrt{u+\epsilon}\,dx\,dt+2\int_\Omega\sqrt{u_0+\epsilon}\,\psi(.,0)\,dx\\
=&-\int_0^T\int_\Omega \sqrt{u}\,\Delta\sqrt{u}\,\frac{\psi}{u+\epsilon}\Delta\sqrt{u+\epsilon}\,dx\,dt\\
&+\int_0^T\int_\Omega \sqrt{u}\,\Delta\sqrt{u}\,\frac{\Delta\psi}{\sqrt{u+\epsilon}}\,dx\,dt\\
&+2\int_0^T\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\frac{\nabla\psi}{\sqrt{u+\epsilon}}\,dx\,dt\\
&-2\int_0^T\int_\Omega \sqrt{u}\,\Delta\sqrt{u}\,\frac{\nabla\psi}{u+\epsilon}\cdot\nabla\sqrt{u+\epsilon}\\
&-2\int_0^T\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\frac{\psi}{u+\epsilon}\nabla\sqrt{u+\epsilon}\,dx\,dt\\
&+2\int_0^T\int_\Omega \sqrt{u}\,\Delta\sqrt{u}\,\frac{\psi}{\sqrt{u+\epsilon}^3}\nabla\sqrt{u+\epsilon}\cdot\nabla\sqrt{u+\epsilon}\,dx\,dt
\end{aligned}$$

which is equivalent to (recall that by Lemma 11 we have $D^2\sqrt{u} \equiv 0$ a.e. on $\{u=0\}$)

$$\begin{aligned}
&2\int_0^T \int_\Omega \psi_t \sqrt{u+\epsilon}\ dx\ dt + 2\int_\Omega \sqrt{u_0+\epsilon}\ \psi(.,0)\ dx \\
=& -\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\ \frac{\psi u^{3/4}}{(u+\epsilon)^{3/4}} \frac{1}{(u+\epsilon)^{1/4}}\Delta\sqrt{u+\epsilon}\ dx\ dt \\
&+ \int_0^T \int_\Omega \Delta\sqrt{u}\Delta\psi \frac{\sqrt{u}}{\sqrt{u+\epsilon}}\ dx\ dt \\
&+ 8\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\nabla u^{1/8} \cdot \frac{u^{5/8}\nabla\psi}{\sqrt{u+\epsilon}}\ dx\ dt \\
&- 8\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u} \frac{u^{3/4}\nabla\psi}{(u+\epsilon)^{5/8}} \cdot \nabla(u+\epsilon)^{1/8}\ dx\ dt \\
&- 32\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\nabla u^{1/8} \cdot \frac{\psi u^{5/8}}{(u+\epsilon)^{5/8}}\nabla(u+\epsilon)^{1/8}\ dx\ dt \\
&+ 32\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\ \frac{\psi u^{3/4}}{(u+\epsilon)^{3/4}}\nabla(u+\epsilon)^{1/8} \cdot \nabla(u+\epsilon)^{1/8}\ dx\ dt\ .
\end{aligned}$$

Note that the equation

$$\frac{1}{u^{1/4}}\Delta\sqrt{u} = 2\ \Delta u^{1/4} + 8\ \nabla u^{1/8} \cdot \nabla u^{1/8} \tag{13}$$

holds pointwise for any smooth strictly positive u; thus the same equation holds pointwise a.e. for any nonnegative u with $u^{1/4} \in L^2(I; H^2(\Omega))$ by approximation (recall Lemma 10 and Lemma 11). In particular, the term $\frac{1}{u^{1/4}}\Delta\sqrt{u}$ belongs to $L^2(I; L^2(\Omega))$. By Lemma 11 and (13), we get convergence of $\frac{1}{(u+\epsilon)^{1/4}}\Delta\sqrt{u+\epsilon}$ to $\frac{1}{u^{1/4}}\Delta\sqrt{u}$ strongly in $L^2(I; L^2(\Omega))$ as $\epsilon \to 0$.

Using Lemma 11 to pass to the limit $\epsilon \to 0$, we get

$$\begin{aligned}
&2\int_0^T \int_\Omega \psi_t \sqrt{u}\ dx\ dt + 2\int_\Omega \sqrt{u_0}\ \psi(.,0)\ dx \\
=& -\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\ \ \psi\ \frac{1}{u^{1/4}}\Delta\sqrt{u}\ dx\ dt \\
&+ \int_0^T \int_\Omega \Delta\sqrt{u}\Delta\psi\ dx\ dt \\
&+ 8\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\nabla u^{1/8} \cdot u^{1/8}\nabla\psi\ dx\ dt \\
&- 8\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\ u^{1/8}\nabla\psi \cdot \nabla u^{1/8}\ dx\ dt \\
&- 32\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\nabla u^{1/8} \cdot \psi\nabla u^{1/8}\ dx\ dt \\
&+ 32\int_0^T \int_\Omega \frac{1}{u^{1/4}}\Delta\sqrt{u}\ \psi\nabla u^{1/8} \cdot \nabla u^{1/8}\ dx\ dt\ .
\end{aligned}$$

The undesireable terms cancel and we see that the remaining terms yield the desired right-hand side of (11). □

Now that we have an evolution equation for $\sqrt{u}$, we can proceed to the actual proof of uniqueness.

Proof of Theorem 5. Obviously, the functions u_1 and u_2 satisfy the conditions of the previous lemma. By the lemma, we see that we have $\sqrt{u_i} \in W^{1,1}(I; H^{-2}(\Omega))$ since $v \in H^2(\Omega)$ implies $v \in L^\infty(\Omega)$ (as $d \leq 3$); moreover, $\sqrt{u_i}(t=0)$ which is a priori the evaluation of the continuous representative of $\sqrt{u} \in W^{1,1}(I; H^{-2}(\Omega))$ at $t=0$ is identified as $\sqrt{u_{0i}}$.

Take a smooth nonnegative test function $\xi \in C_c^\infty([0,T))$ and consider the test function $\psi := \xi \cdot \rho_\delta * (\rho_\delta * \sqrt{u_2})$. Since $\sqrt{u_i} \in L^\infty(I; L^2(\Omega))$ due to conservation of mass, we have $\psi \in L^\infty(I; W^{2,\infty}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ and therefore it constitutes a valid test function. We obtain since $\langle (\rho_\delta * \psi)_t, \phi \rangle = \langle \psi_t, \rho_\delta * \phi \rangle$ which is easily verified using the definition of ∂_t

$$
\begin{aligned}
&-2\int_0^T \int_\Omega (\rho_\delta * \sqrt{u_1})\xi_t(\rho_\delta * \sqrt{u_2})\ dx\ dt - 2\int_\Omega (\rho_\delta * \sqrt{u_{01}})\xi(0)(\rho_\delta * \sqrt{u_{02}})\ dx \\
&-2\int_0^T \langle \xi(\rho_\delta * \sqrt{u_1}), (\rho_\delta * \sqrt{u_2})_t \rangle\ dt \\
=&\int_0^T \int_\Omega \xi \frac{\rho_\delta * \rho_\delta * \sqrt{u_2}}{\sqrt{u_1}} |\Delta \sqrt{u_1}|^2\ dx\ dt - \int_0^T \int_\Omega \xi \Delta \sqrt{u_1} \Delta \rho_\delta * \rho_\delta * \sqrt{u_2}\ dx\ dt\ .
\end{aligned}
$$

Repeating the same calculation with u_1 and u_2 interchanged and adding, we obtain

$$
\begin{aligned}
&-4\int_0^T \int_\Omega (\rho_\delta * \sqrt{u_1})\xi_t(\rho_\delta * \sqrt{u_2})\ dxdt - 4\int_\Omega (\rho_\delta * \sqrt{u_{01}})\xi(0)(\rho_\delta * \sqrt{u_{02}})\ dx \\
&-2\int_0^T \langle \xi(\rho_\delta * \sqrt{u_1}), (\rho_\delta * \sqrt{u_2})_t \rangle\ dt - 2\int_0^T \langle \xi(\rho_\delta * \sqrt{u_2}), (\rho_\delta * \sqrt{u_1})_t \rangle\ dt \\
=&\int_0^T \int_\Omega \xi \frac{\rho_\delta * \rho_\delta * \sqrt{u_2}}{\sqrt{u_1}} |\Delta \sqrt{u_1}|^2\ dx\ dt - \int_0^T \int_\Omega \xi (\rho_\delta * \Delta \sqrt{u_1}) \Delta (\rho_\delta * \sqrt{u_2})\ dx\ dt \\
&+\int_0^T \int_\Omega \xi \frac{\rho_\delta * \rho_\delta * \sqrt{u_1}}{\sqrt{u_2}} |\Delta \sqrt{u_2}|^2\ dx\ dt - \int_0^T \int_\Omega \xi (\rho_\delta * \Delta \sqrt{u_2}) \Delta (\rho_\delta * \sqrt{u_1})\ dx\ dt\ .
\end{aligned}
\tag{14}
$$

The left-hand side can be rewritten as

$$
-2\int_0^T \int_\Omega (\rho_\delta * \sqrt{u_1})\xi_t(\rho_\delta * \sqrt{u_2})\ dx\ dt - 2\int_\Omega \xi(0)(\rho_\delta * \sqrt{u_{01}})(\rho_\delta * \sqrt{u_{02}})\ dx\ .
$$

We know that $\rho_\delta * \rho_\delta * \sqrt{u_i}$ converges to $\sqrt{u_i}$ pointwise a.e.: For a.e. t we know that a.e. point x is a Lebesgue point of $\sqrt{u_i}(.,t)$; moreover, $\rho_\delta * \rho_\delta$ is a function supported in $B_{2\delta}(0)$ with $||\rho_\delta * \rho_\delta||_{L^\infty} \leq C(d)\delta^{-d}$ and $\int (\rho_\delta * \rho_\delta)(y)\ dy = 1$. Thus we get

$$
\begin{aligned}
&|(\rho_\delta * \rho_\delta * \sqrt{u_i})(x,t) - \sqrt{u_i}(x,t)| \\
=&\left| \int (\rho_\delta * \rho_\delta)(x-y)(\sqrt{u_i}(y,t) - \sqrt{u_i}(x,t))\ dy \right| \\
\leq& C(d) \fint_{B_{2\delta}(x)} |\sqrt{u_i}(y,t) - \sqrt{u_i}(x,t)|\ dy
\end{aligned}
$$

which implies the convergence for a.e. x (since a.e. point is a Lebesgue point).

Thus, letting $\delta \to 0$ and using Fatou's lemma, equation (14) becomes

$$\begin{aligned}
&-2\int_0^T \int_\Omega \xi_t \sqrt{u_1}\sqrt{u_2}\,dx\,dt - 2\int_\Omega \xi(0)\sqrt{u_{01}}\sqrt{u_{02}}\,dx \\
&\geq \int_0^T \int_\Omega \xi \frac{\sqrt{u_2}}{\sqrt{u_1}} |\Delta\sqrt{u_1}|^2\,dxdt - \int_0^T \int_\Omega \xi \Delta\sqrt{u_1}\Delta\sqrt{u_2}\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \xi \frac{\sqrt{u_1}}{\sqrt{u_2}} |\Delta\sqrt{u_2}|^2\,dx\,dt - \int_0^T \int_\Omega \xi \Delta\sqrt{u_2}\Delta\sqrt{u_1}\,dx\,dt \\
&= \int_0^T \int_\Omega \xi \left| \left(\frac{u_2}{u_1}\right)^{1/4} \Delta\sqrt{u_1} - \left(\frac{u_1}{u_2}\right)^{1/4} \Delta\sqrt{u_2} \right|^2\,dx\,dt\ .
\end{aligned} \tag{15}$$

In case $t_1 = 0$, set $\xi(t) \equiv 1$ for $t < t_2 - \epsilon$, $\xi(t) \equiv 0$ for $t > t_2$ and let ξ be monotone decreasing on $[t_2 - \epsilon, t_2]$ with $|\xi_t| \leq \frac{C}{\epsilon}$. In case $t_1 > 0$, set $\xi(t) \equiv 0$ for $t < t_1$ and $t > t_2$; set $\xi \equiv 1$ for $t_1 + \epsilon < t < t_2 - \epsilon$ and let ξ be monotone on the remaining intervals with $|\xi_t| \leq \frac{C}{\epsilon}$. Letting $\epsilon \to 0$, we obtain since a.e. $t > 0$ is a Lebesgue point of $\int \sqrt{u_1}\sqrt{u_2}\,dx$

$$2\int_\Omega \sqrt{u_1(.,t_2)}\sqrt{u_2(.,t_2)}\,dx \geq 2\int_\Omega \sqrt{u_1(.,t_1)}\sqrt{u_2(.,t_1)}\,dx$$

for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$. This finishes the proof since we have

$$\begin{aligned}
&\int_\Omega \left|\sqrt{u_1(.,t_2)} - \sqrt{u_2(.,t_2)}\right|^2\,dx \\
&= \int_\Omega u_1(.,t_2)\,dx + \int_\Omega u_2(.,t_2)\,dx - 2\int_\Omega \sqrt{u_1(.,t_2)}\sqrt{u_2(.,t_2)}\,dx \\
&\leq \int_\Omega u_1(.,t_1)\,dx + \int u_2(.,t_1)\,dx - 2\int_\Omega \sqrt{u_1(.,t_1)}\sqrt{u_2(.,t_1)}\,dx \\
&= \int_\Omega \left|\sqrt{u_1(.,t_1)} - \sqrt{u_2(.,t_1)}\right|^2\,dx
\end{aligned}$$

where we have used the fact that mass is conserved. □

4. Regularity of the solutions constructed by Jüngel and Matthes

In this section, we shall gather some results by Jüngel and Matthes regarding the regularity of their solutions to the DLSS equation constructed in [18]. The entropies $\gamma < 1$ are not treated explicitly by Jüngel and Matthes, though this case follows using entirely the same methods. Since the regularity inferred from the zeroth-order entropy for $\gamma = \frac{1}{2}$ is crucial to our uniqueness result, we explicitly treat the case $\gamma < 1$ here.

