SHARP CRITERIA FOR THE WAITING TIME PHENOMENON IN SOLUTIONS TO THE THIN-FILM EQUATION

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Abstract. We establish sharp criteria for the instantaneous propagation of free boundaries in solutions to the thin-film equation. The criteria are formulated in terms of the initial distribution of mass (as opposed to previous almost-optimal results), reflecting the fact that mass is a locally conserved quantity for the thin-film equation. In the regime of weak slippage, our criteria are at the same time necessary and sufficient. The proof of our upper bounds on free boundary propagation is based on a strategy of "propagation of degeneracy" down to arbitrarily small spatial scales: We combine estimates on the local mass and estimates on energies to show that "degeneracy" on a certain space-time cylinder entails "degeneracy" on a spatially smaller space-time cylinder with the same time horizon. The derivation of our lower bounds on free boundary propagation is based on a combination of a monotone quantity and almost optimal estimates established previously by the second author with a new estimate connecting motion of mass to entropy production.

1. Introduction

1.1. The thin-film equation.

The thin-film equation

$$\partial_t u = -\nabla \cdot (u^n \nabla \Delta u) \tag{1}$$

(with the positive real parameter $n > 0$) describes the surface-tension-driven evolution of the height $u(x,t)$ of a viscous thin liquid film on a flat surface. Like its second-order sibling, the porous medium equation

$$\partial_t u = \Delta u^m = m \nabla \cdot (u^{m-1} \nabla u)$$

(with $m > 1$; see e. g. [60] for an overview of the corresponding theory), the thin-film equation gives rise to a free boundary problem, the free boundary being the boundary of the liquid film $\partial\{u(\cdot,t) > 0\}$. The dynamics of the thin-film equation (1) is mostly of interest in the regime $n \in (0,3]$, as for $n \geq 3$ it is conjectured that the support of solutions remains constant in time. Physically, the parameter n is determined by the boundary condition for the flow at the liquid-solid interface: The case $n = 3$ corresponds to a no-slip boundary condition [57]; $n = 2$ takes into account – roughly speaking – the Navier slip condition (see [39, 45]), and various parameters $n \in (1,3)$ have been suggested to model the effects of stronger $(1 < n < 2)$ or weaker $(2 < n < 3)$ slippage [39]. The case $n = 1$ arises in the lubrication approximation of the Darcy's flow in the Hele-Shaw cell [32].

In the present work, we are interested in the qualitative behavior of the free boundary $\partial\{u(\cdot,t) > 0\}$ in the so-called case of complete wetting. Depending on

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the growth of the initial data u_0 near the free boundary, a *waiting time phenomenon* may occur: If the initial data u_0 are "flat enough" near some point x_0 on the initial free boundary – namely, if u_0 grows at most like $|x-x_0|^{4/n}$ near x_0 – , the free boundary will locally remain stationary (or at most move backward) for some time before it finally starts moving forward (see [16, 43, 29]). The amount of time that passes before the free boundary moves beyond its initial location is called the *waiting time*. On the other hand, in the regime of weak slippage $n \in (2,3)$, it is known that the free boundary will start moving forward instantaneously if the initial data u_0 grow steeper than $|x-x_0|^{4/n}$ near the initial free boundary [24, 25]; in the case $n=2$, a similar result holds up to a logarithmic correction term. The restriction $n \geq 2$ in the results of [24, 25] is optimal, as for $n<2$ the stationary state $u(x,t) = (x-x_0)_+^2$ would provide a counterexample. However, even in the regime $n \in (2,3)$ there is a small gap between the sufficient conditions for a waiting time in [16, 43, 29] and the sufficient conditions for instantaneous forward motion in [24, 25]: This gap is not in terms of the critical growth exponent $4/n$ (which is inferred from the scaling of the equation, see [16, Section 7]), but in terms of the norms used to formulate the growth condition. It is the goal of the present work to close this gap, providing a condition for the occurrence of a waiting time phenomenon for a higher-order degenerate parabolic equation which is at the same time necessary and sufficient. Even though the remaining gap is only in terms of norms and not in terms of scaling, closing it requires substantial additional ideas; see Section 1.2 below for a comparison of our new results to the previous ones in the literature, Section 2 for precise statements of our theorems, and Section 3 for a summary of the strategies employed to carry out the proofs.

In contrast to the porous medium equation, due to its fourth-order structure the thin-film equation does not give rise to a comparison principle. In the parameter range $n < \frac{3}{2}$, the support of solutions to the thin-film equation may even shrink as shown for example by the moving front solution $u(x,t) = (x - c_n t)_+^{3/n}$. Furthermore, many techniques for second-order equations – in particular from regularity theory – are not applicable to the thin-film equation. For these reasons, the analysis of the qualitative behavior of the thin-film equation – and in particular the derivation of lower bounds on free boundary propagation, first accomplished in [24, 25] – are substantially more challenging than in the case of the porous medium equation.

Due to the fourth order structure of the thin-film equation – and unlike in the case of the second-order porous medium equation – , it is also necessary to prescribe an additional boundary condition at the free boundary $\partial\{u(\cdot,t)>0\}$ (in addition to the natural boundary condition $u=0$) in order to prevent ill-posedness [2]. Energetic considerations suggest to prescribe the contact angle – that is, the slope $|\nabla u|$ – at the free boundary according to Young's law. The case of zero contact angle $|\nabla u| = 0$ is called the case of "complete wetting", while the case of a fixed positive contact angle $|\nabla u| = b > 0$ is known as the case of "partial wetting".

In the last decades, an extensive theory of weak solution concepts (see [2, 4, 6, 8, 14, 15, 42]) and strong solution concepts (see [28, 30, 31, 37, 38, 35, 36, 46]) has been developed for the case of vanishing contact angle $|\nabla u| = 0$ on $\partial \operatorname{supp} u(\cdot,t)$. However, to date no uniqueness result is known for weak solution concepts in the presence of a free boundary (except for Dirac initial data in the case $n=1$; see [53]), while the strong solution concepts are so far limited to local-in-time existence results or small perturbations of self-similar solutions or steady-states. Nevertheless, there

is a rich theory of qualitative behavior of solutions to the thin-film equation. The long-time behavior of the thin-film equation has been studied e. g. in [12, 55, 59]. Finite speed of propagation results for the free boundary have been proven in [3, 4, 8, 41, 44]. Sufficient conditions for waiting times have been established rigorously in [16, 29]. A formal analysis of the waiting time behavior has been performed in [10]. Based on the discovery of certain new monotonicity formulas, lower bounds on free boundary propagation have been proven in [24, 21, 25]. For more complex (S)PDEs of thin-film type, see for example [1, 7, 26, 27, 33], though this list is far from exhaustive.

In the case of partial wetting $|\nabla u| = b > 0$ on $\partial \operatorname{supp} u(\cdot, t)$ for some constant $b > 0$, the mathematical theory for the thin-film equation is more limited and consists mostly of some existence (and, for strong solution concepts, uniqueness) results; see [9, 56, 58] for weak solution concepts and [19, 50, 51] for strong solution concepts.

Despite the lack of a comparison principle, the thin-film equation is one of the two notable examples of a nonnegativity-preserving fourth-order equation, the other one being the Derrida-Lebowitz-Speer-Spohn equation (DLSS equation) (see e. g. [20, 18, 22, 34, 47, 48, 49]). Recall that the standard linear parabolic equation $\partial_t u = -\Delta^2 u$ fails to preserve positivity. In contrast to the thin-film equation, solutions to the DLSS equation feature infinite speed of propagation [23]. For further classes of nonnegativity-preserving higher-order parabolic equations, see e. g. [11, 52, 54].

1.2. **Informal summary of the results.** In the present work, in the parameter regime $2 < n < 3$ we provide conditions on the initial data u_0 which are both necessary and sufficient for instantaneous forward motion of the free boundary in solutions to the thin-film equation (1) in the case of zero contact angle $|\nabla u| = 0$ on $\partial \operatorname{supp} u(\cdot, t)$.

To give one example of our results, consider the one-dimensional thin-film equation $\partial_t u = -(u^n u_{xxx})_x$ with compactly supported nonnegative initial data $u_0 \in H^1(\mathbb{R})$. Denote by x_0 the leftmost point in the support of u_0. In the regime $2 < n < 3$, we prove that instantaneous forward motion of the free boundary at x_0 occurs if and only if u_0 grows faster than $(x - x_0)_+^{4/n}$ near the free boundary x_0 in the sense of "averages of the mass"

$$\limsup_{r \to 0} r^{-4/n} \fint_{(x_0, x_0+r)} u_0 \, \mathrm{d}x = \infty. \tag{2}$$

In other words, a waiting time phenomenon occurs if and only if the opposite condition

$$\limsup_{r \to 0} r^{-4/n} \fint_{(x_0, x_0+r)} u_0 \, \mathrm{d}x < \infty \tag{3}$$

holds true.

Our new results differ from the previous results in the literature as follows:

- The best previously known sufficient condition for the occurrence of a waiting time phenomenon for the thin-film equation for $n \in [2, 3)$ was

$$\limsup_{r \to 0} r^{-4/n+1} \Big(\fint_{(x_0, x_0+r)} |\nabla u_0|^2 \, \mathrm{d}x \Big)^{1/2} < \infty \tag{4}$$

as derived by Dal Passo, Giacomelli, and Grün in [16]. While for "regular" initial data like $u_0(x) = (x - x_0)_+^\beta$ near x_0 for some $\beta > 0$ the condition (4) is equivalent to our condition (2), it fails to capture cases of "irregular" initial data: For example, the oscillatory initial data

$$u_0(x) := \Big(2 + \sin \frac{1}{x - x_0}\Big)(x - x_0)_+^{4/n} \tag{5}$$

meets our new sufficient criterion for the occurrence of a waiting time (3) but fails to meet the previously known condition (4). For a plot of the example (5), see Figure 1.

- The only previous results guaranteeing instantaneous forward motion of the free boundary in solutions to the thin-film equation – as derived in a series of papers by the second author [24, 25] – required the slightly stronger condition

$$\limsup_{r \to 0} r^{-4/n} \Big(\fint_{(x_0, x_0+r)} u_0^p \, \mathrm{d}x \Big)^{1/p} = \infty \tag{6}$$

for a certain $p = p(n) \in (0, 1)$, with typically $0 < p(n) \leq \frac{1}{2}$. While again for "regular" initial data like $u_0(x) = (x - x_0)_+^\beta$ near x_0 the condition (6) is equivalent to our new condition (3), the two conditions differ in the case of "concentrated" initial data: For example, letting $\varphi : \mathbb{R} \to \mathbb{R}_0^+$ be a bump function supported in $[0, 1]$, for the initial data

$$u_0(x) := (x - x_0)_+^{4/n} + (x - x_0)_+^{4/n - \delta} \cdot \sum_{k=2}^{\infty} k^2 \varphi\Big(k^2 \Big(x - x_0 - \frac{1}{k}\Big)\Big) \tag{7}$$

(for $\delta > 0$ small enough) our new condition (2) for instantaneous forward motion of the free boundary is satisfied, but the previously known condition (6) is not. See Figure 2 for an illustration of the example (7).

- We also obtain optimal upper and lower bounds for waiting times, which are both formulated in terms of the quantity

$$\sup_{r > 0} r^{-4/n} \fint_{(x_0, x_0+r)} u_0 \, \mathrm{d}x \tag{8}$$

and differ from each other only by a constant factor.

Our sufficient criterion for a waiting time (2) is not limited to the regime $n \in (2, 3)$, but holds for the full range $n \in (1, 3)$. However, the stationary state $u(x, t) = (x - x_0)_+^2$ shows that in the regime $n < 2$ one cannot expect a condition like (3) to be sufficient for instantaneous forward motion of the free boundary, as $(x - x_0)_+^2$ grows steeper than $(x - x_0)_+^{4/n}$ in this regime. Nevertheless, the constructions in [25] show that our condition (2) is in fact sharp among all conditions formulated in terms of the growth of the initial data at the free boundary: In [25, Theorem 3] it is shown that there exist initial data with only slightly steeper growth than $(x - x_0)_+^{4/n}$ for which instantaneous forward motion occurs.

Notation. Throughout the paper, we use standard notation for Lebesgue and Sobolev spaces. For a domain $\Omega \subset \mathbb{R}^d$ we denote for $p \geq 1$ by $L^p(\Omega)$ the space of all measurable functions f with finite norm $||f||_{L^p(\Omega)} := (\int_\Omega |f|^p \, \mathrm{d}x)^{1/p}$. The Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, consists of all measurable functions $f \in L^p(\Omega)$ whose distributional derivative ∇f belongs to $L^p(\Omega)$; it is equipped with the norm $||f||_{W^{1,p}} := (\int_\Omega |f|^p + |\nabla f|^p \, \mathrm{d}x)^{1/p}$. Similarly, we define higher-order Sobolev

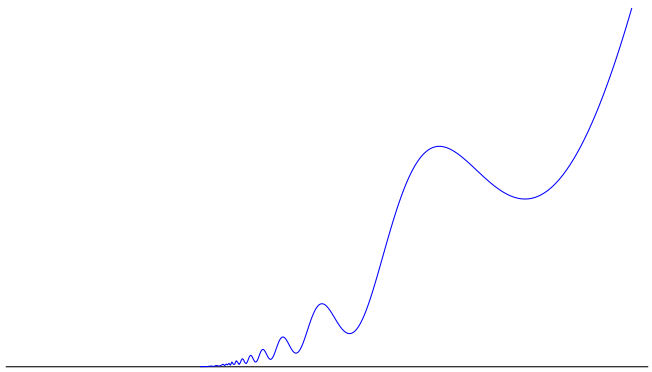

FIGURE 1. Plot of the example (5). While the initial data u_0 are clearly bounded from above and from below by a multiple of $(x - x_0)^{4/n}$, due to the rapid oscillations near the free boundary the limit (4) is infinite. As a result, the sufficient criterion for waiting times from [16] is not applicable. In contrast, our sufficient condition in Theorem 2.2 shows that for this initial data indeed a waiting time phenomenon occurs.

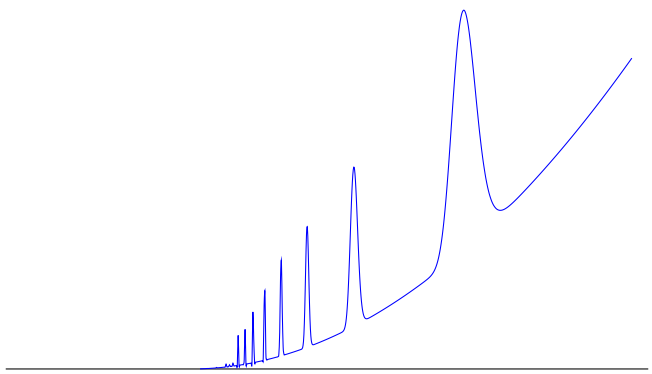

FIGURE 2. Illustration of the example (7). The initial data features infinitely many "bumps" accumulating at x_0. The "bumps" near a point $x > x_0$ have mass of order $(x - x_0)^{4/n-\delta}$ but width of order $|x - x_0|^2$. As a consequence of the mass estimate for the bumps, our sufficient condition for instantaneous forward motion of the free boundary in Theorem 2.3 is applicable. In contrast, the sufficient conditions for instantaneous forward motion from [24, 25] are not applicable for $\delta > 0$ small enough, as the increasingly strong concentration of the bumps cause the limit in (6) to be finite.

spaces $W^{k,p}(\Omega)$, $k \geq 2$, consisting of those functions in $W^{k-1,p}(\Omega)$ whose kth distributional derivatives belongs to $L^p(\Omega)$. We also use the standard abbreviation $H^k(\Omega) := W^{k,2}(\Omega)$. For a function $f : \Omega \times [0, T] \to \mathbb{R}$ depending on space and time, we denote by ∇ and Δ the (weak) gradient and the (weak) Laplacian with respect to spatial coordinates only. The (weak) time derivative of f is denoted by $\partial_t f$. As usual, for a Banach space X we denote by X' its dual. Given a Banach space X, by

$L^p([0,T];X)$ we denote the usual Lebesgue-Bochner space of strongly measurable maps $f:[0,T]\to X$ with $||f||^p_{L^p([0,T];X)} := \int_{[0,T]} |f|^p_X \,\mathrm{d}t < \infty$. By $B_r(x)$ we denote the ball of radius r around the point x.

2. Main Results

The rigorous definition of a *waiting time* which our results refer to is given as follows.

Definition 2.1. *Let $u_0 \in L^1(\mathbb{R}^d)$ and $u \in L^\infty([0,T);L^1(\mathbb{R}^d))$. For any point $x_0 \in \mathbb{R}^d \setminus \operatorname{supp} u_0$ in the complement of the support of u_0, we define the* waiting time T^* *of u at x_0 as*

$$T^* := \operatorname{essinf}\{t>0 : x_0 \in \operatorname{supp} u(\cdot,t)\},$$

where $\operatorname{supp} u(\cdot,t)$ is understood in the sense of support of a distribution.

For any point $x_0 \in \partial \operatorname{supp} u_0$ on the boundary of the initial support, we define the waiting time T^* of u at x_0 as

$$T^* := \operatorname{essinf}\{t>0 : x_0 \notin \overline{\mathbb{R}^d \setminus \operatorname{supp} u(\cdot,t)}\}.$$

In other words, for a point x_0 which lies outside of the support of the initial data, we define the waiting time T^* to be the first time at which the support of the solution u reaches x_0. For a point x_0 on the initial free boundary $\partial \operatorname{supp} u_0$, we define the waiting time to be the first time at which x_0 is contained in the interior of the support of the solution u.

We defer the (rather technical) definitions of solutions to the thin-film equation and first state our main results (Theorem 2.2 and Theorem 2.3). In the regime $n \in (1,3)$, we provide the following sufficient condition for the occurrence of a waiting time phenomenon, along with lower bounds on the waiting time.

Theorem 2.2. *Let $d \in \{1,2,3\}$ and $n \in (1,3)$. Let $u_0 \in H^1(\mathbb{R}^d)$ be compactly supported and nonnegative. In the case $n \in [2,3)$, let $u : \mathbb{R}^d \times [0,T) \to \mathbb{R}$ be an energy-dissipating weak solution to the thin-film equation (1) with zero contact angle and initial data u_0 in the sense of Definition 2.5. In the case $n \in (1,2)$, let $u : \mathbb{R}^d \times [0,T) \to \mathbb{R}$ instead be a weak solution to the thin-film equation (1) with zero contact angle and initial data u_0 in the sense of Definition 2.6, and assume that u has been constructed by the approximation procedure in [8].*

Let $x_0 \in \partial \operatorname{supp} u_0 \bigcup (\mathbb{R}^d \setminus \operatorname{supp} u_0)$ be a point on the boundary or outside of the support of the initial data. Suppose that there exists a constant $\kappa > 0$ such that for all $r > 0$ the estimate

$$\fint_{B_r(x_0)} u_0 \,\mathrm{d}x \le \kappa\, r^{\frac{4}{n}} \tag{9}$$

holds. If $x_0 \in \partial \operatorname{supp} u_0$, suppose furthermore that $\operatorname{supp} u_0$ satisfies an exterior cone condition at x_0 with some positive opening angle $\lambda > 0$.[1]

[1] As usual, we say that a closed set $U \subset \mathbb{R}^d$ satisfies an exterior cone condition at $x_0 \in \partial U$ with opening angle $\lambda > 0$ if its complement $\mathbb{R}^d \setminus U$ contains a cone C_{x_0} with tip x_0, opening angle $\lambda > 0$, and arbitrary axis and height. In the one-dimensional case, the notion of "exterior cone condition" reduces to the requirement that either $(x_0, x_0+\delta) \cap \operatorname{supp} u_0$ or $(x_0-\delta, x_0) \cap \operatorname{supp} u_0$ is empty for some $\delta > 0$ small enough, and the notion of "opening angle" becomes irrelevant.

Then u has a positive waiting time T^* at x_0 (in the sense of Definition 2.1) and there exists a constant c (depending only on d, n, and possibly λ) such that the waiting time T^* is bounded from below by

$$T^* \geq c\,\kappa^{-n}.$$

In the regime of strong slippage $n \in (1,2)$, our preceding sufficient condition for a waiting time phenomenon is not a necessary condition, as the counterexample $u(x,t) = (x-x_0)_+^2$ demonstrates. Nevertheless, the approach of [25] shows that our sufficient condition for a waiting time phenomenon is (at least in one dimension $d=1$) optimal among all conditions formulated in terms of the growth of the initial data near the free boundary.

On the other hand, in the regime $n \in (2,3)$ our preceding sufficient condition for the occurrence of a waiting time phenomenon is also a necessary condition, as our next result shows. Furthermore, the lower bounds on the waiting time in Theorem 2.2 above are optimal up to a universal constant factor.

Theorem 2.3. *Let $d=1$ and let $n \in (2,3)$. Let $u_0 \in H^1(\mathbb{R})$ be compactly supported and nonnegative. Let $u : \mathbb{R}^d \times [0,T) \to \mathbb{R}$ be an energy-dissipating weak solution to the thin-film equation (1) with zero contact angle and initial data u_0 in the sense of Definition 2.5. Let $x_0 \in \partial \operatorname{supp} u_0 \bigcup (\mathbb{R}^d \setminus \operatorname{supp} u_0)$ be a point on the boundary or outside of the support of the initial data. Then there exists a constant C (depending only on n and d) such that the waiting time T^* of u at x_0 (in the sense of Definition 2.1) is bounded from above by*

$$T^* \leq C \left(\sup_{r>0} r^{-\frac{4}{n}} \fint_{(x_0-r,x_0+r)} u_0 \, \mathrm{d}x \right)^{-n}. \tag{10}$$

In particular, if the initial data u_0 satisfy

$$\limsup_{r\to 0} r^{-\frac{4}{n}} \fint_{(x_0-r,x_0+r)} u_0 \, \mathrm{d}x = \infty$$

at a point on the initial free boundary $x_0 \in \partial \operatorname{supp} u_0$, the free boundary starts moving forward immediately at x_0, without waiting time.

Remark 2.4. In the multidimensional case $d \in \{2,3\}$, by combining the ideas of our proof of Theorem 2.1 with the approach used for the multidimensional case in [24], one could prove a similar upper bound on the waiting time for $n \in (2,3)$, namely a bound of the form

$$T^* \leq C(d,n) \Big(\limsup_{\delta\to 0} \limsup_{r\to 0} r^{-4/n} \fint_{(\partial \operatorname{supp} u_0 \cap B_\delta(x_0)) + B_r} u_0 \, \mathrm{d}x \Big)^{-n}$$

for any point $x_0 \in \partial \operatorname{supp} u_0$ near which $\partial \operatorname{supp} u_0$ is a C^4 manifold. However, due to the already substantial length of the present paper we refrain from carrying out the estimates.

Let us now state the precise definitions of solutions to the thin-film equation that our main results are concerned with. For $d \in \{2,3\}$ and the parameter range

$$n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3 \right),$$

in [42] an existence result has been proven for the following class of solutions to the thin-film equation. Earlier results of [4] show the same existence result in $d = 1$ for $n \in \left(\frac{1}{2}, 3\right)$.

Definition 2.5 (Energy-dissipating weak solutions)**.** *Let $d \in \{1, 2, 3\}$ and $n \in (2, 3)$. Let $T > 0$ and let $u_0 \in H^1(\mathbb{R}^d)$ have compact support. We call a nonnegative function $u \in L^\infty([0, T); H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$, $u \geq 0$, an* energy-dissipating weak solution of the thin-film equation with zero contact angle and initial data u_0 *if the following conditions are satisfied:*

a) *We have $\nabla u^{\frac{n+2}{6}} \in L^6(\mathbb{R}^d \times [0, T))$, $u^{\frac{n-2}{2}} \nabla u \otimes D^2 u \in L^2(\mathbb{R}^d \times [0, T))$, and $\chi_{\{u>0\}} u^{\frac{n}{2}} \nabla \Delta u \in L^2(\mathbb{R}^d \times [0, T))$.*
b) *For all $\alpha \in (\max\left\{-1, \frac{1}{2} - n\right\}, 2 - n) \setminus \{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2(\mathbb{R}^d \times [0, T))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4(\mathbb{R}^d \times [0, T))$.*
c) *It holds that $u \in H^1_{loc}([0, T); (W^{1,p}(\mathbb{R}^d))')$ for all $p > \frac{4d}{2d+n(2-d)}$.*
d) *For any $\psi \in L^2([0, T), W^{1,\infty}(\mathbb{R}^d))$ and any $T > 0$, we have*

$$\int_0^T \langle \partial_t u, \psi \rangle_{(W^{1,p}(\mathbb{R}^d))' \times W^{1,p}(\mathbb{R}^d)} \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^n \nabla \Delta u \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t.$$

e) *u attains its initial data u_0 in the sense $\lim_{t \to 0} u(\cdot, t) = u_0(\cdot)$ in $L^1(\mathbb{R}^d)$.*

In the parameter range $n \in (1, 2)$, we need to resort to a different solution concept, at least in case $d \in \{2, 3\}$, as in this case the existence of energy-dissipating weak solutions is unknown.