Theorem 16. *The solutions constructed by Jüngel and Matthes satisfy the zeroth-order entropy estimate (3) for all $\gamma \in C_d$, with C_d defined as below formula (3).*

Proof. We start with the time-discrete regularized formulation in [18] which (for a single timestep of length $\tau > 0$) reads

$$\begin{aligned}
\frac{1}{\tau}(u_\epsilon - w) = &-\frac{1}{2}D^2 : (u_\epsilon D^2 \log u_\epsilon) - \frac{\epsilon}{2}(\Delta^2 \log u_\epsilon + \log u_\epsilon) \\
&+ \frac{\epsilon}{8}\,\mathrm{div}(|\nabla \log u_\epsilon|^2 \nabla \log u_\epsilon)\ .
\end{aligned} \tag{16}$$

For $\gamma > 1$, the estimate

$$\begin{aligned}&\frac{1}{2\tau}\left(\int_\Omega u_\epsilon^\gamma \, dx - \int_\Omega w^\gamma \, dx\right)\\ \leq &-c\int_\Omega |\Delta u_\epsilon^{\gamma/2}|^2 \, dx - \epsilon A_2 - \epsilon A_3\end{aligned}$$

is shown in [18] in the course of the proof of their Lemma 11, where A_2 and A_3 are bounded from below uniformly in $\epsilon > 0$. Passing to the limit $\epsilon \to 0$, by lower semicontinuity of the norm in $L^2(\Omega)$ and the strong convergence $u_\epsilon^{\gamma/2} \to u^{\gamma/2}$ in $L^2(\Omega)$ (note that the inequality provides the necessary compactness) we obtain

$$\frac{1}{2\tau}\left(\int_\Omega u^\gamma \, dx - \int_\Omega w^\gamma \, dx\right) \leq -c\int_\Omega |\Delta u^{\gamma/2}|^2 \, dx$$

For $\gamma = 1$, using the above convergence arguments the estimates by Jüngel and Matthes imply the corresponding inequality

$$\frac{1}{2\tau}\left(\int_\Omega u\log u \, dx - \int_\Omega w\log w \, dx\right) \leq -c\int_\Omega |\Delta u^{1/2}|^2 \, dx$$

The case $0 < \gamma < 1$ is not treated explicitly by Jüngel and Matthes; however, using similar arguments it is easily derived: By concavity of x^γ for $0 < \gamma < 1$, we have $w^\gamma \leq u_\epsilon^\gamma + \gamma u_\epsilon^{\gamma-1}(w - u_\epsilon)$, which implies $u_\epsilon^\gamma - w^\gamma \geq (u_\epsilon - w)\gamma u_\epsilon^{\gamma-1}$. We therefore obtain by plugging in $u_\epsilon^{\gamma-1}$ as a test function in the weak formulation of (16) (this is possible since $u_\epsilon \in H^2(\Omega)$ is bounded away from zero, see [18])

$$\begin{aligned}&\frac{1}{\gamma\tau}\left(\int_\Omega u_\epsilon^\gamma \, dx - \int_\Omega w^\gamma \, dx\right)\\ \geq &-\int_\Omega (u_\epsilon^{1/2}D^2u_\epsilon^{1/2} - \nabla u_\epsilon^{1/2}\otimes\nabla u_\epsilon^{1/2}) : D^2u_\epsilon^{\gamma-1} \, dx\\ &-\frac{\epsilon}{8}\int_\Omega 4\Delta\log u_\epsilon \, \Delta u_\epsilon^{\gamma-1} + |\nabla\log u_\epsilon|^2\nabla(\log u_\epsilon)\cdot\nabla(u_\epsilon^{\gamma-1}) \, dx\\ &-\frac{\epsilon}{2}\int_\Omega u_\epsilon^{\gamma-1}\log u_\epsilon \, dx \; .\end{aligned} \tag{17}$$

A straightforward calculation shows that the second integral can be rewritten as

$$\begin{aligned}&\frac{1}{8}\int_\Omega 4\Delta u_\epsilon^{\gamma-1}\Delta\log u_\epsilon + |\nabla\log u_\epsilon|^2\nabla(\log u_\epsilon)\cdot\nabla(u_\epsilon^{\gamma-1}) \, dx\\ =&2(\gamma-1)\int_\Omega u_\epsilon^{\gamma-1}\left[\left(\frac{\Delta\sqrt{u_\epsilon}}{\sqrt{u_\epsilon}}\right)^2 - 2(2-\gamma)\frac{\Delta\sqrt{u_\epsilon}}{\sqrt{u_\epsilon}}\left|\frac{\nabla\sqrt{u_\epsilon}}{\sqrt{u_\epsilon}}\right|^2 + 2(2-\gamma)\left|\frac{\nabla\sqrt{u_\epsilon}}{\sqrt{u_\epsilon}}\right|^4\right] dx\\ =&2(\gamma-1)\int_\Omega u_\epsilon^{\gamma-1}\left[\left(\frac{\Delta\sqrt{u_\epsilon}}{\sqrt{u_\epsilon}} - (2-\gamma)\left|\frac{\nabla\sqrt{u_\epsilon}}{\sqrt{u_\epsilon}}\right|^2\right)^2 + \gamma(2-\gamma)\left|\frac{\nabla\sqrt{u_\epsilon}}{\sqrt{u_\epsilon}}\right|^4\right] dx \leq 0\end{aligned}$$

which implies nonnegativity of the second term on the right-hand side in (17).

Since u_ϵ is smooth enough and bounded away from zero, the results of Jüngel and Matthes (equation (6) in [18]) imply that the first integral on the right-hand side of (17) can be estimated from below by

$$-\int_\Omega (u_\epsilon^{1/2}D^2u_\epsilon^{1/2} - \nabla u_\epsilon^{1/2}\otimes\nabla u_\epsilon^{1/2}) : D^2u_\epsilon^{\gamma-1} \, dx \geq c(\gamma)\int_\Omega |\Delta u_\epsilon^{\gamma/2}|^2 \, dx$$

as long as $\gamma \in C_d \cap (0,1)$. Using the estimate

$$-\frac{1}{2}\int_\Omega u_\epsilon^{\gamma-1}\log u_\epsilon \, dx \geq -\int_\Omega C(\gamma) \, dx$$

which holds since $\gamma < 1$, we therefore obtain from (17)

$$\frac{1}{\gamma\tau}\left(\int u_\epsilon^\gamma \, dx - \int_\Omega w^\gamma \, dx\right) \geq c(\gamma)\int_\Omega |\Delta u_\epsilon^{\gamma/2}|^2 \, dx - C(\gamma)\epsilon$$

for all $\gamma \in C_d \cap (0,1)$. Passing to the limit, using that $u_\epsilon^\gamma \to u^\gamma$ in $L^2(\Omega)$ since $u_\epsilon^{\tilde\gamma} \to u^{\tilde\gamma}$ for all $\tilde\gamma > 1$, $\tilde\gamma \in C_d$, we get

$$\frac{1}{\tau}\left(\int_\Omega u^\gamma \, dx - \int_\Omega w^\gamma \, dx\right) \geq c(\gamma)\int_\Omega |\Delta u^{\gamma/2}|^2 \, dx$$

This proves the entropy inequality for a single time-step in the time-discrete formulation.

We denote the iterated time-discrete solution with time-step τ by $u^{(\tau)}$. Multiplying the above estimate by τ and summing over all timesteps which correspond to times $\leq T$, we obtain

$$\int_\Omega \left(u^{(\tau)}\left(\,.\,,\tau\left\lfloor\frac{T}{\tau}\right\rfloor\right)\right)^\gamma \, dx \geq \int_\Omega u_0^\gamma \, dx + c(\gamma)\sum_{k=1}^{k\tau\leq T}\tau\int_\Omega |\Delta(u^{(\tau)}(.,k\cdot\tau))^{\gamma/2}|^2 \, dx \;.$$

Note that the solution in [18] is constructed by taking the limit of a subsequence $u^{(\tau)}$ which converges strongly in $L^1([0,T];L^1(\Omega))$ for every $T > 0$; by lower semicontinuity of the $L^2(I;H^2(\Omega))$ norm with respect to convergence in the sense of distributions, this implies that the limit u satisfies

$$\int_\Omega u^\gamma(.,T) \, dx \geq \int_\Omega u_0^\gamma \, dx + c(\gamma)\int_0^T\int_\Omega |\Delta u^{\gamma/2}|^2 \, dx \, dt$$

for a.e. $T > 0$ since for a.e. $T > 0$ we have $u^{(\tau)}(.,T) \to u(.,T)$ in $L^1(\Omega)$. We thus see that the entropy estimate carries over to the limit $\tau \to 0$. □

5. First-order entropy inequalities

Using our uniqueness result, we now show that in case $d = 1$ any weak solution of the DLSS equation which belongs to our class of uniqueness satisfies the first-order entropy estimate for any $\alpha \in A_1 \cap (0,\frac{1}{2}]$.

Jüngel and Violet have constructed solutions which satisfy the α entropy estimate for one value of $\alpha \leq 1$; however, given u with $u \in L^\infty(\Omega)$, $u^{\alpha/2}, u^{\alpha/4} \in L^2(I;H^2(\Omega))$ and setting $w := u^{\alpha/4}$ they use the reformulation

$$\begin{aligned}(u^\alpha)_t = &-2\left(4-\frac{4}{\alpha}\right)\left[ww_{xx}(w^2)_{xx} - 4w|w_x|^2w_{xx} + 4(ww_x(ww_{xx}-|w_x|^2))_x\right]\\ &-4\left(4-\frac{4}{\alpha}\right)\left(3-\frac{4}{\alpha}\right)|w_x|^2(ww_{xx}-|w_x|^2) - 4(w^2(ww_{xx}-|w_x|^2))_{xx}\end{aligned}$$

to define a notion of weak solution of the DLSS equation. We will refer to this definition as α weak solution.

Recall that by Lemma 10 the above regularity implies $u^{\alpha/4}, u^{\alpha/8} \in L^4(I;W^{1,4}(\Omega))$. It is then straightforward to verify that $(u^\alpha)_t \in L^1(I;L^1(\Omega)) + L^{\frac{4}{3}}(I;H^{-2}(\Omega))$ for any α weak solution (i.e. the weak time derivative of u^α can be represented as a sum of one element of $L^1(I;L^1(\Omega))$ and one element of $L^{\frac{4}{3}}(I;H^{-2}(\Omega))$).

Jüngel and Violet [21] prove the following theorem:

Theorem 17. *For any strictly positive $u_0 \in H^1(S^1)$ and any $\alpha \in A_1 \cap (0,1]$, there exists an α weak solution of the DLSS equation with the regularity $u^{\alpha/2} \in L^2(I;H^3(S^1)) \cap L^\infty(I;H^1(S^1))$, $u^{\alpha/6} \in L^6(I;W^{1,6}(S^1))$, $\log u \in L^2(I;H^2(S^1))$.*

The initial data is attained in the sense of $u^\alpha(.,t) \to u_0^\alpha$ strongly in $H^{-2}(S^1)$ as $t \to 0$. Moreover, for $t_2 > t_1 \geq 0$ we have the estimate

$$\int_{S^1} |(u^{\alpha/2})_x|^2 \; dx \Big|_{t_1}^{t_2} \leq -c(\alpha) \int_{t_1}^{t_2} \int_{S^1} |(u^{\alpha/6})_x|^6 + |(u^{\alpha/2})_{xxx}|^2 \; dx \; dt \; .$$

Note that Jüngel and Violet do not explicitly state the full entropy inequality in their theorem, but it follows easily from their proof (pass to the continuum limit in inequality (21) in [21]). They do not answer the question whether an α weak solution conserves mass (at least for $\alpha \neq 1$).

Lemma 18. *Given $\alpha \in (0, \frac{1}{2}]$ and some nonnegative $u_0 \in L^1(S^1)$ with $u_0^{\alpha/2} \in H^1(S^1)$, any α weak solution satisfies $u^{1/2}, u^{1/4} \in L^2(I; H^2(S^1))$ and is also a weak solution of the DLSS equation in the sense of Definition 1. In particular, the solution is mass-preserving.*

Proof. It is obvious that u has the regularity properties required for being a weak solution since we have $u^{\alpha/4} \in L^2(I; H^2(S^1))$, $u^{\alpha/8} \in L^4(I; W^{1,4}(S^1))$ and $u \in L^\infty(S^1 \times I)$ by definition.

Let $\psi \in C_c^\infty(\Omega \times [0,T))$ be smooth. We now plug in $\rho_\delta * \left[\psi(\rho_\delta * u^\alpha)^{(1-\alpha)/\alpha}\right]$ as a test function in the definition of α weak solutions. The left-hand side of the parabolic equation (i.e. the terms involving the time derivative) can be rearranged to yield

$$-\alpha \int_{S^1} (\rho_\delta * u_0^\alpha)^{1/\alpha} \psi(.,0) \; dx - \alpha \int_0^T \int_{S^1} (\rho_\delta * u^\alpha)^{1/\alpha} \psi_t \; dxdt$$

since $u^\alpha \in W^{1,1}(I; H^{-2}(S^1)) \subset C^0(I; H^{-2}(S^1))$ and $u^\alpha(t=0) = u_0^\alpha$ in the sense of $H^{-2}(S^1)$ which implies $(\rho_\delta * u^\alpha)(t=0) = \rho_\delta * u_0^\alpha$, where the first convolution is to be read as convolution with a distribution.