Definition 2.6 (Weak solutions)**.** *Let $d \in \{1, 2, 3\}$ and $n \in (\frac{1}{8}, 2)$. Let $T > 0$ and let $u_0 \in H^1(\mathbb{R}^d)$ have compact support. We say that a nonnegative function $u \in L^\infty([0, T); H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$ is a* weak solution of (1) with zero contact angle and initial data u_0 *if the following conditions are satisfied:*

a) *$u \in H^1_{loc}\left([0, T); (W^{1,p}(\mathbb{R}^d))'\right)$ for all $p > \frac{4d}{2d+n(2-d)}$;*
b) *For any $\alpha \in (\max\left\{-1, \frac{1}{2} - n\right\}, 2 - n) \setminus \{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2(\mathbb{R}^d \times [0, T))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4(\mathbb{R}^d \times [0, T))$.*
c) *for any $\psi \in L^\infty([0, T); C^3_c(\mathbb{R}^d))$ we have for any $T > 0$*

$$\begin{aligned}
&\int_0^T \langle \partial_t u, \psi \rangle_{(W^{1,p}(\Omega))' \times W^{1,p}(\Omega)} \, \mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^n \nabla u \cdot \nabla \Delta \psi \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + n \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^{n-1} \nabla u \cdot D^2 \psi \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \frac{n}{2} \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^{n-1} |\nabla u|^2 \Delta \psi \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \frac{n(n-1)}{2} \int_0^T \int_{\mathbb{R}^d \cap \{u>0\}} u^{n-2} |\nabla u|^2 \nabla u \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

d) *u attains its initial data u_0 in the sense $\lim_{t \to 0} u(\cdot, t) = u_0(\cdot)$ in $L^1(\mathbb{R}^d)$.*

3. Strategies for the proofs of the main results

3.1. **Strategy for the lower bounds on free boundary propagation.** Our argument for the lower bounds on free boundary propagation for the thin-film equation relies in parts crucially on the results and strategies of the previous works by the second author [24, 25]. In the particular case of one dimension $d = 1$, the key results of [24, 25] may be summarized as follows: For $n \in (2, 3)$, for any point x_0 on the boundary or outside of the support of the initial data u_0 the waiting time is bounded from above by

$$T^* \leq C\Big(\sup_{r>0} r^{-4/n}\Big(\fint_{(x_0-r,x_0+r)} u_0^p \,\mathrm{d}x\Big)^{1/p}\Big)^n,$$

where $C > 0$ and $p \in (0, 1)$ depend only on n. The results of [24, 25] are based on the discovery of certain new monotonicity formulas for solutions to the thin-film equation, taking the form of a weighted entropy inequality

$$\partial_t \int_{\mathbb{R}} u^{1+\alpha}|x - x_0|^\gamma \,\mathrm{d}x \geq c \int_{\mathbb{R}} u^{1+\alpha+n}|x - x_0|^{\gamma-4} + |\nabla u^{\frac{1+\alpha+n}{4}}|^4|x - x_0|^\gamma \,\mathrm{d}x \tag{11}$$

and being valid for suitable $-1 < \alpha < 0$ and suitable $\gamma < -1$, as long as the support of the solution $u(\cdot, t)$ does not touch the singularity of the weight at x_0. The monotonicity formula enables one to apply a differential inequality argument due to Chipot and Sideris [13]: Suppose, for the sake of simplicity, that x_0 is the leftmost point in the support of the solution. Using Hölder's inequality and assuming that the support of u remains to the right of x_0, one obtains from the monotonicity formula applied with $x_0 - \delta$ in place of x_0

$$\begin{aligned}&\partial_t \int_{\mathbb{R}} u^{1+\alpha}|x - x_0 + \delta|^\gamma \,\mathrm{d}x\\ &\geq c\delta^{-\frac{(\gamma+1)n}{(1+\alpha)}-4}\Big(\int_{\mathbb{R}} u^{1+\alpha}|x - x_0 + \delta|^\gamma \,\mathrm{d}x\Big)^{\frac{1+\alpha+n}{1+\alpha}}.\end{aligned}$$

This implies finite-time blowup of $\int_{\mathbb{R}} u^{1+\alpha}(\cdot, t)|x - x_0 + \delta|^\gamma \,\mathrm{d}x$ and thereby a contradiction to the assumption that the support of $u(\cdot, T)$ remains to the right of x_0 as soon as

$$T \geq C\delta^{\frac{(1+\gamma)n}{(1+\alpha)}+4}\Big(\int_{\mathbb{R}} u_0^{1+\alpha}|x - x_0 + \delta|^\gamma \,\mathrm{d}x\Big)^{-\frac{n}{(1+\alpha)}},$$

so, in particular, as soon as

$$T \geq C\Big(\delta^{-4(1+\alpha)/n}\fint_{(x_0,x_0+\delta)} u_0^{1+\alpha} \,\mathrm{d}x\Big)^{-n/(1+\alpha)}.$$

The problem for "concentrated" initial data like (7) is that the integral on the right-hand side of the previous formula is much smaller than suggested by the relation

$$\fint_{(x_0,x_0+\delta)} u_0^{1+\alpha} \,\mathrm{d}x \sim \Big(\fint_{(x_0,x_0+\delta)} u_0 \,\mathrm{d}x\Big)^{1+\alpha}$$

which would be valid for initial data like $u_0(x) \sim (x - x_0)_+^\beta$.

The proof of our sharp sufficient condition (2) for instantaneous forward motion of the free boundary is based on the following idea: If initially some amount of mass is present in the interval $(x_0, x_0 + \delta)$, then there are basically two options – either at least half of the mass remains near the interval $(x_0, x_0 + \delta)$ up until

at least time $T/2$, or at least half of the mass "escapes" from the vicinity of the interval before time $T/2$. In the former case, the monotonicity formula (11) entails a lower bound on $\int_{\mathbb{R}} u^{1+\alpha}(x,T/2)|x-x_0|^\gamma \,\mathrm{d}x$ by a simple application of Hölder's inequality, and it turns out that this lower bound is sufficient for the derivation of our result. In the latter case, a combination of the monotonicity formula (11) with a careful estimate based on testing the PDE (1) with a suitable smooth cutoff shows that motion of mass entails entropy production, again yielding a lower bound for $\int_{\mathbb{R}} u^{1+\alpha}(x,T/2)|x-x_0|^\gamma \,\mathrm{d}x$. In both cases, we then use the estimates of [24, 25], starting at time $t_0 = T/2$ instead of $t_0 = 0$, to conclude. The full argument is provided in Section 4.

3.2. **Strategy for the upper bounds on free boundary propagation.** Our strategy for the derivation of upper bounds on free boundary propagation is based on the following concept: In the regime $n \in [2,3)$, we say that a solution to the thin-film equation u is *degenerate* on a parabolic cylinder $B_r(x_0)\times[0,T]$ if it satisfies both

$$\sup_{t\in(0,T)} \fint_{B_r(x_0)} u \,\mathrm{d}x \leq \varepsilon T^{-1/n} r^{4/n} \tag{12a}$$

and

$$\sup_{t\in(0,T)} \fint_{B_r(x_0)} \frac{t^\beta}{T^\beta}|\nabla u|^2 \,\mathrm{d}x + \int_0^T \fint_{B_r(x_0)} \frac{t^\beta}{T^\beta}\big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n|\nabla\Delta u|^2\big) \,\mathrm{d}x \,\mathrm{d}t \leq \varepsilon^\delta T^{-2/n}(r^{4/n-1})^2 \tag{12b}$$

for some appropriately chosen constants $\varepsilon = \varepsilon(d,n) > 0$ and $\delta = \delta(d,n) > 0$ and some suitably chosen $\beta \in (0,1)$. In the regime $n \in (1,2)$, we use a closely related ansatz, which replaces the degeneracy condition in terms of the energy (12b) by a corresponding condition in terms of a (localized) entropy, see (23) below for details. In the remainder of this exposition, we shall focus only on the case $n \in [2,3)$.

The central idea of our proof is to show that – provided that the initial data also satisfy a degeneracy condition of the type (3) – the degeneracy of u on a parabolic cylinder $B_r(x_0)\times[0,T]$ implies the degeneracy of u on the spatially smaller parabolic cylinder $B_{r/2}(x_0)\times[0,T]$ with the same time horizon T. Propagating the degeneracy down to $r \to 0$, this essentially shows $u(x_0,t) = 0$ for $t \leq T$.

The general spirit of the proof is inspired by the approach of [16, 43, 29], one difference being that in our formulation the iteration à la Stampacchia present in [16, 43, 29] is done essentially explicitly by the propagation of degeneracy. However, the key difference of our approach to [16, 43, 29] is that the latter is formulated in terms of the local energy only and does not keep track of the propagation of mass. This substantially simplifies the estimates, but comes at the cost of formulating the degeneracy condition on the initial data in terms of the local energy $\fint_{B_r(x_0)} |\nabla u_0|^2 \,\mathrm{d}x$, making it impossible to derive an optimal result. By keeping track of the propagation of mass via (12a), we are able to eliminate the dependence on the initial energy by introducing a weight $(t/T)^\beta$ in the degeneracy condition for the energy (12b).

The rough idea for the propagation of the first degeneracy condition (12a) is the following: Starting with degenerate initial data u_0 (in the sense that the quantity (8) is finite), after choosing T appropriately (depending on the size of the quantity (8)) it suffices to control the possible influx of mass u into the smaller ball $B_{r/2}(x_0)$ up to

time T. The degeneracy properties (12a) and (12b) on a spatially larger parabolic cylinder in turn ensure that the influx of mass into the smaller ball $B_{r/2}(x_0)$ remains sufficiently limited up to time T; to see this, we test the PDE (1) with a weight and estimate the right-hand side carefully.

In order to propagate the second degeneracy condition (12b) which involves the energy, we cannot rely on the energy of the initial data, as the localized H^1 norms of the initial data do not need to reflect the degeneracy of the initial data near x_0 (recall for instance the counterexample (5)). We instead rely on the regularization properties of the nonlinear fourth-order parabolic operator, reducing the problem to an estimate on the local mass. This idea is close in spirit to the consideration (for the thin-film equation on a bounded domain Ω)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2\,\mathrm{d}x \le -c\int_\Omega |\nabla u^{\frac{n+2}{6}}|^6\,\mathrm{d}x \le -c(\Omega)\Big(\int_\Omega u\,\mathrm{d}x\Big)^{n-4}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^3$$

where the first step is just the energy dissipation property, combined with Bernis-Grün's inequality, and the second step is a simple application of Hölder's inequality. This estimate implies by an elementary ODE argument a bound of the form

$$\int_\Omega |\nabla u(\cdot,t)|^2\,\mathrm{d}x \le C(\Omega)t^{-1/2}\Big(\sup_{s\in[0,t]}\int_\Omega u(\cdot,s)\,\mathrm{d}x\Big)^{2-n/2}, \tag{13}$$

which is now independent of $\int_\Omega |\nabla u_0|^2\,\mathrm{d}x$, but blows up for $t\to 0$. Note that the blowup near initial time is the reason for our choice of including the factor t^β/T^β in our condition (12b). A result of the type (13) has been the basis for the existence theory for the thin film equation with measure-valued initial data in [14].

As for our purposes a global estimate on the energy in terms of the mass like (13) is insufficient – we rather need to estimate a localized energy – , to show the degeneracy condition for the energy (12b) on the spatially smaller parabolic cylinder we additionally need to control the influx of energy into the smaller ball $B_{r/2}(x_0)$ suitably. It turns our that the latter may be achieved using the control on mass and energy provided by the assumptions (12a) and (12b) on the bigger cylinder. In total, we obtain the degeneracy (12b) on the smaller cylinder $B_{r/2}(x_0)\times[0,T]$ as a result of the degeneracies (12a) and (12b) on the bigger cylinder $B_r(x_0)\times[0,T]$.

4. Proof of the necessary condition for the waiting time phenomenon

We now provide the proof of the sharp sufficient criteria for instantaneous forward motion of the free boundary stated in Theorem 2.3.