We leave the right-hand side of the parabolic equation (i.e. the terms associated with the stationary equation) unchanged and pass to the limit $\delta \to 0$. As u^α has enough regularity and $\frac{1-\alpha}{\alpha} \geq 1$, everything converges to the appropriate limit. We obtain

$$\begin{aligned} &- \int_{S^1} u_0 \psi(.,0) \; dx - \int_0^T \int_{S^1} u \psi_t \; dxdt \\ &= \int_0^T \int_{S^1} -2\left(4 - \frac{4}{\alpha}\right) \left[w w_{xx} (w^2)_{xx} - 4w|w_x|^2 w_{xx}\right] u^{(1-\alpha)/\alpha} \psi \\ &\qquad + 2\left(4 - \frac{4}{\alpha}\right) \cdot 4(w w_x (w w_{xx} - |w_x|^2))(u^{(1-\alpha)/\alpha}\psi)_x \\ &\qquad - 4\left(4 - \frac{4}{\alpha}\right)\left(3 - \frac{4}{\alpha}\right) |w_x|^2 (w w_{xx} - |w_x|^2) u^{(1-\alpha)/\alpha} \psi \\ &\qquad - 4w^2 (w w_{xx} - |w_x|^2)(u^{(1-\alpha)/\alpha}\psi)_{xx} \; dx \; dt \; . \end{aligned}$$

For w smooth in space, a straightforward but tedious computation shows that the right-hand side becomes exactly the expression occuring on the right-hand side in the definition of weak solutions (6) (see Jüngel and Violet [21]). For w with $w \in L^2(I; H^2(S^1)) \cap L^4(I; W^{1,4}(S^1)) \cap L^\infty(S^1 \times I)$, i.e. the case we are interested in, the equality follows by approximation: Take the convolution of w with a mollifier and pass to the limit; we obtain convergence of $\rho_\delta * w$ in $L^4(I; W^{1,4}(S^1)) \cap L^2(I; H^2(S^1))$. Note that the expressions $u_\delta^{1/2} = (\rho_\delta * w)^{2/\alpha}$ and $u_\delta^{(1-\alpha)/\alpha} = (\rho_\delta * w)^{4(1-\alpha)/\alpha^2}$ converge in $L^2(I; H^2(S^1)) \cap L^4(I; W^{1,4}(S^1))$ as the exponents are greater or equal to 2; additionally these expressions remain uniformly bounded in $L^\infty(S^1 \times I)$ and converge a.e.. This finishes the proof. $\square$

The existence result by Jüngel and Violet now immediately generalizes to nonnegative initial data as the regularity inferred from the α first-order entropy estimate provides sufficient compactness to pass to the limit:

Lemma 19. *Let $d = 1$, $\alpha \in A_1$, $\alpha \leq \frac{1}{2}$. For any $u_0 \in L^1(S^1)$ with $u_0^{\alpha/2} \in H^1(S^1)$ there exists a weak solution to the DLSS equation with the additional regularity $u^{\alpha/2} \in L^2(I; H^3(S^1)) \cap L^\infty(I; H^1(S^1))$, $u^{\alpha/6} \in L^6(I; W^{1,6}(S^1))$. The initial data is attained in the sense of $u(.,t) \to u_0$ strongly in $H^{-2}(S^1)$ as $t \to 0$. Moreover, for $t_2 > t_1 \geq 0$ we have the estimate*

$$\int_{S^1} |(u^{\alpha/2})_x|^2 \, dx \Big|_{t_1}^{t_2} \leq -c \int_{t_1}^{t_2} \int_{S^1} |(u^{\alpha/6})_x|^6 + |(u^{\alpha/2})_{xxx}|^2 \, dx \, dt \, .$$

Proof. The proof is easy, replacing u_0 by $u_0 + \epsilon$ and then passing to the limit. Note that as $\alpha \leq \frac{1}{2}$, the entropy estimate provide sufficient regularity to obtain the evolution equation for $\sqrt{u}$. Using the Aubin-Lions lemma to deduce strong convergence of $\sqrt{u}$ in $L^2(I; H^1(S^1))$ and reflexivity to obtain weak convergence of $\sqrt{u}$ in $L^2(I; H^2(S^1))$, we may pass to the limit in the weak formulation. For details, the reader is asked to consult the section on construction of solutions with weak initial trace.

Note that the regularity property $\log u \in L^2(I; H^2(S^1))$ which holds for strictly positive initial data may get lost in this process. $\square$

Proof of Corollary 6. This is a consequence of the fact that $u_0^{\beta/2} \in H^1(S^1)$ implies $u_0^{\alpha/2} \in H^1(S^1)$ for any $\alpha > \beta$, Lemma 19, the existence result by Jüngel and Matthes [18], Theorem 16, and the uniqueness result Theorem 5. $\square$

6. Decay estimates for the entropies

We now derive the entropy decay estimates which will provide the necessary compactness for the construction of solutions with weak initial trace.

Lemma 20. *Given any solution of the DLSS equation on $\Omega := S^1$ with $u_0^{\alpha/2} \in H^1(S^1)$ which in addition satisfies the α first order entropy estimate, we have*

$$||\nabla u^{\alpha/2}(.,t)||_{L^2}^2 \leq C||u_0||_{L^1}^\alpha t^{-1/2} \tag{18}$$

for $\alpha \leq 1$ and

$$||\nabla u^{\alpha/2}(.,t)||_{L^2}^2 \leq C||u_0||_{L^1}^\alpha \max(t^{-1/2-d(\alpha-1)/4}, t^{-1/2}) \tag{19}$$

in case $\alpha > 1$.

Proof. By Hölder's inequality we have

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx = 9 \int_\Omega u^{2\alpha/3} |\nabla u^{\alpha/6}|^2 \, dx \leq 9 \left(\int_\Omega u^\alpha \, dx \right)^{2/3} \left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{1/3} .$$

In case $\alpha \leq 1$, employing Hölder's inequality we obtain since S^1 has finite Lebesgue measure

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \leq C \left(\int_\Omega u \, dx \right)^{2\alpha/3} \left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{1/3} . \tag{20}$$

Recall the Gagliardo-Nirenberg-Sobolev inequality

$$||v||_{L^p} \le C||Dv||_{L^r}^{\theta} \cdot ||v||_{L^q}^{1-\theta} + C||v||_{L^q}$$

which holds for any Lipschitz domain (C depending on the domain) when θ is determined by

$$\frac{1}{p} = \theta \left(\frac{1}{r} - \frac{1}{d} \right) + (1-\theta)\frac{1}{q} \ .$$

If $\alpha > 1$, we use the Gagliardo-Nirenberg-Sobolev interpolation inequality with $v = u^{\alpha/6}$, $p = 6$, $r = 6$, $q = \frac{6}{\alpha}$, $d = 1$ and therefore $\theta = \frac{\alpha-1}{\alpha+5}$ to yield

$$\begin{aligned}
&\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \\
\le & C \left[\left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{(\alpha-1)/(\alpha+5)} \left(\int_\Omega u \, dx \right)^{6\alpha/(\alpha+5)} + \left(\int_\Omega u \, dx \right)^{\alpha} \right]^{2/3} \\
& \cdot \left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{1/3} \\
\le & C \left(\int_\Omega u \, dx \right)^{(4\alpha)/(\alpha+5)} \left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{(\alpha+1)/(\alpha+5)} \\
& + C \left(\int_\Omega u \, dx \right)^{2\alpha/3} \left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{1/3} \\
\le & C \max \left[\left(\int_\Omega u \, dx \right)^{4\alpha/(\alpha+5)} \left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{(\alpha+1)/(\alpha+5)} , \right. \\
& \qquad \left. \left(\int_\Omega u \, dx \right)^{2\alpha/3} \left(\int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \right)^{1/3} \right] .
\end{aligned} \tag{21}$$

The entropy estimate states that for $t_2 > t_1 \ge 0$ we have

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \Big|_{t_1}^{t_2} \le -c \int_{t_1}^{t_2} \int_\Omega |\nabla u^{\alpha/6}|^6 \, dx \, dt \ ,$$

which yields in case $\alpha \le 1$ (by 20)

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \Big|_{t_1}^{t_2} \le -c \int_{t_1}^{t_2} \left(\int_\Omega u \, dx \right)^{-2\alpha} \left(\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \right)^3 \, dt$$

and in case $\alpha > 1$ (by 21)

$$\begin{aligned}
\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \Big|_{t_1}^{t_2} \le -c \int_{t_1}^{t_2} \min & \left[\left(\int_\Omega u \, dx \right)^{-4\alpha/(\alpha+1)} \left(\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \right)^{(\alpha+5)/(\alpha+1)} , \right. \\
& \left. \left(\int_\Omega u \, dx \right)^{-2\alpha} \left(\int_\Omega |\nabla u^{\alpha/2}|^2 \, dx \right)^3 \right] dt \ .
\end{aligned}$$

By the comparison principle, this differential inequality implies that the entropy is bounded from above by the solution of the corresponding differential equation. Noting that mass is conserved, we see that in case $\alpha \le 1$ we are looking for a solution of an ODE of the form

$$\frac{d}{dt} f = -a f^b$$

for some $a > 0$, $b > 1$ constant and $f(0) = f_0 > 0$. It is well-known that the solution to this ODE is given by

$$f(t) = \left(a(b-1)t + f_0^{1-b}\right)^{1/(1-b)} \leq (a(b-1)t)^{1/(1-b)} \ .$$

The latter estimate follows since $b > 1$ which implies that $(.)^{1/(1-b)}$ is strictly decreasing.

Thus, in case $\alpha \leq 1$ we have the estimate

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \ dx \leq C||u_0||_{L^1}^\alpha t^{-1/2} \ .$$

In case $\alpha > 1$ the situation is a bit more difficult: as long as $\int_\Omega |\nabla u^{\alpha/2}|^2 \ dx > \left(\int_\Omega u \ dx\right)^\alpha$, we have $b = \frac{\alpha+5}{\alpha+1}$ and $a = c\left(\int_\Omega u \ dx\right)^{-4\alpha/(\alpha+1)}$. Solving the corresponding differential equation, we see that

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \ dx \leq \max(C||u_0||_{L^1}^\alpha t^{-(\alpha+1)/4}, ||u_0||_{L^1}^\alpha) \ . \tag{22}$$

As soon as $\int_\Omega |\nabla u^{\alpha/2}|^2 \ dx \leq \left(\int u \ dx\right)^\alpha$, we get $b = 3$ and $a = c\left(\int_\Omega u \ dx\right)^{-2\alpha}$. Defining t_0 to be the time at which $\int_\Omega |\nabla u^{\alpha/2}|^2 \ dx$ drops below $\left(\int_\Omega u \ dx\right)^\alpha$ and again solving the corresponding differential equation, we get

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \ dx \leq C||u_0||_{L^1}^\alpha (t-t_0)^{-1/2} \tag{23}$$

for $t \geq t_0$. By (22) we get $t_0 \leq C$. Using (22) for $t \leq 2t_0$ and (23) for $t > 2t_0$, we obtain

$$\int_\Omega |\nabla u^{\alpha/2}|^2 \ dx \leq C||u_0||_{L^1}^\alpha \max(t^{-(\alpha+1)/4}, t^{-1/2})$$

for C only depending on α, but independent of u and u_0. □

Remark 21. *Decay estimates of this kind are well-known in the theory of parabolic partial differential equations; in the case of the thin film equation they have been established in* [2] *and* [3] *and subsequently been used in* [11] *to construct solutions for nonnegative Radon measures with finite mass as initial data.*

Lemma 22. *Let $\gamma > 1$. Given any solution to the DLSS equation on $\Omega = [S^1]^d$ with $u_0^\gamma \in L^1(\Omega)$ which satisfies the zeroth-order γ entropy estimate, we have*

$$\int_\Omega u^\gamma(.,t) \ dx \leq C(d,\gamma)||u_0||_{L^1}^\gamma \left(t^{-d(\gamma-1)/4} + 1\right) \ .$$

Proof. By the Gagliardo-Nirenberg-Sobolev inequality with $v = u^{\gamma/4}$, $p = 4$, $r = 4$, $q = \frac{4}{\gamma}$ and thus $\theta = \frac{\gamma-1}{\gamma-1+\frac{4}{d}}$, we have

$$\int_\Omega u^\gamma \ dx \leq C||u||_{L^1}^{\frac{4\gamma}{d(\gamma-1)+4}} \left(\int_\Omega |\nabla u^{\gamma/4}|^4 \ dx\right)^{\frac{d(\gamma-1)}{d(\gamma-1)+4}} + C||u||_{L^1}^\gamma \ .$$

The entropy inequality gives for a.e. $t_2 > t_1$ in case $\gamma > 1$

$$\begin{aligned}
&\int_\Omega u^\gamma(.,t_2) \ dx - \int_\Omega u^\gamma(.,t_1) \ dx \\
\leq& -c\int_{t_1}^{t_2} \int_\Omega |\Delta u^{\gamma/2}|^2 \ dx \ dt \leq -c\int_{t_1}^{t_2} \int_\Omega |\nabla u^{\gamma/4}|^4 \ dx \ dt \\
\leq& -c\int_{t_1}^{t_2} ||u||_{L^1}^{-\frac{4\gamma}{d(\gamma-1)}} \left[\int_\Omega u^\gamma \ dx - C||u||_{L^1}^\gamma\right]_+^{\frac{d(\gamma-1)+4}{d(\gamma-1)}} \ dt \ .
\end{aligned}$$

As long as $\int_\Omega u^\gamma \; dx \geq 2C||u||_{L^1}^\gamma$, we have

$$\int_\Omega u^\gamma(.,t_2) \; dx - \int_\Omega u^\gamma(.,t_1) \; dx \leq -c \int_{t_1}^{t_2} ||u||_{L^1}^{-\frac{4\gamma}{d(\gamma-1)}} \left[\int_\Omega u^\gamma \; dx\right]^{\frac{d(\gamma-1)+4}{d(\gamma-1)}} \; dt \; .$$

Proceeding as in the proof of the decay estimates for first-order entropies, this differential inequality yields the bound

$$\int_\Omega u^\gamma(.,t) \; dx \leq C||u_0||_{L^1}^\gamma \left(t^{-d(\gamma-1)/4} + 1\right) \; .$$

The "1" which appears on the right-hand side is due to the differential inequality only being valid as long as $\int_\Omega u^\gamma(.,t) \; dx \geq 2C||u||_{L^1}^\gamma = 2C||u_0||_{L^1}^\gamma$; afterwards we only know that $\int_\Omega u^\gamma(.,t) \; dx$ is nonincreasing. □

Remark 23. *Our entropy decay estimates are probably optimal in case $\alpha \geq 1$, $\gamma > 1$, at least on small timescales: A self-similar solution to the DLSS equation on $\Omega = \mathbb{R}^d$ is given by*

$$U(x,t) = \frac{1}{(8\pi^2 t)^{\frac{d}{4}}} e^{-\frac{|x|^2}{8\sqrt{t}}}$$

(see e.g. [17]*); a straightforward calculation shows that the self-similar solution displays entropy decay exactly as predicted by our estimates. Since for small timescales and highly concentrated initial data, a bounded domain may be regarded as a "perturbation" of the $\mathbb{R}^d$, it seems likely that our decay estimates are optimal for $\alpha \geq 1$, $\gamma > 1$.*

Remark 24. *For bounded domains, the entropies decay exponentially with time as shown by Jüngel and Toscani* [20]*, Caceres, Carrillo, and Toscani* [6] *as well as by Jüngel and Matthes* [18]*; this of course provides significantly more information regarding large-time behaviour, but almost no information on immediate smoothing effects of the DLSS operator.*

7. Existence of solutions with weak initial trace

We now turn to the construction of solutions with weak initial trace using the estimates from the previous section.