Proof of Theorem 2.3. Let $T^* := \inf\{t\ge 0 : \operatorname{supp} u(\cdot,t)\cap(-\infty,x_0) = \emptyset\}$. Fix $r>0$ and let $\psi_r:\mathbb{R}^d\to\mathbb{R}$ be a non-increasing function supported in $B_r(x_0)$ such that $0\le\psi_r\le 1$, $\psi_r\equiv 1$ on $B_{r/2}(x_0)$, and $|\nabla\psi_r|\le Cr^{-1}$, $|D^2\psi_r|\le Cr^{-2}$, $|D^3\psi_r|\le Cr^{-3}$, $|D^4\psi_r|\le Cr^{-4}$. Let $\varphi=\psi_r^k$, with k to be chosen later large enough. In the proof of the theorem, we shall distinguish two cases:

1. $\displaystyle\int_{B_r(x_0)} u(x,t)\,\varphi(x)\,\mathrm{d}x \ \ge\ \frac{1}{2}\int_{B_r(x_0)} u_0\varphi\,\mathrm{d}x$ for all $t\in(0,T^*/2)$;
2. $\displaystyle\int_{B_r(x_0)} u(x,t)\,\varphi(x)\,\mathrm{d}x \ \le\ \frac{1}{2}\int_{B_r(x_0)} u_0\varphi\,\mathrm{d}x$ for some $t\in(0,T^*/2)$.

Case 1. By applying either [24, Theorem 1] or [25, Theorem 3] – depending on the value of $n \in (2,3)$ – starting at time $t_0 = T^*/2$ instead of $t_0 = 0$, we obtain

$$\left(T^* - \frac{T^*}{2}\right) \leq C r^{4+\frac{n}{\alpha+1}(1+\gamma)} \left(\int_{\mathbb{R}} u^{\alpha+1}\left(x, T^*/2\right) |x - x_0 + r|^{\gamma} \, \mathrm{d}x\right)^{-\frac{n}{\alpha+1}}$$

for certain suitable $-1 < \alpha < 0$ and $\gamma < -1$. This implies by (11)

$$\begin{aligned} r^{-\frac{4}{n}} \Bigg(\fint_{B_r(x_0)} u^{\alpha+1}(x, T^*/2) \, \mathrm{d}x \\ + \int_0^{T^*/2} \fint_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \, \mathrm{d}x \, \mathrm{d}t \Bigg)^{\frac{1}{\alpha+1}} \leq C T^{*-\frac{1}{n}} \end{aligned} \tag{14}$$

and thus in particular

$$T^{*-\frac{1}{n}} \geq C r^{-\frac{4}{n}} \left(\int_0^{T^*/2} \fint_{B_r(x_0)} r^{-4} u^{\alpha+n+1} \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{\alpha+1}}.$$

Jensen's inequality yields

$$T^{*-\frac{1}{n}} \geq C r^{-\frac{4}{n}} \left(r^{-4} \int_0^{T^*/2} \left(\fint_{B_r(x_0)} u \, \mathrm{d}x \right)^{\alpha+n+1} \mathrm{d}t \right)^{\frac{1}{\alpha+1}}.$$

Using the assumption

$$\int_{B_r(x_0)} u(\cdot, t) \varphi \, \mathrm{d}x \geq \frac{1}{2} \int_{B_r(x_0)} u_0 \varphi \, \mathrm{d}x \quad \text{for} \quad t \in (0, T^*/2),$$

we obtain

$$T^{*-\frac{1}{n}} \geq C r^{-\frac{4}{n}} \, T^{*\frac{1}{\alpha+1}} r^{-\frac{4}{\alpha+1}} \left(\fint_{B_r(x_0)} u_0 \varphi \, \mathrm{d}x \right)^{\frac{\alpha+n+1}{\alpha+1}}.$$

This implies

$$T^{*-\frac{\alpha+n+1}{n(\alpha+1)}} \geq C r^{-\frac{4(1+\alpha+n)}{n(\alpha+1)}} \left(\fint_{B_r(x_0)} u_0 \varphi \, \mathrm{d}x \right)^{\frac{\alpha+n+1}{\alpha+1}},$$

which directly yields the desired estimate

$$T^* \leq C \left(r^{-\frac{4}{n}} \fint_{B_r(x_0)} u_0 \, \mathrm{d}x \right)^{-n}.$$

Case 2. For a smooth cut-off function φ and any $T \in (0, T^*/2)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(x,T)\varphi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} u^n \nabla\Delta u \cdot \nabla\varphi \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x - \int_0^T \int_{\mathbb{R}^d} u^n D^2 u : D^2\varphi + n u^{n-1}\nabla u \cdot D^2 u \cdot \nabla\varphi \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} u^n \nabla u \cdot \nabla\Delta\varphi + n u^{n-1}\nabla u \cdot D^2\varphi \cdot \nabla u \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \frac{n}{2} u^{n-1}|\nabla u|^2 \Delta\varphi + \frac{n(n-1)}{2} u^{n-2}|\nabla u|^2 \nabla u \cdot \nabla\varphi \,\mathrm{d}x \,\mathrm{d}t \\
&\geq \int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x - C\int_0^T \int_{B_r(x_0)} u^{\frac{n+1-3\alpha}{4}} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^3 |\nabla\varphi| \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - C\int_0^T \int_{B_r(x_0)} u^{n+1}\left(|\Delta^2\varphi| + \frac{|D^2\varphi|^3}{|\nabla\varphi|^2}\right) \,\mathrm{d}x \,\mathrm{d}t \,.
\end{aligned}$$

This implies, by Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(x,T)\varphi \,\mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x - C\int_0^T \left(\int_{B_r(x_0)} \varphi^{4-\varepsilon}\, u^{n+1-3\alpha} \,\mathrm{d}x\right)^{\frac{1}{4}} \\
&\qquad \times \left(\int_{B_r(x_0)} \frac{|\nabla\varphi|^{\frac{4}{3}}}{\varphi^{\frac{4-\varepsilon}{3}}} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 + \left(\frac{|\Delta^2\varphi|^{\frac{4}{3}}}{\varphi^{\frac{4-\varepsilon}{3}}} + \frac{|D^2\varphi|^4}{|\nabla\varphi|^{\frac{8}{3}}\varphi^{\frac{4-\varepsilon}{3}}}\right) u^{\alpha+n+1} \,\mathrm{d}x\right)^{\frac{3}{4}} \,\mathrm{d}t.
\end{aligned}$$

From Lemma 4.1, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(x,T)\varphi \,\mathrm{d}x \\
&\quad \geq \int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x - C\int_0^T \left(\int_{B_r(x_0)} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 + r^{-4}u^{\alpha+n+1} \,\mathrm{d}x\right)^{\frac{\vartheta(n+1-3\alpha)}{4(1+\alpha+n)}} \\
&\qquad \times \left(\int_{B_r(x_0)} \varphi u \,\mathrm{d}x\right)^{\frac{(1-\vartheta)(n+1-3\alpha)}{4}} \\
&\qquad \times \left(\int_{B_r(x_0)} \frac{|\nabla\varphi|^{\frac{4}{3}}}{\varphi^{\frac{4-\varepsilon}{3}}} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 + \left(\frac{|\Delta^2\varphi|^{\frac{4}{3}}}{\varphi^{\frac{4-\varepsilon}{3}}} + \frac{|D^2\varphi|^4}{|\nabla\varphi|^{\frac{8}{3}}\varphi^{\frac{4-\varepsilon}{3}}}\right) u^{\alpha+n+1} \,\mathrm{d}x\right)^{\frac{3}{4}} \,\mathrm{d}t.
\end{aligned}$$

Let $\psi_r : \mathbb{R}^d \to \mathbb{R}$ be a function supported in $B_r(x_0)$ such that $0 \leq \psi_r \leq 1$, $\psi_r \equiv 1$ on $B_{r/2}(x_0)$, and $|\nabla\psi_r| \leq Cr^{-1}$, $|D^2\psi_r| \leq Cr^{-2}$, $|D^3\psi_r| \leq Cr^{-3}$, $|D^4\psi_r| \leq Cr^{-4}$.

By choosing $\varphi := \psi_r^k$ and setting k large enough (depending on ε), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(x,T)\varphi \,\mathrm{d}x \\
&\quad \geq \int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x - Cr^{-1}\int_0^T \left(\int_{B_r(x_0)} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 + r^{-4}u^{\alpha+n+1} \,\mathrm{d}x\right)^{\frac{\vartheta(n+1-3\alpha)}{4(1+\alpha+n)}+\frac{3}{4}} \\
&\qquad\qquad \times \left(\int_{B_r(x_0)} u\varphi \,\mathrm{d}x\right)^{\frac{(1-\vartheta)(n+1-3\alpha)}{4}} \,\mathrm{d}t.
\end{aligned}$$

Since the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = q(t)\cdot [z(t)]^m$$

is given by

$$z(t) = \left(z(0)^{1-m} - (m-1)\int_0^t q(s)\,\mathrm{d}s\right)^{\frac{1}{1-m}},$$

a comparison argument yields (note that we have $(1-\vartheta)(n+1-3\alpha) < 4$ by Lemma 4.1 and we have $\vartheta(n+1-3\alpha)/(1+\alpha+n) = (n-3\alpha)/(4+\alpha+n) < 1$ by $\alpha > -1$)

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} u(x,T)\varphi \,\mathrm{d}x\right)^{\frac{4-(1-\vartheta)(n+1-3\alpha)}{4}} \\
&\quad \geq \left(\int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x\right)^{\frac{4-(1-\vartheta)(n+1-3\alpha)}{4}} - Cr^{-1}T^{\frac{1}{4}-\frac{\vartheta(n+1-3\alpha)}{4(1+\alpha+n)}} \\
&\qquad \times \left(\int_0^T\int_{B_r(x_0)} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 + r^{-4}u^{\alpha+n+1} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\vartheta(n+1-3\alpha)}{4(1+\alpha+n)}+\frac{3}{4}}.
\end{aligned}$$

By making use of (14), we infer for $T \leq T^*/2$

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} u(x,T)\varphi \,\mathrm{d}x\right)^{\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4}} \\
&\quad \geq \left(\int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x\right)^{\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4}} \\
&\qquad - CT^* r^{-1}\left(CT^{*-\frac{\alpha+n+1}{n}} r^{\frac{4(\alpha+1)+n}{n}}\right)^{\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4(1+\alpha+n)}+\frac{n}{\alpha+n+1}} \\
&\quad \geq \left(\int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x\right)^{\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4}} \\
&\qquad - CT^{*-\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4n}} \\
&\qquad\quad \times \left(r^{\frac{(4+n)(1+\alpha+n)}{n}-4-\alpha-n}\right)^{\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4(1+\alpha+n)}+\frac{n}{\alpha+n+1}} \\
&\quad \geq \left(\int_{\mathbb{R}^d} u_0\varphi \,\mathrm{d}x\right)^{\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4}} \\
&\qquad - CT^{*-\frac{3-n+3\alpha+\vartheta(n+1-3\alpha)}{4n}} r^{\frac{(4+n)(3-n+3\alpha+\vartheta(n+1-3\alpha))}{4n}},
\end{aligned}$$

i. e.

$$\int_{B_r(x_0)} u(x,T)\varphi \,\mathrm{d}x \geq \int_{B_r(x_0)} u_0\varphi \,\mathrm{d}x - CT^{*-\frac{1}{n}} r^{\frac{4}{n}+1}$$

for any $T \leq T^*/2$. Combining this lower bound with the assumption

$$\int_{B_r(x_0)} u(\cdot,t)\varphi \,\mathrm{d}x \leq \frac{1}{2}\int_{B_r(x_0)} u_0\varphi \,\mathrm{d}x \quad \text{for some } \ t \in (0,T^*/2)\,,$$

we obtain the desired estimate

$$T^* \leq \left(C r^{-\frac{4}{n}} \fint_{B_r(x_0)} u_0\varphi \,\mathrm{d}x \right)^{-n} \leq C \left(r^{-\frac{4}{n}} \fint_{B_r(x_0)} u_0 \,\mathrm{d}x \right)^{-n}.$$

□

In the proof of Theorem 2.3, we have used the following technical interpolation lemma.

Lemma 4.1. *Let $d = 1$, $n \in (2,3)$, and $\alpha \in (-1,0)$ satisfying $\alpha + n < 2$. Let $u \in L^1(\mathbb{R})$ be a nonnegative function such that $u^{\frac{\alpha+n+1}{4}} \in W^{1,4}(\mathbb{R})$. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth cut-off function which symmetric around x_0, monotone decreasing in $|x - x_0|$, and satisfies $0 \leq \varphi \leq 1$ as well as*

$$\varphi(x) = \begin{cases} 1, & x \in B_{r/2}(x_0)\,, \\ 0, & x \in \mathbb{R} \setminus B_r(x_0), \end{cases}$$

and $|\nabla\varphi| \leq C$. For any $0 < \varepsilon \ll 1$ small enough (depending only on α, n, and d), there exists a constant $C > 0$ (depending also only on α, n, and d) such that the estimate

$$\begin{aligned} &\int_{B_r(x_0)} \varphi^{4-\varepsilon} u^{n+1-3\alpha} \,\mathrm{d}x \\ &\qquad \leq C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\ &\qquad\quad \times \left(\int_{B_r(x_0)} \varphi u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)} \end{aligned}$$

holds, where $\vartheta \in (0,1)$ is given by

$$\vartheta = \frac{(\alpha+n+1)(n-3\alpha)}{(n+1-3\alpha)(4+\alpha+n)}.$$

Furthermore, ϑ satisfies $(1-\vartheta)(n+1-3\alpha) < 4$.

Proof. The Gagliardo-Nirenberg-Sobolev interpolation inequality (applied to $v := u^{\frac{\alpha+n+1}{4}}$ with $p = \frac{4(n+1-3\alpha)}{\alpha+n+1}$, $m = 4$, $q = \frac{4}{\alpha+n+1}$) implies, for $s \in (r/2, r)$,

$$\begin{aligned} \int_{B_s(x_0)} u^{n+1-3\alpha} \,\mathrm{d}x \leq C &\left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\ &\times \left(\int_{B_s(x_0)} u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)}, \end{aligned}$$

where

$$\vartheta = \frac{\frac{\alpha+n+1}{4(n+1-3\alpha)} - \frac{\alpha+n+1}{4}}{\frac{1}{4} - \frac{1}{d} - \frac{\alpha+n+1}{4}} = \frac{(\alpha+n+1)(n-3\alpha)}{(n+1-3\alpha)(4+\alpha+n)}.$$

It is immediate that $0 < \vartheta < 1$. Note also that the constant C does not depend on $s \in (r/2, r)$.

Fix $S \in (r/2, r)$; choosing $s(h) := \min\{\sup\{|x| : \varphi(x) \geq h\}, S\}$ and integrating with respect to h, we infer

$$\begin{aligned}
\int_{B_S(x_0)} \varphi u^{n+1-3\alpha} \,\mathrm{d}x &= \int_0^1 \int_{B_{s(h)}(x_0)} u^{n+1-3\alpha} \,\mathrm{d}x \,\mathrm{d}h \\
&\leq C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\
&\qquad \times \int_0^1 \left(\int_{B_{s(h)}(x_0)} u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)} \mathrm{d}h \\
&\leq C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\
&\qquad \times \left(\int_{B_S(x_0)} u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)-1} \int_0^1 \int_{B_{s(h)}(x_0)} u \,\mathrm{d}x \,\mathrm{d}h \\
&\leq C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\
&\qquad \times \left(\int_{B_S(x_0)} u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)-1} \int_{B_r(x_0)} \varphi u \,\mathrm{d}x \,.
\end{aligned}$$

Repeating this procedure, we get for $k = 2$ and subsequently $k = 3$ (as long as $(1-\vartheta)(n+1-3\alpha) > 3$)

$$\begin{aligned}
\int_{B_S(x_0)} \varphi^k u^{n+1-3\alpha} \,\mathrm{d}x &= \int_0^1 \int_{B_{s(h)}(x_0)} \varphi^{k-1} u^{n+1-3\alpha} \,\mathrm{d}x \,\mathrm{d}h \\
&\leq \int_0^1 C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\
&\qquad \times \left(\int_{B_{s(h)}(x_0)} u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)-(k-1)} \left(\int_{B_r(x_0)} \varphi u \,\mathrm{d}x \right)^{k-1} \mathrm{d}h \\
&\leq C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\
&\qquad \times \left(\int_{B_S(x_0)} u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)-k} \left(\int_{B_r(x_0)} \varphi u \,\mathrm{d}x \right)^{k} .
\end{aligned}$$

Set $\lambda := (1-\vartheta)(n+1-3\alpha)$ and $k = \lfloor \lambda \rfloor$; note that we have $1 < \lambda < 4$ and therefore $k \in \{1,2,3\}$. By making use of the estimates above, we get

$$\begin{aligned}
\int_{B_r(x_0)} \varphi^{(k+1)-\varepsilon} u^{n+1-3\alpha} \,\mathrm{d}x &= \int_0^r |\nabla \varphi^{1-\varepsilon}(s)| \int_{B_s(x_0)} \varphi^k u^{n+1-3\alpha} \,\mathrm{d}x \,\mathrm{d}s \\
&\le C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\
&\quad \times \left(\int_0^r |\nabla \varphi^{1-\varepsilon}(s)| \left(\int_{B_s(x_0)} u \,\mathrm{d}x \right)^{(1-\vartheta)(n+1-3\alpha)-k} \mathrm{d}s \right) \left(\int_{B_r(x_0)} \varphi u \,\mathrm{d}x \right)^k \\
&\le C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \left(\int_{B_r(x_0)} \varphi u \,\mathrm{d}x \right)^k \\
&\quad \times \int_0^r \left| \nabla \varphi^{\frac{1-\lambda+k-\varepsilon}{1-\lambda+k}} \right|^{1-\lambda+k} |\nabla \varphi|^{\lambda-k} \left(\int_{B_s(x_0)} u \,\mathrm{d}x \right)^{\lambda-k} \mathrm{d}s \\
&\le C \left(\int_{B_r(x_0)} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 + r^{-4} u^{\alpha+n+1} \,\mathrm{d}x \right)^{\frac{\vartheta(n+1-3\alpha)}{\alpha+n+1}} \\
&\quad \times \left(\int_{B_r(x_0)} \varphi u \,\mathrm{d}x \right)^k \left(\int_0^r |\nabla \varphi(s)| \int_{B_s(x_0)} \varphi u \,\mathrm{d}x \,\mathrm{d}s \right)^{\lambda-k} \\
&\quad \times \left(\int_0^r \left| \nabla \varphi^{\frac{1-\lambda+k-\varepsilon}{1-\lambda+k}} \right| \,\mathrm{d}s \right)^{1-\lambda+k}
\end{aligned}$$

(where we used Hölder's inequality in the last step). This proves the lemma since $\int_{-\infty}^{\infty} |\nabla \varphi^{\frac{1-\lambda+k-\varepsilon}{1-\lambda+k}}| \,\mathrm{d}s = 2$. □

5. Proof of the sufficient condition for the waiting time phenomenon

We split the proof in two cases: In the regime of strong slippage, i. e. $n \in (1,2)$, the "propagation of degeneracy argument" is based on the interplay between a localized mass estimate and a time-weighted localized entropy estimate; on the other hand, in the regime of weak slippage, i. e. $n \in [2,3)$, we employ a localized mass estimate and a time-weighted localized energy estimate.

Proof of Theorem 2.2, case $n \in (1,2)$. We will prove the following statement: The assumption (9) implies that for $T := c\kappa^{-n}$ and for $R > 0$ large enough the estimate

$$\fint_0^T \fint_{B_{\frac{R}{2^k}}(x_0)} u \,\mathrm{d}x \,\mathrm{d}t \le \sup_{t \in (0,T)} \fint_{B_{\frac{R}{2^k}}(x_0)} u \,\mathrm{d}x \le C T^{-1/n} \left(\frac{R}{2^k} \right)^{4/n} \tag{15}$$

holds for all $k \in \mathbb{N}$. To see that this implication entails our lower bound on waiting times, we refer to the discussion of the same issue in the case $n \in [2,3)$ provided at the beginning of the proof of Theorem 2.2 in the case $n \in [2,3)$.

Step 1. Choice of test functions. Fix $R > 0$ and let $r_k := 2^{-k}R$, with $k \geq 1$, and let $\varphi_{r_k} \in C_c^\infty(\mathbb{R}^d)$ be a smooth function such that $\mathrm{supp}(\varphi_{r_k}) \subset B_{r_k}(x_0)$, $0 \leq \varphi_{r_k} \leq 1$,

$$\varphi_{r_k}(x) = \begin{cases} 1, & x \in B_{r_{k+1}}(x_0), \\ 0, & x \in \mathbb{R}^d \setminus B_{r_k}(x_0), \end{cases}$$

and $|\nabla \varphi_{r_k}| \leq C(r_{k+1})^{-1}$, $|D^2\varphi_{r_k}| \leq C(r_{k+1})^{-2}$, $|D^3\varphi_{r_k}| \leq C(r_{k+1})^{-3}$, $|D^4\varphi_{r_k}| \leq C(r_{k+1})^{-4}$.

Step 2. Time-weighted localized entropy estimate. Let $T > 0$ and $\varphi \in C^\infty(\mathbb{R}^d \times [0,T])$ be a smooth nonnegative cut-off function. By arguing as in [8, Theorem 3.1], for weak solution to the thin-film equation (1) with zero contact angle in the sense of Definition 2.6 constructed with the approximation procedure in [8] it is possible to prove the time-weighted localized α-entropy estimate

(16)
$$\begin{aligned} &\sup_{t\in(0,T)} \int_{\mathbb{R}^d} \psi^4 u^{\alpha+1} \,\mathrm{d}x\Big|_0^T \\ &\quad + C\left(\int_0^T \int_{\mathbb{R}^d} \psi^4 \left|D^2 u^{\frac{\alpha+n+1}{2}}\right|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \psi^4 \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \,\mathrm{d}x\,\mathrm{d}t\right) \\ &\quad \leq C \int_0^T \int_{\{\psi>0\}} u^{\alpha+n+1}(|\nabla\psi|^4 + \psi^2|D^2\psi|^2) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} |\partial_t\psi| u^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for any $\alpha \in \left(\frac{1}{2} - n, 2 - n\right) \setminus \{-1, 0\}$, $\alpha > 0$, and a.e. $T \geq 0$. This implies by taking $\psi = \varphi_{r_k} t^\beta$ (with $0 < \beta < 1$),

$$\begin{aligned} &\sup_{t\in(0,T)} \int_{B_{r_{k+1}}(x_0)} t^\beta u^{\alpha+1} \,\mathrm{d}x + C\int_0^T \int_{B_{r_{k+1}}(x_0)} t^\beta \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq C\int_0^T \int_{B_{r_k}(x_0)} t^{\beta-1} u^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-4} \int_0^T \int_{B_{r_k}(x_0)} t^\beta u^{\alpha+n+1} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

The Gagliardo-Nirenberg-Sobolev interpolation inequality (34) (applied to $v = u^{\frac{\alpha+n+1}{4}}$ with $p = 4$, $q = \frac{4}{\alpha+n+1}$, $r = 4$) yields

$$\begin{aligned} \int_{B_{r_k}(x_0)} u^{\alpha+n+1} \,\mathrm{d}x \leq & C\left(\int_{B_{r_k}(x_0)} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \,\mathrm{d}x\right)^\sigma \left(\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{(\alpha+n+1)(1-\sigma)} \\ & + C\left(\frac{R}{2^{k+1}}\right)^{-d(\alpha+n)} \left(\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{\alpha+n+1} \end{aligned}$$

with

$$\sigma = \frac{d(\alpha+n)}{(d\alpha + dn + 4)} \,.$$

Again by the Gagliardo-Nirenberg-Sobolev interpolation inequality (34) (applied to $v = u^{\frac{\alpha+n+1}{4}}$ with $p = \frac{4(\alpha+1)}{\alpha+n+1}$, $q = \frac{4}{\alpha+n+1}$, $r = 4$), we also have

$$\int_{B_{r_k}(x_0)} u^{\alpha+1}\ \mathrm{d}x \le C\left(\int_{B_{r_k}(x_0)} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\ \mathrm{d}x\right)^{\frac{\nu(\alpha+1)}{\alpha+n+1}} \left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{(\alpha+1)(1-\nu)} + C\left(\frac{R}{2^{k+1}}\right)^{-d\alpha}\left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{\alpha+1}$$

with

$$\nu = \frac{d\alpha(1+\alpha+n)}{(d\alpha+dn+4)(\alpha+1)}\ .$$

Putting these considerations together, we obtain

$$\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} t^\beta u^{\alpha+1}\ \mathrm{d}x + C\int_0^T\int_{B_{r_{k+1}}(x_0)} t^\beta\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\ \mathrm{d}x\ \mathrm{d}t \\
&\le C\int_0^T t^{\beta-1}\left(\int_{B_{r_k}(x_0)}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\ \mathrm{d}x\right)^{\frac{\nu(\alpha+1)}{\alpha+n+1}} \\
&\qquad\times\left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{(\alpha+1)(1-\nu)}\ \mathrm{d}t \\
&\quad + C\left(\frac{R}{2^{k+1}}\right)^{-d\alpha}\int_0^T t^{\beta-1}\left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{\alpha+1}\ \mathrm{d}t \\
&\quad + C\left(\frac{R}{2^{k+1}}\right)^{-4}\int_0^T t^\beta\left(\int_{B_{r_k}(x_0)}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\ \mathrm{d}x\right)^\sigma \\
&\qquad\qquad\times\left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{(\alpha+n+1)(1-\sigma)}\ \mathrm{d}t \\
&\quad + C\left(\frac{R}{2^{k+1}}\right)^{-4-d(\alpha+n)}\int_0^T t^\beta\left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{\alpha+n+1}\ \mathrm{d}t.
\end{aligned}$$