Proof of Theorem 7. The proof relies on the existence result by Jüngel and Matthes for initial data in the $L \log L$ Orlicz class. We replace our initial data μ by $\rho_\epsilon * \mu$ where ρ_ϵ is the usual smoothing kernel. Let us denote the unique solution of the DLSS equation with initial data $\rho_\epsilon * \mu$ by u_ϵ.

The entropy decay estimates from the previous section in conjunction with the regularity inferred from the entropy estimates then provide sufficient compactness for the passage to the limit $\epsilon \to 0$: For $\delta > 0$ small enough such that $1 + \delta \in C_\gamma$, we have

$$\int_\Omega u_\epsilon^{1+\delta}(.,t) \; dx \leq C(d,\delta)[\mu(\Omega)]^{1+\delta} \left(t^{-\delta d/4} + 1\right) \tag{24}$$

since $||\rho_\epsilon * \mu||_{L^1} = \mu(\Omega)$. Thus we have

$$\int_{t_0}^T \int_\Omega |\Delta u_\epsilon^{(1+\delta)/2}|^2 + |\nabla u_\epsilon^{(1+\delta)/4}|^4 \; dx \; dt \leq C(d,\delta)[\mu(\Omega)]^{1+\delta} \left({t_0}^{-\delta d/4} + 1\right) \; .$$

We obtain by the entropy estimate for $\gamma = \frac{1}{2}$ and Hölder's inequality since Ω is bounded as well as Lemma 10

$$\int_0^T \int_\Omega |\Delta u_\epsilon^{1/4}|^2 + |\nabla u_\epsilon^{1/8}|^4 \ dx \ dt \leq C \int_\Omega u_\epsilon^{1/2}(.,T) \ dx \leq C||u_\epsilon(.,T)||_{L^1}^{1/2} = C[\mu(\Omega)]^{1/2} \ .$$

Note that $\Delta u_\epsilon^{1/2} = 2u_\epsilon^{1/4}\Delta u_\epsilon^{1/4} + 8u_\epsilon^{1/4}|\nabla u_\epsilon^{1/8}|^2$. Using the formula $\Delta u_\epsilon^{(1+\delta)/2} = (2+2\delta)u_\epsilon^{(1+2\delta)/4}\Delta u^{1/4} + (2+2\delta)(1+2\delta)\frac{1}{(1+\delta)^2}|\nabla u_\epsilon^{(1+\delta)/4}|^2$ in case $u > 1$, by Young's inequality we get a bound of the form

$$\begin{aligned}&\int_{t_0}^T \int_\Omega |\Delta u_\epsilon^{1/2}|^2 + |\nabla u_\epsilon^{1/4}|^4 \ dx \ dt \\ \leq & C(d,\delta) \int_{t_0}^T \int_\Omega |\Delta u_\epsilon^{1/4}|^2 + |\nabla u_\epsilon^{1/8}|^4 + |\Delta u_\epsilon^{(1+\delta)/2}|^2 + |\nabla u_\epsilon^{(1+\delta)/4}|^4 \ dx \ dt \ .\end{aligned}$$

From the equation

$$-\int_{t_0}^T \langle \sqrt{u_\epsilon}, \psi_t \rangle \ dt = \int_{t_0}^T \int_\Omega \frac{\psi}{\sqrt{u_\epsilon}} |\Delta \sqrt{u_\epsilon}|^2 \ dx \ dt - \int_{t_0}^T \int_\Omega \Delta \sqrt{u_\epsilon} \Delta \psi \ dx \ dt$$

which holds for all $\psi \in C_c^\infty(\Omega \times (t_0,T))$ (see (11)), we infer that (since $d \leq 3$)

$$||\partial_t \sqrt{u_\epsilon}||_{L^2((t_0,T);H^{-2})} \leq \int_{t_0}^T \int_\Omega |\Delta u_\epsilon^{1/4}|^2 + |\nabla u_\epsilon^{1/8}|^4 \ dx \ dt$$

and therefore

$$||\partial_t \nabla \sqrt{u_\epsilon}||_{L^2((t_0,T);H^{-3})} \leq \int_{t_0}^T \int_\Omega |\Delta u_\epsilon^{1/4}|^2 + |\nabla u_\epsilon^{1/8}|^4 \ dx \ dt \ .$$

We now pass to the limit $\epsilon \to 0$. By the Aubin-Lions Lemma, we infer that for a subsequence $\sqrt{u_\epsilon}$ converges strongly in $L^2((t_0,T);L^2)$; let us denote the limit by $\sqrt{u}$. Passing to a further subsequence we infer (again by the Aubin-Lions Lemma) that $\nabla\sqrt{u_\epsilon} \to w$ strongly in $L^2((t_0,T);L^2)$; the latter limit is immediately identified as $\nabla\sqrt{u}$. In addition we may assume that $\sqrt{u_\epsilon} \rightharpoonup \sqrt{u}$ weakly in $L^2((t_0,T);H^2)$.

We see immediately that these convergence properties are sufficient to pass to the limit in the equation

$$-\int_{t_0}^T \int_\Omega u_\epsilon \psi_t \ dx \ dt + \int_{t_0}^T \int_\Omega (\sqrt{u_\epsilon} D^2 \sqrt{u_\epsilon} - \nabla\sqrt{u_\epsilon} \otimes \nabla\sqrt{u_\epsilon}) : D^2\psi \ dx \ dt = 0$$

for any $\psi \in C_c^\infty(\Omega \times (t_0,T))$. The properties $u_\epsilon^{1/2} \in L^2((t_0,T);H^2)$ and $u_\epsilon^{1/4} \in L^2((t_0,T);H^2)$ carry over to the limit by reflexivity of $L^2((t_0,T);H^2)$, strong convergence of u in $L^1((t_0,T);L^1)$, and the uniform bounds on the norms.

Thus, setting $t_0 := \frac{1}{k}$, $T := k$, $k \in \mathbb{N}$, and applying a diagonalization argument we see that we can enforce that $u^{1/2} \in L^2_{loc}((0,\infty);H^2)$, $u^{1/4} \in L^2(I;H^2)$; furthermore, u is a weak solution of the DLSS equation on every time interval (t_0,∞) with $t_0 > 0$.

It remains to show that the initial trace of u coincides with our measure μ.

To this aim, we rearrange the weak formulation of the DLSS equation to yield for $\psi \in C_c^\infty(\Omega \times [0,1))$

$$\begin{aligned}
&\int_0^1 \int_\Omega u_\epsilon \psi_t \ dx \ dt + \int_\Omega (\rho_\epsilon * \mu)\psi(.,0) \ dx \\
=&\int_0^1 \int_\Omega \frac{1}{1-\delta} u_\epsilon^{(1+\delta)/2} D^2 u_\epsilon^{(1-\delta)/2} : D^2\psi \ dx \ dt \\
&+ \int_0^1 \int_\Omega \frac{1}{1-\delta} u_\epsilon^{1/2} (\nabla u_\epsilon^{\delta/2} \otimes \nabla u_\epsilon^{(1-\delta)/2}) : D^2\psi \ dx \ dt \\
&- \int_0^1 \int_\Omega (\nabla u_\epsilon^{1/2} \otimes \nabla u_\epsilon^{1/2}) : D^2\psi \ dx \ dt
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\int_0^1 \int_\Omega u_\epsilon \psi_t \ dx \ dt + \int_\Omega (\rho_\epsilon * \mu)\psi(.,0) \ dx \\
=&\int_0^1 \int_\Omega \frac{1}{1-\delta} u_\epsilon^{(1+\delta)/2} D^2 u_\epsilon^{(1-\delta)/2} : D^2\psi \ dx \ dt \\
&+ \int_0^1 \int_\Omega \left(\frac{\delta}{(1-\delta)(1+\delta)} - \frac{1}{(1+\delta)(1-\delta)} \right) (\nabla u_\epsilon^{(1+\delta)/2} \otimes \nabla u_\epsilon^{(1-\delta)/2}) : D^2\psi \ dx \ dt \ .
\end{aligned}$$

Integrating by parts twice we obtain

$$\begin{aligned}
&\int_0^1 \int_\Omega u_\epsilon \psi_t \ dx \ dt + \int (\rho_\epsilon * \mu)\psi(.,0) \ dx \\
&= \int_0^1 \int_\Omega c_1(\delta) u_\epsilon^{(1+\delta)/2} D^2 u_\epsilon^{(1-\delta)/2} : D^2\psi \ dx \ dt - \int_0^1 \int_\Omega c_2(\delta) u_\epsilon \Delta^2 \psi \ dx \ dt \ .
\end{aligned} \tag{25}$$

This implies $u_\epsilon \in W^{1,1}([0,1); H^{-4})$. We estimate using (24)

$$\int_0^1 \int_\Omega u_\epsilon^{1+2\delta} \ dxdt \le \int_0^1 C(d,\delta)[\mu(\Omega)]^{1+2\delta} (t^{-2\delta d/4} + 1) \ dt \le C(\delta)[\mu(\Omega)]^{1+2\delta}$$

if $\delta < \frac{2}{d}$. Thus $u_\epsilon^{(1+2\delta)/2}$ is bounded uniformly in $L^2([0,1); L^2)$ if δ is sufficiently small. Using the fact that $\sqrt{u_\epsilon} \to \sqrt{u}$ strongly in $L^2_{loc}((0,1); L^2)$, we deduce that $u_\epsilon^{(1+\delta)/2} \to u^{(1+\delta)/2}$ strongly in $L^2([0,1); L^2)$. Knowing that $u_\epsilon^{(1-\delta)/2} \rightharpoonup u^{(1-\delta)/2}$ weakly in $L^2([0,1); H^2)$ due to the entropy estimates (backward in time), we see that the terms on the right-hand side of (25) converge when passing to the limit $\epsilon \to 0$. The terms on the left-hand side also converge. We therefore obtain

$$\begin{aligned}
&\int_0^1 \int_\Omega u \psi_t \ dx \ dt + \int \psi(.,0) \ d\mu(x) \\
&= \int_0^1 \int_\Omega c_1(\delta) u^{(1+\delta)/2} D^2 u^{(1-\delta)/2} : D^2\psi \ dx \ dt - \int_0^1 \int_\Omega c_2(\delta) u \Delta^2 \psi \ dx \ dt \ .
\end{aligned}$$

In particular, we have $u \in W^{1,1}([0,1]; H^{-4})$ which implies $u \in C^0([0,1]; H^{-4})$. Moreover, we see that the equality $u(.,0) = \mu$ as elements of H^{-4} holds. Thus, $u(.,t) \to \mu$ strongly in H^{-4} as $t \to 0$; moreover, $u(.,t)$ is bounded uniformly (with respect to t) in RM (since mass is conserved and since u is nonnegative). This implies $u(.,t) \overset{*}{\rightharpoonup} \mu$ as $t \to 0$. □

8. Uniqueness for the quantum drift-diffusion equation

In this section, we shall show how our method of proving uniqueness extends to the case of quantum drift-diffusion systems, where several species of charge carriers are coupled via the electric field. In order not to overburden notation we present the proof of uniqueness for a single species; the reader will check immediately that the arguments generalize to systems, deriving estimates for $\sum_i \int \left|\sqrt{u_1^i} - \sqrt{u_2^i}\right|^2 \, dx$ instead of $\int \left|\sqrt{u_1} - \sqrt{u_2}\right|^2 \, dx$.

Proof of Theorem 9 in case of periodic boundary conditions. As the proof is mostly analogous to the case of the DLSS equation, we only describe the differences.

In the derivation of the evolution equation for $\sqrt{u}$ (proof of equation (11)) we see that on the right-hand side we get the additional term

$$
\begin{aligned}
& -\int_0^T \int_\Omega \left(\vartheta \nabla u + \tilde{Q} u \nabla V_{el}\right) \cdot \nabla \rho_\delta * \left(\frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right) \, dx \, dt \\
= & -\int_0^T \int_\Omega \vartheta \left(\rho_\delta * \nabla u\right) \cdot \frac{\nabla \psi}{\sqrt{\rho_\delta * u + \epsilon}} \, dx \, dt \\
& + 2 \int_0^T \int_\Omega \vartheta \left(\rho_\delta * \nabla u\right) \cdot \frac{\psi}{(\rho_\delta * u + \epsilon)^{3/4}} \nabla (\rho_\delta * u + \epsilon)^{1/4} \, dx \, dt \\
& + \tilde{Q} \int_0^T \int_\Omega \nabla u \cdot \nabla V_{el} \left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right) \, dx \, dt \\
& + \tilde{Q} \int_0^T \int_\Omega u \, \Delta V_{el} \left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right) \, dx \, dt \, .
\end{aligned}
$$

We now intend to pass to the limit $\delta \to 0$. The term $\vartheta \nabla u$ belongs to $L^{\frac{4}{3}}_{loc}(I; L^{\frac{4}{3}}(\Omega))$ since $\sqrt{u} \in L^2_{loc}(I; H^2(\Omega))$ and $\sqrt{u} \in L^2_{loc}(I; L^\infty(\Omega)) \cap L^\infty_{loc}(I; L^2(\Omega))$ which implies $\sqrt{u} \in L^4_{loc}(I; L^4(\Omega))$. Thus $\rho_\delta * \vartheta \nabla u$ converges to ∇u strongly in $L^{\frac{4}{3}}_{loc}(I; L^{\frac{4}{3}}(\Omega))$. By Lemma 14, we know that $\nabla(\rho_\delta * u + \epsilon)^{1/4}$ converges to $\nabla(u+\epsilon)^{1/4}$ strongly in $L^4(\Omega \times I)$.

Since $\Delta V_{el} = Qu$, we see that $\Delta V_{el} \in L^2_{loc}(I; L^2(\Omega))$. Thus we get $u \Delta V_{el} \in L^1_{loc}(I; L^1(\Omega))$.

We have $\nabla u = 2\sqrt{u} \nabla \sqrt{u}$ and therefore $\nabla u \in L^2_{loc}(I; L^{\frac{3}{2}}(\Omega))$ (due to $\nabla \sqrt{u} \in L^2_{loc}(I; H^1(\Omega)) \subset L^2_{loc}(I; L^6(\Omega))$ and $\sqrt{u} \in L^\infty_{loc}(I; L^2(\Omega))$). Knowing that $\nabla V_{el} \in L^2_{loc}(I; L^6(\Omega))$ (by regularity theory and $d \leq 3$), we deduce that $\nabla u \cdot \nabla V_{el} \in L^1_{loc}(I; L^1(\Omega))$.