Finally, by using Hölder's inequality, we infer under the assumption $(1+\beta)(1-\frac{\nu(\alpha+1)}{\alpha+n+1})-1>0$ (which is satisfied for $\alpha>0$ small enough, the required smallness depending on $\beta>0$)

$$\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} t^\beta u^{\alpha+1}\ \mathrm{d}x + C\int_0^T\int_{B_{r_{k+1}}(x_0)} t^\beta\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\ \mathrm{d}x\ \mathrm{d}t \\
&\le CT^{(1+\beta)(1-\frac{(\alpha+1)\nu}{\alpha+n+1})-1}\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\ \mathrm{d}x\ \mathrm{d}t\right)^{\frac{\nu(\alpha+1)}{\alpha+n+1}} \\
&\qquad\times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{(\alpha+1)(1-\nu)}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{R}{2^{k+1}}\right)^{-d\alpha} T^{\beta}\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\alpha+1}\\
&+C\left(\frac{R}{2^{k+1}}\right)^{-4} T^{(1-\sigma)(1+\beta)}\left(\int_0^T\int_{B_{r_k}(x_0)} t^{\beta}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{\sigma}\\
&\qquad\times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{(\alpha+n+1)(1-\sigma)}\\
&+C\left(\frac{R}{2^{k+1}}\right)^{-4-d(\alpha+n)} T^{\beta+1}\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\alpha+n+1},
\end{aligned}
$$

i. e. for $\alpha>0$ small enough we have (plugging in the definition of ν and σ)

(17)

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} t^{\beta}u^{\alpha+1}\,\mathrm{d}x + C\int_0^T\int_{B_{r_{k+1}}(x_0)} t^{\beta}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\,\mathrm{d}t\\
&\le CT^{\beta-\frac{d\alpha}{d\alpha+dn+4}-\beta\frac{d\alpha}{d\alpha+dn+4}}\left(\int_0^T\int_{B_{r_k}(x_0)} t^{\beta}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{d\alpha}{d\alpha+dn+4}}\\
&\qquad\times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{dn+4+4d}{d\alpha+dn+4}}\\
&\quad+\left(\frac{R}{2^{k+1}}\right)^{-d\alpha} T^{\beta}\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\alpha+1}\\
&\quad+C\left(\frac{R}{2^{k+1}}\right)^{-4} T^{(1+\beta)\frac{4}{d\alpha+dn+4}}\left(\int_0^T\int_{B_{r_k}(x_0)} t^{\beta}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{d\alpha+dn}{d\alpha+dn+4}}\\
&\qquad\times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{4(\alpha+n+1)}{d\alpha+dn+4}}\\
&\quad+C\left(\frac{R}{2^{k+1}}\right)^{-4-d(\alpha+n)} T^{\beta+1}\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\alpha+n+1}\\
&=: E_1+E_2+E_3+E_4.
\end{aligned}
$$

Step 3. Localized mass estimate. Starting from the weak formulation of the thin-film equation (see Definition 2.6c), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} u(x,T)\varphi\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^d} u_0\varphi\,\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d\cap\{u>0\}} u^n\nabla u\cdot\nabla\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+n\int_0^T\int_{\mathbb{R}^d\cap\{u>0\}} u^{n-1}\nabla u\cdot D^2\varphi\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n}{2}\int_0^T\int_{\mathbb{R}^d\cap\{u>0\}} u^{n-1}|\nabla u|^2\Delta\varphi \,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{n(n-1)}{2}\int_0^T\int_{\mathbb{R}^d\cap\{u>0\}} u^{n-2}|\nabla u|^2\nabla u\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t\\
\le&\int_{\mathbb{R}^d} u_0\varphi\,\mathrm{d}x + C\int_0^T\int_{\mathbb{R}^d} u^{\frac{n+1-3\alpha}{4}}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^3|\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t\\
&+C\int_0^T\int_{\mathbb{R}^d} u^{n+1}\left(|\Delta^2\varphi|+\frac{|D^2\varphi|^3}{|\nabla\varphi|^2}\right)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
$$

Choosing $\varphi=\varphi_{r_k}$ as a test function, the previous inequality implies

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} u\,\mathrm{d}x\\
&\quad\le\int_{B_{r_k}(x_0)} u_0\,\mathrm{d}x + C\left(\frac{R}{2^{k+1}}\right)^{-1}\int_0^T\int_{B_{r_k}(x_0)} u^{\frac{n+1-3\alpha}{4}}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^3\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+C\left(\frac{R}{2^{k+1}}\right)^{-4}\int_0^T\int_{B_{r_k}(x_0)} u^{n+1}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By Hölder's inequality, we obtain

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} u\,\mathrm{d}x\\
&\le\int_{B_{r_k}(x_0)} u_0\,\mathrm{d}x\\
&\quad+C\left(\frac{R}{2^{k+1}}\right)^{-1}\int_0^T\left(\int_{B_{r_k}(x_0)} u^{n+1-3\alpha}\,\mathrm{d}x\right)^{\frac14}\left(\int_{B_{r_k}(x_0)}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\right)^{\frac34}\,\mathrm{d}t\\
&\quad+C\left(\frac{R}{2^{k+1}}\right)^{-4}\int_0^T\int_{B_{r_k}(x_0)} u^{n+1}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

The Gagliardo-Nirenberg-Sobolev interpolation inequality (34) (applied to $v=u^{\frac{\alpha+n+1}{4}}$ with $p=\frac{4(n+1)}{\alpha+n+1}$, $q=\frac{4}{\alpha+n+1}$, $r=4$) yields

$$
\begin{aligned}
\int_{B_{r_k}(x_0)} u^{n+1}\,\mathrm{d}x \le{}& C\left(\int_{B_{r_k}(x_0)}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\right)^{\frac{\vartheta(n+1)}{\alpha+n+1}}\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{(1-\vartheta)(n+1)}\\
&+C\left(\frac{R}{2^{k+1}}\right)^{-nd}\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{n+1}
\end{aligned}
$$

with

$$
\vartheta=\frac{nd(\alpha+n+1)}{(nd+\alpha d+4)(n+1)}.
$$

Note that for all $\alpha\in(0,2-n)$, all $d\in\{1,2,3\}$, and all $1<n<2$ we have $0<\vartheta<1$. By the Gagliardo-Nirenberg-Sobolev interpolation inequality (34)

(applied to $v = u^{\frac{\alpha+n+1}{4}}$ with $p = \frac{4(n+1-3\alpha)}{\alpha+n+1}$, $q = \frac{4}{\alpha+n+1}$, $r = 4$), we also have

$$\begin{aligned}\left(\int_{B_{r_k}(x_0)} u^{n+1-3\alpha}\,\mathrm{d}x\right)^{\frac{1}{4}} &\leq C\left(\int_{B_{r_k}(x_0)} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\right)^{\frac{\mu(n+1-3\alpha)}{4(\alpha+n+1)}} \\ &\quad\times\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{(1-\mu)(n+1-3\alpha)}{4}} \\ &\quad + C\left(\frac{R}{2^{k+1}}\right)^{-d\frac{(n-3\alpha)}{4}}\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{n+1-3\alpha}{4}},\end{aligned}$$

with

$$\mu = \frac{d(n-3\alpha)(\alpha+n+1)}{(nd+\alpha d+4)(n+1-3\alpha)}.$$

Putting these considerations together, we obtain

$$\begin{aligned}&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} u\,\mathrm{d}x \\ &\quad\leq \int_{B_{r_k}(x_0)} u_0\,\mathrm{d}x \\ &\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-1}\int_0^T\left(\int_{B_{r_k}(x_0)}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\right)^{\frac{\mu(n+1-3\alpha)}{4(\alpha+n+1)}+\frac{3}{4}} \\ &\qquad\qquad\times\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{(1-\mu)(n+1-3\alpha)}{4}}\mathrm{d}t \\ &\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-1-d\frac{(n-3\alpha)}{4}}\int_0^T\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{n+1-3\alpha}{4}} \\ &\qquad\qquad\times\left(\int_{B_{r_k}(x_0)}|\nabla u|^4\,\mathrm{d}x\right)^{\frac{3}{4}}\mathrm{d}t \\ &\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-4}\int_0^T\left(\int_{B_{r_k}(x_0)}\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4\,\mathrm{d}x\right)^{\frac{\vartheta(n+1)}{\alpha+n+1}} \\ &\qquad\qquad\times\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{(1-\vartheta)(n+1)}\mathrm{d}t \\ &\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-4-dn}\int_0^T\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{n+1}\mathrm{d}t\end{aligned}$$

This implies by Hölder's inequality, assuming that $1-(1+\beta)(\frac{\mu(n+1-3\alpha)}{4(\alpha+n+1)}+\frac{3}{4})>0$ and $\beta<\frac{1}{3}$ as well as $1-\frac{\vartheta(n+1)}{\alpha+n+1}-\beta\frac{\vartheta(n+1)}{\alpha+n+1}>0$,

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} u \,\mathrm{d}x \\
&\quad\le \int_{B_{r_k}(x_0)} u_0 \,\mathrm{d}x \\
&\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-1} T^{1-\frac{\mu(n+1-3\alpha)}{4(\alpha+n+1)}-\frac{3}{4}-\beta\left(\frac{\mu(n+1-3\alpha)}{4(\alpha+n+1)}+\frac{3}{4}\right)} \\
&\qquad\quad \times\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\mu(n+1-3\alpha)}{4(\alpha+n+1)}+\frac{3}{4}} \\
&\qquad\quad \times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{\frac{(1-\mu)(n+1-3\alpha)}{4}} \\
&\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-1-d\frac{(n-3\alpha)}{4}} T^{\frac{1}{4}-\frac{3}{4}\beta}\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{3}{4}} \\
&\qquad\quad \times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{\frac{n+1-3\alpha}{4}} \\
&\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-4} T^{1-\frac{\vartheta(n+1)}{\alpha+n+1}-\beta\frac{\vartheta(n+1)}{\alpha+n+1}} \\
&\qquad\quad \times\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\vartheta(n+1)}{(\alpha+n+1)}} \\
&\qquad\quad \times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{(1-\vartheta)(n+1)} \\
&\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-4-dn} T\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{n+1}.
\end{aligned}
$$

Plugging in μ and ϑ, we deduce

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} u \,\mathrm{d}x \\
&\quad\le \int_{B_{r_k}(x_0)} u_0 \,\mathrm{d}x \\
&\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-1} T^{\frac{\alpha d+1}{dn+\alpha d+4}-\beta\frac{dn+3}{dn+\alpha d+4}} \\
&\qquad\quad \times\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta\left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{dn+3}{dn+\alpha d+4}}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\times \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \, \mathrm{d}x \right)^{\frac{\alpha d+n+1-3\alpha}{dn+\alpha d+4}} \\
&+ C \left(\frac{R}{2^{k+1}} \right)^{-1-d\frac{(n-3\alpha)}{4}} T^{\frac{1}{4}-\frac{3}{4}\beta} \left(\int_0^T \int_{B_{r_k}(x_0)} t^\beta \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{3}{4}} \\
&\times \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \, \mathrm{d}x \right)^{\frac{n+1-3\alpha}{4}} \\
&+ C \left(\frac{R}{2^{k+1}} \right)^{-4} T^{\frac{\alpha d+4}{dn+\alpha d+4} - \beta \frac{dn}{dn+\alpha d+4}} \\
&\times \left(\int_0^T \int_{B_{r_k}(x_0)} t^\beta \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{dn}{dn+\alpha d+4}} \\
&\times \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \, \mathrm{d}x \right)^{\frac{\alpha d+4n+4}{nd+\alpha d+4}} \\
&+ C \left(\frac{R}{2^{k+1}} \right)^{-4-dn} T \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \, \mathrm{d}x \right)^{n+1} \\
&=: M_1 + M_2 + M_3 + M_4 + M_5
\end{aligned}
$$

under the assumptions $(\alpha d+1) - \beta(dn+3) > 0$, $\beta < \frac{1}{3}$, and $\alpha d + 4 - \beta dn > 0$. Note that for $\beta < \frac{1}{9}$ these assumptions are satisfied.