In the limit $\delta \to 0$ our additional terms therefore become

$$
\begin{aligned}
& -\int_0^T \int_\Omega \vartheta \nabla u \cdot \frac{\nabla \psi}{\sqrt{u+\epsilon}} \, dx \, dt \\
& + 2 \int_0^T \int_\Omega \vartheta \nabla u \cdot \frac{\psi}{(u+\epsilon)^{3/4}} \nabla (u+\epsilon)^{1/4} \, dx \, dt \\
& + \tilde{Q} \int_0^T \int_\Omega \nabla u \cdot \nabla V_{el} \, \frac{\psi}{\sqrt{u+\epsilon}} \, dx \, dt \\
& + \tilde{Q} \int_0^T \int_\Omega u \, \Delta V_{el} \, \frac{\psi}{\sqrt{u+\epsilon}} \, dx \, dt \\
= & -\int_0^T \int_\Omega 2\vartheta \frac{\sqrt{u}}{\sqrt{u+\epsilon}} \nabla \sqrt{u} \cdot \nabla \psi - 2\vartheta \frac{\sqrt{u}}{u+\epsilon} \nabla \sqrt{u} \cdot \nabla \sqrt{u+\epsilon} \, \psi \, dx \, dt \\
& - \tilde{Q} \int_0^T \int_\Omega \frac{u}{\sqrt{u+\epsilon}} \nabla V_{el} \cdot \nabla \psi - \frac{u}{u+\epsilon} \nabla V_{el} \cdot \nabla \sqrt{u+\epsilon} \, \psi \, dx \, dt \, .
\end{aligned}
$$

Letting $\epsilon \to 0$, we see that in the analogue of equation (11) we get the following additional terms on the right-hand side:

$$
\begin{aligned}
&-\int_0^T \int_\Omega 2\vartheta\nabla\sqrt{u}\cdot\nabla\psi - 8\vartheta\nabla u^{1/4}\cdot\nabla u^{1/4}\ \psi\ dx\ dt \\
&-\tilde{Q}\int_0^T \int_\Omega \sqrt{u}\ \nabla V_{el}\cdot\nabla\psi - \nabla V_{el}\cdot\nabla\sqrt{u}\ \psi\ dx\ dt
\end{aligned}
$$

In the proof of the stability estimate (the analogue of Theorem 5), the following differences occur: Inserting $\xi\cdot\rho_\delta * \big(\rho_\delta * \sqrt{u_2}\big)$ as a test function in the equation for $\sqrt{u_1}$, we get the additional terms

$$
\begin{aligned}
&+2\int_0^T \xi(t)\int_\Omega -\vartheta\nabla(\rho_\delta*\sqrt{u_1})\cdot\nabla(\rho_\delta*\sqrt{u_2}) + \vartheta\frac{\rho_\delta*\rho_\delta*\sqrt{u_2}}{\sqrt{u_1}}|\nabla\sqrt{u_1}|^2\ dx\ dt \\
&-\tilde{Q}\int_0^T \xi(t)\int_\Omega \sqrt{u_1}\ \nabla V_{el\,1}\cdot\nabla(\rho_\delta*\rho_\delta*\sqrt{u_2}) - \nabla V_{el\,1}\cdot\nabla\sqrt{u_1}\ (\rho_\delta*\rho_\delta*\sqrt{u_2})\ dx\ dt
\end{aligned}
$$

on the right-hand side.

Adding the equation with 1 and 2 interchanged and passing to the limit $\delta\to 0$, using Fatou's Lemma we see that in the analogue of inequality (15) the additional terms

$$
\begin{aligned}
&+2\int_0^T \xi(t)\int_\Omega \vartheta\left|\left(\frac{u_2}{u_1}\right)^{1/4}\nabla\sqrt{u_1} - \left(\frac{u_1}{u_2}\right)^{1/4}\nabla\sqrt{u_2}\right|^2\ dx\ dt \\
&-\tilde{Q}\int_0^T \xi(t)\int_\Omega (\nabla V_{el\,1}-\nabla V_{el\,2})\cdot(\sqrt{u_1}\ \nabla\sqrt{u_2} - \sqrt{u_2}\ \nabla\sqrt{u_1})\ dx\ dt
\end{aligned}
$$

appear on the right-hand side. The first term is nonnegative. Choosing ξ as in the proof of Theorem 5 and passing to the limit, we obtain for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$

$$
\begin{aligned}
I + II :=& \int_\Omega |\sqrt{u_1(.,t_2)} - \sqrt{u_2(.,t_2)}|^2\ dx \\
&+\tilde{Q}\int_{t_1}^{t_2}\int_\Omega (\nabla V_{el\,1}-\nabla V_{el\,2})\cdot(\sqrt{u_1}\ \nabla\sqrt{u_2} - \sqrt{u_2}\ \nabla\sqrt{u_1})\ dx\ dt \\
\le& \int_\Omega |\sqrt{u_1(.,t_1)} - \sqrt{u_2(.,t_1)}|^2\ dx\ .
\end{aligned}
\tag{26}
$$

It remains to derive a bound on the second term on the left-hand side. We rearrange

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega (\nabla V_{el\,1}-\nabla V_{el\,2})\cdot(\sqrt{u_1}\ \nabla\sqrt{u_2} - \sqrt{u_2}\ \nabla\sqrt{u_1})\ dx\ dt \\
=&\int_{t_1}^{t_2}\int_\Omega (\nabla V_{el\,1}-\nabla V_{el\,2})\cdot\big(\sqrt{u_1}\ \nabla\sqrt{u_2} - \sqrt{u_1}\ \nabla\sqrt{u_1} \\
&\qquad\qquad + \sqrt{u_1}\ \nabla\sqrt{u_1} - \sqrt{u_2}\ \nabla\sqrt{u_1}\big)\ dx\ dt \\
=&2\int_{t_1}^{t_2}\int_\Omega \nabla\sqrt{u_1}\cdot(\nabla V_{el\,1}-\nabla V_{el\,2})\,(\sqrt{u_1}-\sqrt{u_2})\ dx\ dt \\
&-\int_{t_1}^{t_2}\int_\Omega (\Delta V_{el\,1}-\Delta V_{el\,2})\,\sqrt{u_1}\,(\sqrt{u_2}-\sqrt{u_1})\ dx\ dt
\end{aligned}
$$

which implies

$$\begin{aligned}
|II| \leq& 2|\tilde{Q}| \int_{t_1}^{t_2} \left(||\sqrt{u_2(.,t)}||_{L^\infty}^2 + ||\sqrt{u_1(.,t)}||_{L^\infty}^2 + 1\right) \int_\Omega |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt \\
&+ |\tilde{Q}| \int_{t_1}^{t_2} \int_\Omega |\Delta V_{el1} - \Delta V_{el2}|^2 \ dx \ dt \\
&+ |\tilde{Q}| \int_{t_1}^{t_2} \frac{1}{1 + ||\sqrt{u_1(.,t)}||_{L^\infty}^2 + ||\sqrt{u_2(.,t)}||_{L^\infty}^2} \int_\Omega |\nabla\sqrt{u_1}|^2 |\nabla V_{el1} - \nabla V_{el2}|^2 \ dx \ dt \\
=:& |\tilde{Q}| \cdot (III + IV + VI) \ .
\end{aligned}$$

By the equation satisfied by $V_{el\,i}$ we have

$$\begin{aligned}
IV =& \int_{t_1}^{t_2} \int_\Omega |u_1 - u_2|^2 \ dx \ dt = \int_0^T \int_\Omega \left|(\sqrt{u_1} - \sqrt{u_2})\,(\sqrt{u_1} + \sqrt{u_2})\right|^2 \ dx \ dt \\
\leq& C \int_{t_1}^{t_2} \left(||\sqrt{u_1}(.,t)||_{L^\infty}^2 + ||\sqrt{u_2}(.,t)||_{L^\infty}^2\right) \int_\Omega |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt \ .
\end{aligned}$$

The reader will check that this estimate also works (with minor changes) in case of multiple species with different charges, yielding a bound of the form

$$IV \leq C \int_{t_1}^{t_2} \max_i \left(||\sqrt{u_1^i(.,t)}||_{L^\infty} + ||\sqrt{u_2^i(.,t)}||_{L^\infty}\right) \cdot \sum_i \int_\Omega \left|\sqrt{u_1^i} - \sqrt{u_2^i}\right|^2 \ dx \ dt \ .$$

It remains to derive a bound on VI. By classical theory of elliptic equations (see e.g. [15]), we have

$$||\nabla\left(V_{el1} - V_{el2}\right)||_{L^p} \leq C||u_1 - u_2||_{L^2}$$

as long as $p \leq \frac{2d}{d-2}$. Thus

$$\begin{aligned}
VI \leq& \int_{t_1}^{t_2} \frac{||\nabla\sqrt{u_1}||_{L^q}^2}{1 + ||\sqrt{u_1}||_{L^\infty}^2 + ||\sqrt{u_2}||_{L^\infty}^2} ||\nabla(V_{el1} - V_{el2})||_{L^p}^2 \ dt \\
\leq& \int_{t_1}^{t_2} \frac{||\nabla\sqrt{u_1}||_{L^q}^2}{1 + ||\sqrt{u_1}||_{L^\infty}^2 + ||\sqrt{u_2}||_{L^\infty}^2} \int_\Omega |u_1 - u_2|^2 \ dx \ dt \\
\leq& C \int_{t_1}^{t_2} ||\nabla\sqrt{u_1}||_{L^q}^2 \int_\Omega |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt
\end{aligned}$$

for q with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Since $p \leq \frac{2d}{d-2}$ it follows that all $q \geq d$ are admissible. Again, a corresponding equation holds in the case of multiple species.

Putting these results together, we have

$$\begin{aligned}
&\int_\Omega |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \Big|_{t_1}^{t_2} \\
\leq& C \int_{t_1}^{t_2} \left(1 + ||\sqrt{u_1}||_{L^\infty}^2 + ||\sqrt{u_2}||_{L^\infty}^2 + ||\nabla\sqrt{u_1}||_{L^d}^2\right) \int_\Omega |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt \qquad (27) \\
\leq& C \int_{t_1}^{t_2} \left(1 + ||\sqrt{u_1}||_{H^2}^2 + ||\sqrt{u_2}||_{H^2}^2\right) \int_\Omega |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt
\end{aligned}$$

where the latter inequality follows by the Sobolev embedding theorem. We note that for $d \leq 3$ we have $\frac{2d}{d-2} \geq 6 \geq d$. Gronwall's inequality now implies the assertion of the theorem. □

9. Nonperiodic boundary conditions

The uniqueness result extends to the case of combined Dirichlet-Neumann boundary conditions. We shall need the following analogue of Lemma 11:

Lemma 25. *Given $u \in H^2(\Omega)$ with $u \geq 0$ and $\phi \in C_c^\infty(\Omega)$, we have the estimate*

$$\int_\Omega \phi^4 |\nabla u^{1/2}|^4 \, dx \leq C(d,\Omega) \int_\Omega \phi^4 |\Delta u|^2 \, dx + C(d) \int_\Omega u^2 |\nabla \phi|^4 \, dx \, .$$

Proof. We calculate for smooth strictly positive u

$$\begin{aligned}\int_\Omega |\partial_i u^{1/2}|^4 \phi^4 \, dx &= \frac{1}{16} \int_\Omega u^{-2} |\partial_i u|^4 \phi^4 \, dx \\ &= \frac{3}{16} \int_\Omega u^{-1} |\partial_i u|^2 \partial_{ii}^2 u \; \phi^4 \, dx + \frac{1}{4} \int_\Omega u^{-1} |\partial_i u|^2 \partial_i u \; \partial_i \phi \; \phi^3 \, dx \, .\end{aligned}$$

By Young's inequality we obtain

$$\int_\Omega |\partial_i u^{1/2}|^4 \phi^4 \, dx \leq C \int_\Omega |\partial_{ii} u|^2 \phi^4 \, dx + C \int_\Omega u^2 |\partial_i \phi|^4 \, dx \, . \tag{28}$$

For smooth u we have

$$\begin{aligned}\int_\Omega |\Delta u|^2 \phi^4 \, dx &= - \int_\Omega \nabla \Delta u \cdot \nabla u \; \phi^4 \, dx - 4 \int_\Omega \Delta u \nabla u \cdot \nabla \phi \; \phi^3 \, dx \\ &= \int_\Omega |D^2 u|^2 \; \phi^4 \, dx + 4 \int_\Omega \nabla u \cdot D^2 u \cdot \nabla \phi \; \phi^3 \, dx - 4 \int_\Omega \Delta u \nabla u \cdot \nabla \phi \; \phi^3 \, dx\end{aligned}$$

which gives using Young's inequality

$$\int_\Omega |D^2 u|^2 \; \phi^4 \, dx \leq C(d) \int_\Omega |\Delta u|^2 \phi^4 \, dx + C(d) \int_\Omega |\nabla u|^2 |\nabla \phi|^2 \phi^2 \, dx \, . \tag{29}$$

Taking the sum with respect to i in (28) and using (29), an application of Young's inequality to treat the last term yields the desired result for smooth u. For general u, the inequality again follows by approximation. ☐

Lemma 26. *Given any weak solution of the DLSS equation with Dirichlet-Neumann boundary data in the sense of Definition 3 with the additional regularity $u^{1/4} \in L^2_{loc}(I; H^2(\Omega))$, for any $\psi \in C_c^\infty(\Omega \times [0,T))$ the equation*

$$\begin{aligned}&-2 \int_0^T \int_\Omega \sqrt{u} \psi_t \, dx \, dt - 2 \int_\Omega \sqrt{u_0} \psi \, dx \\ &= - \int_0^T \int_\Omega \Delta \sqrt{u} \Delta \psi \, dx \, dt + \int_0^T \int_\Omega \frac{|\Delta \sqrt{u}|^2}{\sqrt{u}} \psi \, dx \, dt\end{aligned} \tag{30}$$

is satisfied.