Step 3. Down-propagation of the degeneracy. Let us consider the following functions:

$$
\begin{aligned}
G_1(k) &:= \sup_{t\in(0,T)} \fint_{B_{r_k}(x_0)} u \, \mathrm{d}x, \\
G_2(k) &:= \sup_{t\in(0,T)} \fint_{B_{r_k}(x_0)} t^\beta u^{\alpha+1} \, \mathrm{d}x + \int_0^T \fint_{B_{r_k}(x_0)} t^\beta \left| \nabla u^{\frac{n+\alpha+4}{4}} \right|^4 \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

We want to prove that, for every $k \in \mathbb{N}$, the following bounds hold:

$$
G_1(k) \le \varepsilon \, T^{-1/n} \left(\frac{R}{2^k} \right)^{4/n}, \tag{19a}
$$

$$
G_2(k) \le \varepsilon^\delta \, T^\beta \, T^{-(1+\alpha)/n} \left(\frac{R}{2^k} \right)^{4(\alpha+1)/n}, \tag{19b}
$$

where $\varepsilon, \delta > 0$ are constants that will be chosen suitably below and where α and β are arbitrary within the bounds given above. Note that these estimates directly entail the desired result (15) for all $k \in \mathbb{N}$.

We will prove this claim by induction. It is immediate to check that the base step ($k = 1$) is verified provided that we fix $R > 0$ large enough. Indeed,

$$
\sup_{t\in(0,T)} \int_{B_{r_1}(x_0)} u \, \mathrm{d}x \le \varepsilon \, T^{-1/n} \left(\frac{R}{2} \right)^{\frac{4}{n}+d}
$$

follows from the fact that

$$\int_{\mathbb{R}^d} u \, \mathrm{d}x = \int_{\mathbb{R}^d} u_0 \, \mathrm{d}x < \infty.$$

On the other hand, to prove

$$\begin{aligned}\sup_{t\in(0,T)} \int_{B_{r_1}(x_0)} t^\beta u^{\alpha+1} \, \mathrm{d}x + \int_0^T \int_{B_{r_1}(x_0)} t^\beta \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \, \mathrm{d}x \, \mathrm{d}t \\ \leq \varepsilon^\delta \, T^\beta \, T^{-(\alpha+1)/n} \left(\frac{R}{2}\right)^{\frac{4}{n}(\alpha+1)+d}\end{aligned}$$

we argue as follows. By the property of finite speed of propagation for the solutions to the thin-film equation (see [8, Theorem 5.2]), there exists a ball $B_{\bar{R}}(x_0)$ that contains $\operatorname{supp} u(\cdot,t)$ for $t \in [0,T)$. We consider a smooth cut-off function φ such that $\operatorname{supp}\varphi \subset B_R(x_0)$, $0 \leq \varphi \leq 1$, and $\varphi \equiv 1$ in $B_{\bar{R}}$. Then, from the weighted entropy estimate it follows that

$$\begin{aligned}&\sup_{t\in(0,T)} \int_{B_{r_1}(x_0)} t^\beta u^{\alpha+1} \, \mathrm{d}x + C \int_0^T \int_{B_{r_1}(x_0)} t^\beta \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \leq C T^{\beta - \frac{\beta(\alpha+1)\nu}{\alpha+n+1} - \nu\frac{(\alpha+1)}{\alpha+n+1}} \left(\int_0^T \int_{B_{\bar{R}}(x_0)} t^\beta \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \, \mathrm{d}x \, \mathrm{d}t\right)^{\frac{\nu(\alpha+1)}{\alpha+n+1}} \\ &\qquad \times \left(\sup_{t\in(0,T)} \int_{B_{\bar{R}}(x_0)} u \, \mathrm{d}x\right)^{(\alpha+1)(1-\nu)} \\ &\qquad + C\bar{R}^{-d\alpha} T^\beta \left(\sup_{t\in(0,T)} \int_{B_{\bar{R}}(x_0)} u \, \mathrm{d}x\right)^{\alpha+1}.\end{aligned}$$

Young's inequality yields with $\zeta := \frac{\alpha+n+1}{\nu(\alpha+1)}$ and ζ' subject to $\frac{1}{\zeta} + \frac{1}{\zeta'} = 1$

$$\begin{aligned}&\sup_{t\in(0,T)} \int_{B_{\bar{R}}(x_0)} t^\beta u^{\alpha+1} \, \mathrm{d}x + C \int_0^T \int_{B_{\bar{R}}(x_0)} t^\beta \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \leq C T^\beta T^{\zeta'(\beta - \frac{\beta(\alpha+1)\nu}{\alpha+n+1} - \nu\frac{(\alpha+1)}{\alpha+n+1}) - \beta} \left(\sup_{t\in(0,T)} \int_{B_{\bar{R}}(x_0)} u \, \mathrm{d}x\right)^{(\alpha+1)(1-\nu)\zeta'} \\ &\qquad + C\bar{R}^{-d\alpha} T^\beta \left(\sup_{t\in(0,T)} \int_{B_{\bar{R}}(x_0)} u \, \mathrm{d}x\right)^{\alpha+1}.\end{aligned}$$

This implies the claim, if we choose R and $\bar{R}$ large enough, since

$$\int_{\mathbb{R}^d} u \, \mathrm{d}x = \int_{\mathbb{R}^d} u_0 \, \mathrm{d}x < \infty.$$

Having proved the base step of the induction, we now show that the bounds are propagated down to smaller scales: Assuming

$$\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \, \mathrm{d}x \leq \varepsilon \, T^{-1/n} \left(\frac{R}{2^k}\right)^{\frac{4}{n}+d}, \tag{20}$$

$$\begin{aligned} \sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} t^\beta u^{\alpha+1} \, \mathrm{d}x + \int_0^T \int_{B_{r_k}(x_0)} t^\beta \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \, \mathrm{d}x \, \mathrm{d}t \\ \leq \varepsilon^\delta \, T^\beta \, T^{-(\alpha+1)/n} \left(\frac{R}{2^k}\right)^{\frac{4}{n}(\alpha+1)+d}, \end{aligned} \tag{21}$$

we claim

$$\sup_{t\in(0,T)} \int_{B_{r_{k+1}}(x_0)} u \, \mathrm{d}x \leq \varepsilon \, T^{-1/n} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d}, \tag{22}$$

$$\begin{aligned} \sup_{t\in(0,T)} \int_{B_{r_{k+1}}(x_0)} t^\beta u^{\alpha+1} \, \mathrm{d}x + \int_0^T \int_{B_{r_{k+1}}(x_0)} t^\beta \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 \, \mathrm{d}x \, \mathrm{d}t \\ \leq \varepsilon^\delta \, T^\beta \, T^{-(\alpha+1)/n} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}(\alpha+1)+d}. \end{aligned} \tag{23}$$

Plugging the induction hypothesis (20)-(21) as well as the assumption (9) into the localized entropy and mass estimates (17) and (18), we obtain

$$E_1 \leq \underset{\cdots\cdots\cdots\cdots\cdots\cdots}{C} \, \varepsilon^{\frac{4+nd+4\alpha-\delta nd-4\delta}{dn+\alpha n+4}} \, \varepsilon^\delta \, T^\beta \, T^{-\frac{\alpha+1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}(\alpha+1)+d};$$

$$E_2 \leq \underset{\cdots\cdots\cdots}{C} \, \varepsilon^{\alpha+1-\delta} \, \varepsilon^\delta \, T^\beta \, T^{-\frac{\alpha+1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}(\alpha+1)+d};$$

$$E_3 \leq \underset{\cdots\cdots\cdots\cdots\cdots}{C} \, \varepsilon^{\frac{4(\alpha+n+1-\delta)}{nd+\alpha n+4}} \, \varepsilon^\delta \, T^\beta \, T^{-\frac{\alpha+1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}(\alpha+1)+d};$$

$$E_4 \leq \underset{\cdots\cdots\cdots\cdots}{C} \, \varepsilon^{\alpha+n+1-\delta} \, \varepsilon^\delta \, T^\beta \, T^{-\frac{\alpha+1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}(\alpha+1)+d};$$

$$M_1 \leq C \, \varepsilon^{-1} \, \kappa \, T^{\frac{1}{n}} \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d};$$

$$M_2 \leq \underset{\cdots\cdots\cdots\cdots\cdots\cdots}{C} \, \varepsilon^{\frac{\delta nd+3\delta-nd+n-3-3\alpha}{nd+\alpha d+4}} \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d};$$

$$M_3 \leq \underset{\cdots\cdots\cdots}{C} \, \varepsilon^{\frac{n-3-3\alpha+3\delta}{4}} \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d};$$

$$M_4 \leq \underset{\cdots\cdots\cdots\cdots}{C} \, \varepsilon^{\frac{4n-nd+\delta dn}{nd+\alpha d+4}} \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d};$$

$$M_5 \leq \underset{\cdots\cdots}{C} \, \varepsilon^n \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d}.$$

Putting these estimates together, we conclude that (22) and (23) hold if ε, δ, and α are chosen in a suitable way (i. e. ε small enough and δ and α in such a way that the exponents in the underlined factors are positive, for example $\delta := 1$ and $\alpha > 0$ small enough) and if we suppose that T satisfies

$$C\,\varepsilon^{-1}\,T^{1/n}\,\kappa \leq 1.$$

This completes the induction and shows (19) for all k. □

Proof of Theorem 2.2, case $n \in [2,3)$. In case $x_0 \in \partial \operatorname{supp} u_0$, denote by $\mathcal{C}$ a cone with the same apex and orientation as the cone from the exterior cone condition but with half the opening angle. Our main assumption (9) entails the existence of some $\rho > 0$ such that for any point $\tilde{x}_0 \in B_\rho(x_0)$ (if $x_0 \notin \operatorname{supp} u_0$) respectively for any point $\tilde{x}_0 \in B_\rho(x_0) \cap \mathcal{C}$ (if $x_0 \in \partial \operatorname{supp} u_0$) the estimate

$$\fint_{B_r(\tilde{x}_0)} u_0 \,\mathrm{d}x \leq C(d,n,\lambda)\kappa\, r^{\frac{4}{n}}$$

holds for all $r > 0$. In other words, for all points $\tilde{x}_0$ near x_0 respectively all points $\tilde{x}_0$ near x_0 in the smaller cone $\mathcal{C}$, the initial data u_0 satisfy a growth condition analogous to (9), just with a different constant κ.

We will prove that the assumption (9) implies that for $T := c\kappa^{-n}$ and for $R > 0$ large enough the estimate

$$\fint_0^T \fint_{B_{\frac{R}{2^k}}(x_0)} u \,\mathrm{d}x\,\mathrm{d}t \leq \sup_{t\in(0,T)} \fint_{B_{\frac{R}{2^k}}(x_0)} u \,\mathrm{d}x\,\mathrm{d}t \leq CT^{-1/n}\left(\frac{R}{2^k}\right)^{4/n} \tag{24}$$

holds for any $k \in \mathbb{N}$. In view of the previous discussion, the same implication (up to adjusting the constants) then holds for all points $\tilde{x}_0$ in a neighborhood of x_0, respectively for all $\tilde{x}_0$ near x_0 which belong to the cone $\mathcal{C}$. Letting $k \to \infty$, this implies $u(\tilde{x}_0, t) = 0$ for almost all such points $\tilde{x}_0$ and almost every $t \in (0,T)$. Thus, in view of Definition 2.1 this implies the lower bound $T^* \geq c\kappa^{-n}$ on the waiting time T^* at x_0.

Step 1. Choice of test functions. Fix $R > 0$ and let $r_k := R/2^k$, with $k \geq 1$, and let $\varphi_{r_k} \in C_c^\infty(\mathbb{R}^d)$ be a smooth cut-off function such that $\operatorname{supp}(\varphi_{r_k}) \subset B_{r_k}(x_0)$, $0 \leq \varphi_{r_k} \leq 1$,

$$\varphi_{r_k}(x) = \begin{cases} 1, & x \in B_{r_{k+1}}(x_0), \\ 0, & x \in \mathbb{R}^d \setminus B_{r_k}(x_0), \end{cases}$$

and $|\nabla\varphi_{r_k}| \leq C\,(r_{k+1})^{-1}$, $|D^2\varphi_{r_k}| \leq C\,(r_{k+1})^{-2}$, $|D^3\varphi_{r_k}| \leq C\,(r_{k+1})^{-3}$, $|D^4\varphi_{r_k}| \leq C\,(r_{k+1})^{-4}$.