Proof. The proof is analogous to the case of periodic boundary conditions, since ψ is assumed to be compactly supported in Ω. ☐

Lemma 27. *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ domain. Then there exist $C > 0$ and $C(\Omega) > 0$ such that for all $\tau > 0$ the following Poincare type inequality holds for any $v \in H_0^2(\Omega)$:*

$$\int_{\Omega \cap \{x : \mathrm{dist}(x, \partial\Omega) < \tau\}} \tau^{-2} |\nabla v|^2 + \tau^{-4} |v|^2 \, dx \leq C(\Omega) \int_{\Omega \cap \{x : \mathrm{dist}(x, \partial\Omega) < C\tau\}} |D^2 v|^2 \, dx \, .$$

Moreover, for any $\tau > 0$ and any $v \in W_0^{1,4}(\Omega)$ we have

$$\int_{\Omega \cap \{x : \mathrm{dist}(x, \partial\Omega) < \tau\}} \tau^{-4} |v|^4 \, dx \leq C(\Omega) \int_{\Omega \cap \{x : \mathrm{dist}(x, \partial\Omega) < C\tau\}} |\nabla v|^4 \, dx \, .$$

Proof. Let U be an open subset of $\mathbb{R}^{d-1}$. Assume that (possibly after a rotation) we have a function $g : \mathbb{R}^{d-1} \to \mathbb{R}$ of class $C^{1,1}$ and some $\mu > 0$ such that

$$\Omega_U := \{x : (x_1, \ldots, x_{d-1}) \in U \wedge 0 < x_d - g(x_1, \ldots, x_{d-1}) < \mu\} \subset \Omega$$

and

$$\{(x_1, \ldots, x_{d-1}, g(x_1, \ldots, x_{d-1})) : (x_1, \ldots, x_{d-1}) \in U\} \subset \partial\Omega\ .$$

Then it follows easily (by integrating the one-dimensional Poincare inequality with respect to x_1, $\ldots$, x_{d-1}) that for every $\tau > 0$ and every $v \in H^2_0(\Omega)$ the estimate

$$\begin{aligned} &\int_{\Omega_U \cap \{x : x_d - g(x_1, \ldots, x_{d-1}) < \tau\}} \tau^{-2}|\nabla v|^2 + \tau^{-4}|v|^2 \, dx \\ \leq &C(U) \int_{\Omega_U \cap \{x : x_d - g(x_1, \ldots, x_{d-1}) < \tau\}} |D^2 v|^2 \, dx \end{aligned} \tag{31}$$

holds.

Our domain Ω being $C^{1,1}$, we see that for every $z \in \partial\Omega$ there exists an open neighbourhood V of z such that (after possibly a rotation) the above conditions are satisfied for some U and some g, where $\Omega_U \subset V$ and $z \in \{x : (x_1, \ldots, x_{d-1}) \in U \wedge x_d = g(x_1, \ldots, x_{d-1})\}$. Moreover, for $\delta > 0$ small enough we have for every $x \in B_\delta(z)$

$$\operatorname{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega \cap B_{2\delta}(z)} \sqrt{|x_d - g(y_1, \ldots, y_{d-1})|^2 + \sum_{i=1}^{d-1} |x_i - y_i|^2}\ .$$

Setting $h(x) := |x_d - g(x_1, \ldots, x_{d-1})|$ and $r(x, y) := \sqrt{\sum_{i=1}^{d-1} |x_i - y_i|^2}$ and taking into account that $\operatorname{dist}(x, \partial\Omega) \leq h(x)$, we get for any $\epsilon > 0$ (w.l.o.g. we may assume that $\epsilon < 1$)

$$\begin{aligned} &\operatorname{dist}(x, \partial\Omega) \\ = &\min_{y \in \partial\Omega \cap B_{2\delta}(z), r(x,y) \leq \operatorname{dist}(x, \partial\Omega)} \sqrt{|x_d - g(y_1, \ldots, y_{d-1})|^2 + r(x, y)^2} \\ \geq &\min \Bigg[\min_{y \in \partial\Omega \cap B_{2\delta}(z), \epsilon h(x) \leq r(x,y)} r(x, y), \\ &\qquad \min_{y \in \partial\Omega \cap B_{2\delta}(z), r(x,y) \leq \epsilon h(x)} \sqrt{|x_d - g(y_1, \ldots, y_{d-1})|^2} \Bigg] \\ \geq &\min \left[\epsilon h(x), \left(1 - \epsilon \sup_{y \in B_{2\delta}(z)} |\nabla g(y_1, \ldots, y_{d-1})|\right) h(x) \right]\ . \end{aligned}$$

Using the fact that $g \in C^{1,1}$ (which implies that ∇g is bounded on bounded subsets of $\mathbb{R}^{d-1}$) and choosing ϵ small enough, we get

$$|x_d - g(x_1, \ldots, x_{d-1})| \leq C(z, \delta) \operatorname{dist}(x, \partial\Omega)$$

for any $x \in B_\delta(z)$, if $\delta > 0$ has been chosen small enough.

Combining this estimate with (31) we see that for any $z \in \partial\Omega$ we have some $\delta > 0$ and some neighbourhood V of z such that for any $\tau > 0$ the estimate

$$\begin{aligned} &\int_{\Omega \cap B_\delta(z) \cap \{x : \operatorname{dist}(x, \partial\Omega) < \tau\}} \tau^{-2}|\nabla v|^2 + \tau^{-4}|v|^2 \, dx \\ \leq &C(z, \Omega) \int_{\Omega \cap V \cap \{x : \operatorname{dist}(x, \partial\Omega) < C\tau\}} |D^2 v|^2 \, dx \end{aligned}$$

holds. As this assertion holds for any z, using a covering argument our lemma follows.

The second inequality is proven analogously. $\square$

We are now in position to prove uniqueness of solutions for the DLSS equation with Dirichlet-Neumann boundary conditions. As the proof is mostly analogous to the case of periodic boundary conditions, we only indicate the relevant differences.

Proof of Theorem 8. Take some nonnegative cutoff $\phi \in C_c^\infty(\Omega)$ and some nonnegative $\xi \in C_c^\infty([0,\infty))$. We insert $\xi \cdot (\rho_\delta * (\phi(\rho_\delta * \sqrt{u_2})))$ into (30) to obtain

$$\begin{aligned}
&2\int_0^t \int_\Omega (\rho_\delta * \sqrt{u_1})_t \; \xi\phi \; (\rho_\delta * \sqrt{u_2}) \; dx \; dt \\
=&-\int_0^T \int_\Omega \Delta\sqrt{u_1} \cdot \xi \cdot \Delta(\rho_\delta * (\phi(\rho_\delta * \sqrt{u_2}))) \; dx \; dt \\
&+\int_0^T \int_\Omega \frac{|\Delta\sqrt{u_1}|^2}{\sqrt{u_1}} \cdot \xi \cdot (\rho_\delta * (\phi(\rho_\delta * \sqrt{u_2}))) \; dx \; dt \; .
\end{aligned}$$

Interchanging u_1 and u_2 and adding, we get

$$\begin{aligned}
&-2\int_0^T \int_\Omega \xi_t \phi \; (\rho_\delta * \sqrt{u_1})(\rho_\delta * \sqrt{u_2}) \; dx \; dt \\
&-2\int_\Omega \xi(0)\phi \; (\rho_\delta * \sqrt{u_1(.,0)})(\rho_\delta * \sqrt{u_2(.,0)}) \; dx \\
=&-\int_0^T \int_\Omega \Delta(\rho_\delta * \sqrt{u_1}) \cdot \xi \cdot \Delta(\phi(\rho_\delta * \sqrt{u_2})) \; dx \; dt \\
&-\int_0^T \int_\Omega \Delta(\rho_\delta * \sqrt{u_2}) \cdot \xi \cdot \Delta(\phi(\rho_\delta * \sqrt{u_1})) \; dx \; dt \\
&+\int_0^T \int_\Omega \xi \frac{|\Delta\sqrt{u_1}|^2}{\sqrt{u_1}} \; (\rho_\delta * (\phi(\rho_\delta * \sqrt{u_2}))) \; dx \; dt \\
&+\int_0^T \int_\Omega \xi \frac{|\Delta\sqrt{u_2}|^2}{\sqrt{u_2}} \; (\rho_\delta * (\phi(\rho_\delta * \sqrt{u_1}))) \; dx \; dt \; .
\end{aligned}$$

Again passing to the limit $\delta \to 0$, we obtain by the usual convergence properties of mollifications and Fatou's lemma

$$\begin{aligned}
&-2\int_0^T \int_\Omega \xi_t \phi \sqrt{u_1}\sqrt{u_2} \; dx \; dt - 2\int_\Omega \xi(0)\phi\sqrt{u_1(.,0)}\sqrt{u_2(.,0)} \; dx \\
\geq&-\int_0^T \int_\Omega \Delta\sqrt{u_1} \cdot \xi \cdot \Delta(\phi\sqrt{u_2}) \; dx \; dt - \int_0^T \int_\Omega \Delta\sqrt{u_2} \cdot \xi \cdot \Delta(\phi\sqrt{u_1}) \; dx \; dt \\
&+\int_0^T \int_\Omega \frac{|\Delta\sqrt{u_1}|^2}{\sqrt{u_1}} \; \xi \; \phi\sqrt{u_2} \; dx \; dt + \int_0^T \int_\Omega \frac{|\Delta\sqrt{u_2}|^2}{\sqrt{u_2}} \; \xi \; \phi\sqrt{u_1} \; dx \; dt \\
=&-\int_0^T \int_\Omega \Delta\sqrt{u_1} \cdot \xi \cdot (\Delta\phi \; \sqrt{u_2} + 2\nabla\phi \cdot \nabla\sqrt{u_2}) \; dx \; dt \qquad (32) \\
&-\int_0^T \int_\Omega \Delta\sqrt{u_2} \cdot \xi \cdot (\Delta\phi \; \sqrt{u_1} + 2\nabla\phi \cdot \nabla\sqrt{u_1}) \; dx \; dt \\
&+\int_0^T \int_\Omega \left| \sqrt[4]{\frac{u_2}{u_1}}\Delta\sqrt{u_1} - \sqrt[4]{\frac{u_1}{u_2}}\Delta\sqrt{u_2} \right|^2 \; \xi \; \phi \; dx \; dt \; .
\end{aligned}$$

Inserting $\phi \cdot \xi$ in (7) and integrating by parts, we obtain

$$\begin{aligned}
&-\int_0^T \int_\Omega \xi_t \ \phi \ u_1 \ dx \ dt - \int_\Omega \xi(0)\phi \ u_1(.,0) \ dx \\
=&-\int_0^T \int_\Omega \xi \ (-\sqrt{u_1} \ \nabla\sqrt{u_1} \cdot \nabla\Delta\phi - 2(\nabla\sqrt{u_1} \otimes \nabla\sqrt{u_1}) : D^2\phi) \ dx \ dt \\
=&-\int_0^T \int_\Omega \xi \ (\nabla\sqrt{u_1} \cdot \nabla\sqrt{u_1} \ \Delta\phi + \sqrt{u_1}\Delta\sqrt{u_1}\Delta\phi \\
&\qquad + 2\Delta\sqrt{u_1} \ \nabla\sqrt{u_1} \cdot \nabla\phi + 2\nabla\sqrt{u_1} \cdot D^2\sqrt{u_1} \cdot \nabla\phi) \ dx \ dt \\
=&-\int_0^T \int_\Omega \xi \ (\nabla\sqrt{u_1} \cdot \nabla\sqrt{u_1}\Delta\phi + \sqrt{u_1}\Delta\sqrt{u_1}\Delta\phi \\
&\qquad + 2\Delta\sqrt{u_1} \ \nabla\sqrt{u_1} \cdot \nabla\phi - \nabla\sqrt{u_1} \cdot \nabla\sqrt{u_1} \ \Delta\phi) \ dx \ dt \\
=&-\int_0^T \int_\Omega \xi \ (\sqrt{u_1}\Delta\sqrt{u_1}\Delta\phi + 2\Delta\sqrt{u_1} \ \nabla\sqrt{u_1} \cdot \nabla\phi) \ dx \ dt \ .
\end{aligned}$$

Adding the corresponding equation for u_2 and substracting inequality (32), we get

$$\begin{aligned}
&-\int_0^T \int_\Omega \xi_t\phi \ |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt - \int_\Omega \xi(0)\phi \ \left|\sqrt{u_1(.,0)} - \sqrt{u_2(.,0)}\right|^2 \ dx \\
\leq &-\int_0^T \int_\Omega \Delta\sqrt{u_1} \cdot \xi \cdot (\Delta\phi \ (\sqrt{u_1} - \sqrt{u_2}) + 2\nabla\phi \cdot \nabla(\sqrt{u_1} - \sqrt{u_2})) \ dx \ dt \\
&-\int_0^T \int_\Omega \Delta\sqrt{u_2} \cdot \xi \cdot (\Delta\phi \ (\sqrt{u_2} - \sqrt{u_1}) + 2\nabla\phi \cdot \nabla(\sqrt{u_2} - \sqrt{u_1})) \ dx \ dt \ .
\end{aligned} \tag{33}$$

Let $\tau \in (0,1)$. We now choose $\phi^\tau \in C_c^\infty(\Omega)$ with $0 \leq \phi^\tau \leq 1$ such that $|\nabla\phi^\tau| \leq \frac{C(\Omega)}{\tau}$ and $|\Delta\phi^\tau| \leq \frac{C(\Omega)}{\tau^2}$ as well as $\phi^\tau(x) = 1$ for any $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) \geq \tau$ hold. Such a family of functions exists since Ω has $C^{1,1}$ boundary (one can construct such functions in each coordinate chart of $\overline{\Omega}$ which contains a part of $\partial\Omega$ and glue them together using a partition of unity).

As by our definition of solution we have $\sqrt{u_1} - \sqrt{u_2} \in L^2_{loc}(I; H^2_0(\Omega))$, we therefore obtain from (33), the properties of ϕ^τ, Hölder's inequality, and Lemma 27

$$\begin{aligned}
&-\int_0^T \int_\Omega \xi_t\phi^\tau \ |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt - \int_\Omega \xi(0)\phi^\tau \ \left|\sqrt{u_1(.,0)} - \sqrt{u_2(.,0)}\right|^2 \ dx \\
\leq & C(\Omega) \int_0^T \int_{\Omega\cap\{x:\operatorname{dist}(x,\partial\Omega)<\tau\}} \xi \cdot (|\Delta\sqrt{u_1}| + |\Delta\sqrt{u_1}|) \\
&\qquad \cdot (\tau^{-2}|\sqrt{u_2} - \sqrt{u_1}| + \tau^{-1} \, |\nabla(\sqrt{u_2} - \sqrt{u_1})|) \ dx \ dt \\
\leq & C(\Omega) \int_0^T \xi \cdot \left(\int_\Omega |D^2\sqrt{u_1}|^2 + |D^2\sqrt{u_2}|^2 \ dx\right)^{1/2} \\
&\qquad \cdot \left(\int_{\Omega\cap\{x:\operatorname{dist}(x,\partial\Omega)<C\tau\}} |D^2(\sqrt{u_1} - \sqrt{u_2})|^2 \ dx\right)^{1/2} \ dt \ .
\end{aligned}$$

We know that $\sqrt{u_i} \in L^2_{loc}(I; H^2(\Omega))$. The second factor of the integrand thus tends to zero as $\tau \to 0$. The ϕ^τ converge to 1 pointwise a.e. on Ω and are bounded by 1. We obtain by dominated convergence (applied to both sides of the inequality)

$$-\int_0^T \int_\Omega \xi_t \ |\sqrt{u_1} - \sqrt{u_2}|^2 \ dx \ dt - \int_\Omega \xi(0) \ \left|\sqrt{u_1(.,0)} - \sqrt{u_2(.,0)}\right|^2 \ dx \leq 0 \ .$$

The result now follows by an approximation argument analogous to the one used in the case of periodic boundary conditions. □

Proof of Theorem 9 for Dirichlet-Neumann boundary conditions. The case of quantum drift-diffusion equations with Dirichlet-Neumann boundary conditions is dealt with using the methods from the proof of Theorem 8 and Theorem 9 in case of periodic boundary data. However, in the derivation of the analogue of (26), besides some terms similar to the terms on the right-hand side of (33), the additional terms

$$
\begin{aligned}
&+2\vartheta \int_{t_1}^{t_2} \int_\Omega \nabla\sqrt{u_1}\cdot\nabla\phi^\tau \sqrt{u_2} + \nabla\sqrt{u_2}\cdot\nabla\phi^\tau \sqrt{u_1} \; dx\; dt \\
&+\tilde{Q}\int_{t_1}^{t_2} \int_\Omega \sqrt{u_1}\sqrt{u_2}\,(\nabla V_{el1}+\nabla V_{el2})\cdot\nabla\phi^\tau \; dx\; dt \\
&-\int_{t_1}^{t_2} \int_\Omega \vartheta\nabla(u_1+u_2)\cdot\nabla\phi^\tau \; dx\; dt - \tilde{Q}\int_{t_1}^{t_2} \int_\Omega (u_1\nabla V_{el1}+u_2\nabla V_{el2})\cdot\nabla\phi^\tau \; dx\; dt
\end{aligned}
$$

appear. These terms can be rewritten as

$$
\begin{aligned}
&+2\vartheta \int_{t_1}^{t_2} \int_\Omega \nabla\sqrt{u_1}\cdot\nabla\phi^\tau \,(\sqrt{u_2}-\sqrt{u_1}) + \nabla\sqrt{u_2}\cdot\nabla\phi^\tau \,(\sqrt{u_1}-\sqrt{u_2}) \; dx\; dt \\
&-\tilde{Q}\int_{t_1}^{t_2} \int_\Omega \sqrt{u_1}\,(\sqrt{u_1}-\sqrt{u_2})\nabla V_{el1}\cdot\nabla\phi^\tau \; dx\; dt \\
&-\tilde{Q}\int_{t_1}^{t_2} \int_\Omega \sqrt{u_2}\,(\sqrt{u_2}-\sqrt{u_1})\nabla V_{el2}\cdot\nabla\phi^\tau \; dx\; dt\ .
\end{aligned}
$$

By our Poincare type argument, the first term dissappears in the limit $\tau \to 0$. Regarding the second and the third term, we may estimate using Hölder's inequality and the properties of ϕ^τ

$$
\begin{aligned}
&\left|\int_{t_1}^{t_2} \int_\Omega \sqrt{u_1}\,(\sqrt{u_1}-\sqrt{u_2})\nabla V_{el1}\cdot\nabla\phi^\tau \; dx\; dt\right| \\
\le &C(\Omega)||\sqrt{u_1}||_{L^\infty([t_1,t_2];L^2)}||\nabla V_{el1}||_{L^2([t_1,t_2];L^4)}\left|\left|\frac{\sqrt{u_1}-\sqrt{u_2}}{\tau}\right|\right|_{L^2([t_1,t_2];L^4)} .
\end{aligned}
$$

Knowing that $\nabla\sqrt{u_i}\in L^2_{loc}(I;L^4(\Omega))$ (by $\sqrt{u_i}\in L^2_{loc}(I;H^2(\Omega))$, the Sobolev embedding, and $d\le 3$) and that $\nabla V_{el}\in L^2_{loc}(I;L^4(\Omega))$ (since $V_{el}\in L^2_{loc}(I;H^2(\Omega))$ by $\Delta V_{el}\in L^2_{loc}(I;L^2(\Omega))$ and the homogenous Neumann boundary condition for V_{el}), the term seen to vanish in the limit $\tau\to 0$ due to the Poincare type inequality in Lemma 27 and our condition $\sqrt{u_i}-\sqrt{u_B}\in L^2_{loc}(I;H^2_0(\Omega))\subset L^2_{loc}(I;W^{1,4}_0(\Omega))$.

In particular, for our solution we obtain precisely inequality (26). From this point on, we may proceed precisely as in the proof of Theorem 9 in case of periodic boundary data (the only integration by parts in the process is possible since homogenous Neumann boundary conditions are imposed on V_{el}). □

10. Concluding remarks and open problems

We have seen that the question of uniqueness of solutions of the DLSS equation can in fact be decoupled from the question of preservation of strict positivity, as conjectured by Jüngel and Matthes [17]. The regularity inferred from the entropy estimates for $\gamma = 1$ and $\gamma = \frac{1}{2}$ is sufficient for uniqueness.

Using our uniqueness result, we have seen that the weak solutions constructed by Jüngel and Matthes satisfy most entropy estimates which are known to hold for smooth strictly positive solutions. Furthermore we have shown existence of weak solutions of the DLSS equation with weak initial trace which satisfy all known entropy estimates and therefore only at $t = 0$ may fail to have the regularity required for uniqueness.

Finally, we have sketched how to extend our methods to cover the case of the full quantum drift-diffusion equation and how to treat certain non-periodic boundary conditions.

As a last point, we would like to mention a few selected problems which have been left open:

- It would be interesting to know whether the solutions obtained by Gianazza, Savare and Toscani using methods of optimal transport belong to our class of uniqueness.
- We currently do not know how to find a notion of solution strong enough to guarantee uniqueness of solutions with weak initial trace.
- It is not clear whether the condition for uniqueness $u^{1/2}, u^{1/4} \in L^2(I; H^2(\Omega))$ can be weakened, e.g. to $u^{1/2} \in L^2(I; H^2(\Omega))$. Note that the counterexample by Jüngel and Matthes fails to have the latter regularity.
- For $d > 1$ and Dirichlet boundary conditions for both $\sqrt{u}$ and the quantum Bohm potential $\frac{\Delta\sqrt{u}}{\sqrt{v}}$, the question of existence of solutions is open as attempts to modify the entropy inequalities in a straightforward way to cover this case fail. Moreover, the question of uniqueness also remains open in this case since it is not clear how the mollification arguments could be extended; the cutoff argument used for Dirichlet-Neumann boundary conditions is obviously inappropriate here.

Appendix A

We provide a sketch of the formal computations leading to an entropy estimate for the DLSS equation in case of nonhomogeneous Dirichlet-Neumann boundary data, at least for sufficiently smooth strictly positive boundary data u_B and sufficiently smooth domains Ω. Formally inserting $\log u - \log u_B$ into the weak formulation of the equation, we obtain

$$\frac{d}{dt}\int_\Omega u(\log u - 1) - u\log u_B \; dx + \int_\Omega \frac{u}{u_B}\frac{d}{dt}u_B \; dx =$$
$$-\int_\Omega (\sqrt{u}D^2\sqrt{u} - \nabla\sqrt{u}\otimes\nabla\sqrt{u}) : \left(\frac{2}{\sqrt{u}}D^2\sqrt{u} - \frac{2}{u}\nabla\sqrt{u}\otimes\nabla\sqrt{u} - D^2\log u_B\right) \; dx \; .$$

The derivation of entropies for the DLSS equation by Jüngel and Matthes in [18] now proceeds by adding a constant multiple of the following expressions (note that both expressions are zero) to the right-hand side of the previous equation:

$$\int_\Omega \operatorname{div}\left(\frac{1}{\sqrt{u}}|\nabla\sqrt{u}|^2\nabla\sqrt{u}\right) \; dx - \int_{\partial\Omega} \frac{1}{\sqrt{u}}|\nabla\sqrt{u}|^2\nabla\sqrt{u}\cdot\vec{n} \; d\mathcal{H}^{d-1} \tag{34}$$

and

$$\int_\Omega \operatorname{div}\left(D^2\sqrt{u}\nabla\sqrt{u} - \Delta\sqrt{u}\nabla\sqrt{u}\right) \; dx$$
$$- \int_{\partial\Omega} \vec{n}\cdot D^2\sqrt{u}\cdot\nabla\sqrt{u} - \Delta\sqrt{u}\;\vec{n}\cdot\nabla\sqrt{u} \; d\mathcal{H}^{d-1} \; . \tag{35}$$

Note that the boundary terms contained in both expressions vanish in the case of periodic boundary conditions. In case of sufficiently smooth strictly positive Dirichlet-Neumann boundary data, the boundary terms are still well-behaved:

- The boundary term in (34) obviously only depends on the boundary data, not on the solution (as the tangential components of ∇u on $\partial\Omega$ coincide with the tangential components of ∇u_B on $\partial\Omega$; the normal component of ∇u on $\partial\Omega$ is prescribed to match the normal component of ∇u_B by the Neumann boundary condition).
- The boundary term in (35) also only depends on the boundary data u_B. To see this, the key observation is that the terms involving second derivatives of $\sqrt{u}$ in direction perpendicular to $\partial\Omega$ cancel.

  To be specific, choose an orthonormal base $\vec{t_i}$ of the tangent space of $\partial\Omega$ at a fixed point $x \in \partial\Omega$. We rewrite $\vec{n} \cdot D^2\sqrt{u}$ as $(\vec{n} \cdot D^2\sqrt{u} \cdot \vec{n})\vec{n} + \sum_i(\vec{n} \cdot D^2\sqrt{u} \cdot \vec{t_i})\vec{t_i}$ and $\Delta\sqrt{u}$ as $\vec{n} \cdot D^2\sqrt{u} \cdot \vec{n} + \sum_i \vec{t_i} \cdot D^2\sqrt{u} \cdot \vec{t_i}$.

  We then see that the terms involving the second spatial derivative of u in direction perpendicular to $\partial\Omega$ (i.e. the terms involving the factor $\vec{n} \cdot D^2\sqrt{u} \cdot \vec{n}$) in the boundary integral cancel; more precisely, the integrand at x becomes

$$\sum_i (\vec{n} \cdot D^2\sqrt{u} \cdot \vec{t_i})\vec{t_i} \cdot \nabla\sqrt{u} - \sum_i (\vec{t_i} \cdot D^2\sqrt{u} \cdot \vec{t_i})\vec{n} \cdot \nabla\sqrt{u} \ .$$

  Terms of the form $\vec{t_i} \cdot D^2\sqrt{u} \cdot \vec{t_i}$ can be expressed in terms of derivatives of $u|_{\partial\Omega}$ and in terms of $\vec{n} \cdot \nabla u|_{\partial\Omega}$ only; finally terms of the form $\vec{n} \cdot D^2\sqrt{u} \cdot \vec{t_i}$ can be expressed in terms of derivatives of $u|_{\partial\Omega}$ and $\vec{n} \cdot \nabla u|_{\partial\Omega}$.

  Since we have $u|_{\partial\Omega} = u_B|_{\partial\Omega}$ and $\vec{n} \cdot \nabla u|_{\partial\Omega} = \vec{n} \cdot \nabla u_B|_{\partial\Omega}$, the boundary integral thus only depends upon u_B.

Therefore the procedure by Jüngel and Matthes can (at least formally) be carried out, only resulting in some additional inhomogeneities on the right-hand side. The terms involving products of u and u_B can be dealt with using an absorption argument and Gronwall's lemma (if u_B is strictly positive and regular enough). The entropy estimate for $\gamma = \frac{1}{2}$ is derived in a similar fashion, testing the equation formally with $\frac{1}{\sqrt{u}} - \frac{1}{\sqrt{u_B}}$.

Appendix B

Proof of Lemma 10. For smooth stricly positive u we obtain integrating by parts and using Hölder's inequality

$$\begin{aligned}\frac{1}{3}\int_{[S^1]^d} u^{-2}|\partial_i u|^4 \ dx &= \int_{[S^1]^d} u^{-1}|\partial_i u|^2 \partial_i\partial_i u \ dx \\ &\leq \left(\int_{[S^1]^d} u^{-2}|\partial_i u|^4 \ dx\right)^{1/2} \left(\int_{[S^1]^d} |\partial_i\partial_i u|^2 \ dx\right)^{1/2}\end{aligned}$$

which gives

$$\int_{[S^1]^d} |\nabla u^{1/2}|^4 \ dx \leq C(d) \sum_{i=1}^d \int_{[S^1]^d} |\partial_i\partial_i u|^2 \ dx \leq C(d) \int_{[S^1]^d} |D^2 u|^2 \ dx \ .$$

Thus, for smooth strictly positive u the result follows using the property of the Laplacian $\int \left|D^2 u\right|^2 \ dx = \int |\Delta u|^2 \ dx$ (which follows integrating by parts twice).

For general stricly positive $u \in H^2([S^1]^d)$ the result follows by approximation (mollification of u). For nonnegative $u \in H^2([S^1]^d)$, we replace u by $u + \epsilon$ and pass to the limit $\epsilon \to 0$. $\square$

Proof of Lemma 11. We see that $u^{1/4} \in W^{1,4}([S^1]^d)$ (by Lemma 10). If $\sqrt{u}$ is smooth, we obtain

$$\partial_i\partial_j\sqrt{u+\epsilon} = \frac{\sqrt{u}}{\sqrt{u+\epsilon}}\partial_i\partial_j\sqrt{u} + 4\frac{\epsilon\sqrt{u}}{\sqrt{u+\epsilon}^3}\partial_i u^{1/4}\partial_j u^{1/4} \ .$$

Considering $\rho_\delta * \sqrt{u}$ and passing to the limit $\delta \to 0$, we see that the previous equation holds true for any nonnegative u with $\sqrt{u} \in H^2([S^1]^d)$ (note that we may pass to the limit in the last term on the right-hand side using the previous lemma). We see that the last term on the right-hand side tends to zero a.e. as $\epsilon \to 0$. By dominated convergence we see that it converges to zero in $L^2([S^1]^d)$ as $\epsilon \to 0$.

On the other hand, the first term on the right-hand side converges to $\chi_{u>0}\partial_i\partial_j\sqrt{u}$ pointwise a.e.; by dominated convergence it is seen to converge in $L^2([S^1]^d)$. This implies that $\partial_i\partial_j\sqrt{u} \equiv 0$ a.e. on $\{u = 0\}$: otherwise, we would obtain the estimate $\liminf_{\epsilon\to 0} ||\partial_i\partial_j\sqrt{u+\epsilon}||_{L^2} = \liminf_{\epsilon\to 0} ||\frac{\sqrt{u}}{\sqrt{u+\epsilon}}\partial_i\partial_j\sqrt{u}||_{L^2} = ||\chi_{u>0}\partial_i\partial_j\sqrt{u}||_{L^2} < ||\partial_i\partial_j\sqrt{u}||_{L^2}$ which clearly is in conflict with the lower semicontinuity of the L^2 norm with respect to convergence in the sense of distributions.

Thus, $\chi_{u>0}\partial_i\partial_j\sqrt{u} = \partial_i\partial_j\sqrt{u}$ a.e.. This finishes the proof of the first part of our lemma.

Regarding the second part, assume that u is nonnegative and that $u^{1/4}$ is smooth. We then calculate

$$\partial_i\partial_j(u+\epsilon)^{1/4} = \frac{u^{3/4}}{(u+\epsilon)^{3/4}}\partial_i\partial_j u^{1/4} + 12\frac{\epsilon u^{1/2}}{(u+\epsilon)^{3/2}}\partial_i u^{1/8}\partial_j u^{1/8} \ .$$

This equation holds for general $u^{1/4} \in H^2([S^1]^d)$, as seen by considering $\rho_\delta * u^{1/4}$ and passing to the limit $\delta \to 0$ (using again the previous lemma). Finally, an argument analogous to the proof of the first part of the lemma yields the desired convergence. □

Proof of Lemma 12. We estimate

$$\begin{aligned}||\rho_\delta * f_\delta - f||_{L^p} \leq& ||\rho_\delta * (f_\delta - f)||_{L^p} + ||\rho_\delta * f - f||_{L^p}\\ \leq& ||f_\delta - f||_{L^p} + ||\rho_\delta * f - f||_{L^p} \ ,\end{aligned}$$

where in the second step we have used the fact that mollification does not increase the L^p norms. Passing to the limit $\delta \to 0$, we obtain the desired result. □

Proof of Lemma 13. For $\xi \in C_c^\infty(\Omega_\delta \times (0,\infty))$, we calculate

$$\begin{aligned}&\int_0^\infty \int_\Omega (\rho_\delta * u(.,t))(x)\frac{d}{dt}\xi(x,t)\ dx\ dt\\ =&\int_0^\infty \int_\Omega \int_\Omega \rho_\delta(x-y)u(y,t)\frac{d}{dt}\xi(x,t)\ dy\ dx\ dt\\ =&\int_0^\infty \int_\Omega u(x,t)\left(\rho_\delta * \frac{d}{dt}\xi(.,t)\right)(x)\ dx\ dt\\ =&\int_0^\infty \int_\Omega u(x,t)\frac{d}{dt}(\rho_\delta * \xi)(x,t)\ dx\ dt\\ =&-\int_0^\infty \langle u_t, \rho_\delta * \xi\rangle\ dt \ ,\end{aligned}$$

where we have used the symmetry of ρ_δ. Thus $\rho_\delta * u$ is weakly differentiable with respect to time and the stated representation of $(\rho_\delta * u)_t$ holds. As the mollification of a distribution is smooth and as we have

$$\int_0^\infty \langle u_t, \rho_\delta * \xi \rangle \ dt = \int_0^\infty \langle \rho_\delta * u_t, \xi \rangle \ dt \ ,$$

we see that $\rho_\delta * u \in W^{1,1}_{loc}(I; C^2(\Omega_\delta))$. ☐

Proof of Lemma 14 b,c,d. The proof of assertion b) is similar to the proof of assertion a): we first rewrite

$$\begin{aligned}
&D^2(\rho_\delta * u + \epsilon)^{1/2} \\
=&\frac{1}{2}(\rho_\delta * u + \epsilon)^{-1/2} D^2(\rho_\delta * u + \epsilon) - 4\nabla \left[(\rho_\delta * u + \epsilon)^{1/4}\right] \otimes \nabla \left[(\rho_\delta * u + \epsilon)^{1/4}\right]
\end{aligned}$$

and notice that it only remains to deal with the first term on the right-hand side, since convergence of the last term has already been established. We see that

$$\begin{aligned}
&\left|D^2(\rho_\delta * u + \epsilon)\right|(x,t) = \left|\int \rho_\delta(x-y) D^2 u(y,t) \ dy\right| \\
=&\left|\int \rho_\delta(x-y)\left[2\sqrt{u}(y,t) D^2\sqrt{u}(y,t) + 8\sqrt{u}(y,t)\nabla u^{1/4}(y,t) \otimes \nabla u^{1/4}(y,t)\right] \ dy\right| \\
\le&\left(\int \rho_\delta(x-y)\cdot C \cdot \left(|D^2 u^{1/2}| + |\nabla u^{1/4}|^2\right)^2 (y,t) \ dy\right)^{1/2} \\
&\cdot\left(\int \rho_\delta(x-y) u(y,t) \ dy\right)^{1/2} \ .
\end{aligned}$$

Defining

$$\begin{aligned}
S^\delta_\tau(t) := \Bigg\{ x : \Big| &(u(.,t)+\epsilon)^{-1/2} D^2(u(.,t)+\epsilon) \\
&- (\rho_\delta * u(.,t) + \epsilon)^{-1/2} D^2(\rho_\delta * u(.,t) + \epsilon)\Big| > \tau \Bigg\} \ ,
\end{aligned}$$

we get

$$\begin{aligned}
&\int \chi_{S^\delta_\tau(t)}(x) \left|(\rho_\delta * u + \epsilon)^{-1/2} D^2(\rho_\delta * u + \epsilon)\right|^2 \ dx \\
\le&\int \chi_{S^\delta_\tau(t)}(x) \int \rho_\delta(x-y)\cdot C\cdot\left(|D^2 u^{1/2}| + |\nabla u^{1/4}|^2\right)^2 (y,t) \ dy \ dx \\
=&C \int \left(|D^2 u^{1/2}| + |\nabla u^{1/4}|^2\right)^2 (y,t) \int \chi_{S^\delta_\tau(t)}(x) \rho_\delta(x-y) \ dx \ dy \ .
\end{aligned}$$

Now arguments analogous to the first case lead to the prove of the second assertion.

The proof of assertion c) is similar: we estimate

$$\begin{aligned}
&|\rho_\delta * \left(\sqrt{u}\partial_i\partial_i\sqrt{u}\right)|(x,t) = \int \rho_\delta(x-y)\left(\sqrt{u}\partial_i\partial_i\sqrt{u}\right)(y,t) \ dy \\
\le&\left(\int u(y,t)\rho_\delta(x-y) \ dy\right)^{1/2}\left(\int |\partial_i\partial_i\sqrt{u}(y,t)|^2 \rho_\delta(x-y) \ dy\right)^{1/2} \\
\le&\left(\int u(y,t)\rho_\delta(x-y) \ dy + \epsilon\right)^{1/2}\left(\int |\partial_i\partial_i\sqrt{u}(y,t)|^2 \rho_\delta(x-y) \ dy\right)^{1/2} \ .
\end{aligned}$$

Defining $S_\tau^\delta(t)$ analogous to the definition in the proof of statements a) and b), we see that

$$
\begin{aligned}
&\int \chi_{S_\tau^\delta(t)}(x) \left| \frac{1}{\sqrt{\rho_\delta * u + \epsilon}} \left(\rho_\delta * \left(\sqrt{u}\partial_i\partial_i\sqrt{u}\right)\right)\right|^2 (x,t)\ dx \\
\leq &\int \chi_{S_\tau^\delta(t)}(x) \int \rho_\delta(x-y)|\partial_i\partial_i\sqrt{u}|^2(y,t)\ dy\ dx \\
= &\int |\partial_i\partial_i\sqrt{u}|^2(y,t) \int \chi_{S_\tau^\delta(t)}(x)\rho_\delta(x-y)\ dx\ dy\ .
\end{aligned}
$$

Again using the arguments from the proof of the first assertion, the statement is proved.

Assertion d) again is proven analogously: we have

$$
\begin{aligned}
&\left|\rho_\delta * \left(u^{1/4}\partial_i\partial_i\sqrt{u}\partial_j u^{1/4}\right)\right|(x,t) = \left|\int \rho_\delta(x-y)\left(u^{1/4}\partial_i\partial_i\sqrt{u}\partial_j u^{1/4}\right)(y,t)\ dy\right| \\
\leq &\left(\int u(y,t)\rho_\delta(x-y)\ dy\right)^{1/4} \left(\int |\partial_i\partial_i\sqrt{u}(y,t)|^{4/3}|\partial_j u^{1/4}(y,t)|^{4/3}\rho_\delta(x-y)\ dy\right)^{3/4} \\
\leq &\left(\int u(y,t)\rho_\delta(x-y)\ dy + \epsilon\right)^{1/4} \left(\int |\partial_i\partial_i\sqrt{u}(y,t)|^{4/3}|\partial_j u^{1/4}(y,t)|^{4/3}\rho_\delta(x-y)\ dy\right)^{3/4}\ .
\end{aligned}
$$

Defining $S_\tau^\delta(t)$ analogously, we obtain

$$
\begin{aligned}
&\int \chi_{S_\tau^\delta(t)}(x) \left| \frac{1}{(\rho_\delta * u + \epsilon)^{1/4}} \left(\rho_\delta * \left(u^{1/4}\partial_i\partial_i\sqrt{u}\partial_j u^{1/4}\right)\right)\right|^{4/3} (x,t)\ dx \\
\leq &\int \chi_{S_\tau^\delta(t)}(x) \int \rho_\delta(x-y)\left(|\partial_i\partial_i\sqrt{u}|^{4/3}|\partial_j u^{1/4}|^{4/3}\right)(y,t)\ dy\ dx \\
= &\int \left(|\partial_i\partial_i\sqrt{u}|^{4/3}|\partial_j u^{1/4}|^{4/3}\right)(y,t) \int \chi_{S_\tau^\delta(t)}(x)\rho_\delta(x-y)\ dx\ dy\ .
\end{aligned}
$$

Again, using arguments analogous to the above ones, the fourth assertion is shown. Note that $|\partial_i\partial_i\sqrt{u}|\cdot|\partial_j u^{1/4}| \in L^{\frac{4}{3}}(I;L^{\frac{4}{3}}(\Omega))$ by the assumptions of the lemma and Hölder's inequality. $\square$

Formula B1

$$
\begin{aligned}
&\int_\Omega \left(\sqrt{u}D^2\sqrt{u} - \nabla\sqrt{u}\otimes\nabla\sqrt{u}\right) : D^2\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\\
=&\int_\Omega \Big(- \nabla\sqrt{u}\cdot D^2\sqrt{u} - \sqrt{u}\nabla\Delta\sqrt{u}\\
&\qquad + \Delta\sqrt{u}\nabla\sqrt{u} + \nabla\sqrt{u}\cdot D^2\sqrt{u}\Big)\cdot\nabla\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\\
=&-\int_\Omega \left(\nabla\sqrt{u}\nabla\Delta\sqrt{u} + \Delta\sqrt{u}\Delta\sqrt{u}\right)\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\\
&+\int_\Omega \sqrt{u}\Delta^2\sqrt{u}\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\\
&+\int_\Omega \nabla\sqrt{u}\cdot\nabla\Delta\sqrt{u}\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\\
=&-\int_\Omega \Delta\sqrt{u}\Delta\sqrt{u}\left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\\
&+\int_\Omega \Delta\sqrt{u}\Delta\left(\sqrt{u}\ \left(\rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\right)\\
=&\dots
\end{aligned}
$$

Formula B2

$$
\begin{aligned}
&2\int_0^T\int_\Omega \psi_t\sqrt{\rho_\delta * u + \epsilon}\ dx\ dt\ + 2\int_\Omega \sqrt{\rho_\delta * u_0 + \epsilon}\ \psi(.,0)\ dx\\
=&-\int_0^T \left\langle (\rho_\delta * u)_t, \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right\rangle\ dt\\
=&-\int_0^T \left\langle u_t, \rho_\delta * \frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right\rangle\ dt\\
=&2\int_0^T\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\left(\rho_\delta * \frac{\nabla\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\ dx\ dt\\
&+\int_0^T\int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \frac{\Delta\psi}{\sqrt{\rho_\delta * u + \epsilon}}\right)\ dx\ dt\\
&-2\int_0^T\int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \left(\frac{\nabla\psi}{\rho_\delta * u + \epsilon}\cdot\nabla\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx\ dt\\
&-2\int_0^T\int_\Omega \Delta\sqrt{u}\nabla\sqrt{u}\cdot\left(\rho_\delta * \left(\frac{\psi}{\rho_\delta * u + \epsilon}\nabla\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx\ dt\\
&-\int_0^T\int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \left(\frac{\psi}{\rho_\delta * u + \epsilon}\Delta\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx\ dt\\
&+2\int_0^T\int_\Omega \Delta\sqrt{u}\sqrt{u}\ \left(\rho_\delta * \left(\frac{\psi}{\sqrt{\rho_\delta * u + \epsilon}^3}\nabla\sqrt{\rho_\delta * u + \epsilon}\cdot\nabla\sqrt{\rho_\delta * u + \epsilon}\right)\right)\ dx\ dt\\
=&\dots
\end{aligned}
$$

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