Step 2. Time-weighted localized energy estimate. Let $T > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative smooth cut-off function. In the appendix (Theorem A.3), we prove for any energy-dissipating weak solution of the thin-film equation (1) with zero contact

angle in the sense of Definition 2.5 the time-weighted localized energy estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^d} t^\beta |\nabla u|^2 \varphi^6 \,\mathrm{d}x\Big|_0^T + C\int_0^T \int_{\mathbb{R}^d} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla \Delta u|^2\big)\varphi^6 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\int_0^T \int_{\mathbb{R}^d} t^\beta u^{n+2} \left(|\nabla\varphi|^6 + |D^2\varphi|^2|\nabla\varphi|^2\varphi^2 + |D^2\varphi|^3\varphi^3\right) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \beta \int_0^T t^{-1} \int_{\mathbb{R}^d} t^\beta |\nabla u|^2 \varphi^6 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{25}
$$

for any $\beta \in (0,1)$. Hölder's inequality yields

$$
\int_{\mathbb{R}^d} |\nabla u|^2 \varphi^6 \,\mathrm{d}x \le C\left(\int_{\mathbb{R}^d} |\nabla u^{\frac{n+2}{6}}|^6 \varphi^6 \,\mathrm{d}x\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^d} u\varphi^6 \,\mathrm{d}x\right)^{\frac{4-n}{3}} \left(\int_{\mathbb{R}^d} \varphi^6 \,\mathrm{d}x\right)^{\frac{n-2}{3}} .
$$

Hence we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} t^\beta |\nabla u|^2 \varphi^6 \,\mathrm{d}x\Big|_0^T + C\int_0^T \int_{\mathbb{R}^d} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla \Delta u|^2\big)\varphi^6 \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int_0^T \int_{\mathbb{R}^d} t^\beta u^{n+2} \left(|\nabla\varphi|^6 + |D^2\varphi|^2|\nabla\varphi|^2\varphi^2 + |D^2\varphi|^3\varphi^3\right) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_0^T t^{\beta-1} \left(\int_{\mathbb{R}^d} \varphi^6 \,\mathrm{d}x\right)^{\frac{n-2}{3}} \left(\int_{\mathbb{R}^d} u\varphi^6 \,\mathrm{d}x\right)^{\frac{4-n}{3}} \left(\int_{\mathbb{R}^d} \left|\nabla u^{\frac{n+2}{6}}\right|^6 \varphi^6 \,\mathrm{d}x\right)^{\frac{1}{3}} \,\mathrm{d}t.
\end{aligned}
$$

By applying Hölder's inequality again, we obtain (assuming $\beta > 1/2$)

$$
\begin{aligned}
&\int_{\mathbb{R}^d} t^\beta |\nabla u|^2 \varphi^6 \,\mathrm{d}x\Big|_0^T + C\int_0^T \int_{\mathbb{R}^d} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla \Delta u|^2\big)\varphi^6 \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int_0^T \int_{\mathbb{R}^d} t^\beta u^{n+2} \left(|\nabla\varphi|^6 + |D^2\varphi|^2|\nabla\varphi|^2\varphi^2 + |D^2\varphi|^3\varphi^3\right) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\left(\int_0^T \int_{\mathbb{R}^d} t^\beta |\nabla u^{\frac{n+2}{6}}|^6 \varphi^6 \,\mathrm{d}x\right)^{\frac{1}{3}} \\
&\qquad \times \left(\int_0^T t^{\beta-\frac{3}{2}} \left(\int_{\mathbb{R}^d} \varphi^6 \,\mathrm{d}x\right)^{\frac{n-2}{2}} \left(\int_{\mathbb{R}^d} u\varphi^6 \,\mathrm{d}x\right)^{\frac{4-n}{2}} \,\mathrm{d}t\right)^{\frac{2}{3}} .
\end{aligned}
$$

From Young's inequality (applied with suitably chosen constants), it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} t^\beta |\nabla u|^2 \varphi^6 \,\mathrm{d}x\Big|_0^T + C\int_0^T \int_{\mathbb{R}^d} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla \Delta u|^2\big)\varphi^6 \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int_0^T \int_{\mathbb{R}^d} t^\beta u^{n+2} \left(|\nabla\varphi|^6 + |D^2\varphi|^2|\nabla\varphi|^2\varphi^2 + |D^2\varphi|^3\varphi^3\right) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_0^T t^{\beta-\frac{3}{2}} \left(\int_{\mathbb{R}^d} \varphi^6 \,\mathrm{d}x\right)^{\frac{n-2}{2}} \left(\int_{\mathbb{R}^d} u\varphi^6 \,\mathrm{d}x\right)^{\frac{4-n}{2}} \,\mathrm{d}t .
\end{aligned}
$$

Choosing $\varphi = \varphi_{r_k}$, the previous inequality reduces to

$$\begin{aligned}\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} t^\beta|\nabla u|^2\,\mathrm{d}x + C\int_0^T\int_{B_{r_{k+1}}(x_0)} t^\beta\big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n|\nabla\Delta u|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\ \le C\left(\frac{R}{2^{k+1}}\right)^{-6}\int_0^T\int_{B_{r_k}(x_0)} t^\beta u^{n+2}\,\mathrm{d}x\,\mathrm{d}t \\ + C\left(\frac{R}{2^{k+1}}\right)^{d\frac{n-2}{2}}\int_0^T t^{\beta-\frac{3}{2}}\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{4-n}{2}}\mathrm{d}t\ .\end{aligned}$$

The Gagliardo-Nirenberg-Sobolev interpolation inequality (applied to $v = u^{\frac{n+2}{6}}$, $p = 6$, $q = \frac{6}{n+2}$, $r = 6$) yields

$$\begin{aligned}\int_{B_{r_k}(x_0)} u^{n+2}\,\mathrm{d}x \le & C\left(\int_{B_{r_k}(x_0)} |\nabla u^{\frac{n+2}{6}}|^6\,\mathrm{d}x\right)^\mu\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{(1-\mu)(n+2)} \\ & + C\left(\frac{R}{2^{k+1}}\right)^{-d(n+1)}\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{n+2}\end{aligned}$$

with

$$\mu = \frac{d(n+1)}{dn+d+6}\ .$$

We thus obtain

$$\begin{aligned}&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} t^\beta|\nabla u|^2\,\mathrm{d}x + C\int_0^T\int_{B_{r_{k+1}}(x_0)} t^\beta\big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n|\nabla\Delta u|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\left(\frac{R}{2^{k+1}}\right)^{-6}\int_0^T t^{(1-\mu)\beta}\left(\int_{B_{r_k}(x_0)} t^\beta|\nabla u^{\frac{n+2}{6}}|^6\,\mathrm{d}x\right)^\mu\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{(1-\mu)(n+2)}\mathrm{d}t \\ &\quad + C\left(\frac{R}{2^{k+1}}\right)^{-6}\left(\frac{R}{2^{k+1}}\right)^{-d(n+1)}\int_0^T t^\beta\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{n+2}\mathrm{d}t \\ &\quad + C\left(\frac{R}{2^{k+1}}\right)^{d\frac{n-2}{2}}\int_0^T t^{\beta-\frac{3}{2}}\left(\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{\frac{4-n}{2}}\mathrm{d}t\ .\end{aligned}$$

This implies by Hölder's inequality

$$\begin{aligned}&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} t^\beta|\nabla u|^2\,\mathrm{d}x + C\int_0^T\int_{B_{r_{k+1}}(x_0)} t^\beta\big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n|\nabla\Delta u|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\left(\frac{R}{2^{k+1}}\right)^{-6} T^{1-\mu+\beta(1-\mu)}\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta|\nabla u^{\frac{n+2}{6}}|^6\,\mathrm{d}x\,\mathrm{d}t\right)^\mu \\ &\qquad\times\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\,\mathrm{d}t\right)^{(n+2)(1-\mu)} \\ &\quad + C\left(\frac{R}{2^{k+1}}\right)^{-6-d(n+1)} T^{\beta+1}\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\,\mathrm{d}x\right)^{n+2}\end{aligned}$$

$$+ C\left(\frac{R}{2^{k+1}}\right)^{d\frac{n-2}{2}} T^{\beta-\frac{1}{2}} \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{\frac{4-n}{2}}$$

i. e. (inserting the expression for μ)

(26)

$$\begin{aligned}
&\sup_{t\in(0,T)} \int_{B_{r_{k+1}}(x_0)} t^\beta |\nabla u|^2 \,\mathrm{d}x + C\int_0^T \int_{B_{r_{k+1}}(x_0)} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla\Delta u|^2\big) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad \le C\left(\frac{R}{2^{k+1}}\right)^{-6} T^{1-\mu+\beta(1-\mu)} \left(\int_0^T \int_{B_{r_k}(x_0)} t^\beta |\nabla u^{\frac{n+2}{6}}|^6 \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{nd+d}{nd+d+6}} \\
&\qquad \times \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{6(n+2)}{nd+d+6}} \\
&\qquad + C\left(\frac{R}{2^{k+1}}\right)^{-6-d(n+1)} T^{\beta+1} \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{n+2} \\
&\qquad + C\left(\frac{R}{2^{k+1}}\right)^{d\frac{n-2}{2}} T^{\beta-\frac{1}{2}} \left(\sup_{t\in(0,T)} \int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{\frac{4-n}{2}} \\
&\quad =: E_1 + E_2 + E_3.
\end{aligned}$$

Step 3. Weighted mass estimate. On the other hand, we have by choosing $\psi = \varphi_{r_k}$ in Definition 2.5d

$$\begin{aligned}
&\sup_{t\in(0,T)} \int_{B_{r_{k+1}}(x_0)} u \,\mathrm{d}x \\
&\quad \le \int_{B_{r_k}(x_0)} u_0 \,\mathrm{d}x + C\left(\frac{R}{2^{k+1}}\right)^{-1} \int_0^T \int_{B_{r_k}(x_0)} u^{\frac{n}{2}} |\nabla\Delta u| u^{\frac{n}{2}} \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

The Gagliardo-Nirenberg-Sobolev interpolation inequality (34) (applied to $v = u^{\frac{n+2}{6}}$, $p = \frac{6n}{n+2}$, $q = \frac{6}{n+2}$, $r = 6$) yields

$$\begin{aligned}
\Big(\int_{B_{r_k}(x_0)} u^n \,\mathrm{d}x\Big)^{\frac{1}{2}} \le{}& C\left(\int_{B_{r_k}(x_0)} |\nabla u^{\frac{n+2}{6}}|^6 \,\mathrm{d}x\right)^{\frac{\vartheta n}{2(n+2)}} \left(\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{\frac{(1-\vartheta)n}{2}} \\
&+ C\left(\frac{R}{2^{k+1}}\right)^{-\frac{d(n-1)}{2}} \left(\int_{B_{r_k}(x_0)} u \,\mathrm{d}x\right)^{\frac{n}{2}},
\end{aligned}$$

with

$$\vartheta = \frac{d(n+2)(n-1)}{n(dn+d+6)} \ .$$

Putting these two estimates together, we deduce

$$\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} u\ \mathrm{d}x\\
&\le \int_{B_{r_k}(x_0)} u_0\ \mathrm{d}x\\
&\quad + C\left(\frac{R}{2^{k+1}}\right)^{-1}\int_0^T\left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{\frac{n(1-\vartheta)}{2}}\left(\int_{B_{r_k}(x_0)}|\nabla u^{\frac{n+2}{6}}|^6\ \mathrm{d}x\right)^{\frac{\vartheta n}{2(n+2)}}\\
&\qquad\qquad\times\Big(\int_{B_{r_k}(x_0)} u^n|\nabla\Delta u|^2\ \mathrm{d}x\Big)^{\frac12}\ \mathrm{d}t\\
&\quad + C\left(\frac{R}{2^{k+1}}\right)^{-1-\frac{d(n-1)}{2}}\int_0^T\left(\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{\frac{n}{2}}\left(\int_{B_{r_k}(x_0)} u^n|\nabla\Delta u|^2\ \mathrm{d}x\right)^{\frac12}\ \mathrm{d}t.
\end{aligned}$$

Choosing $\beta<\frac{3+d}{3+nd}$ (note that this is possible in view of the only other condition $\beta>\frac12$), this implies by Hölder's inequality and the formula for ϑ

$$\begin{aligned}
&\sup_{t\in(0,T)}\int_{B_{r_{k+1}}(x_0)} u\ \mathrm{d}x\\
&\le \int_{B_{r_k}(x_0)} u_0\ \mathrm{d}x\\
&\quad + C\left(\frac{R}{2^{k+1}}\right)^{-1} T^{\frac{3+d-\beta dn-3\beta}{dn+d+6}}\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{\frac{3n+d}{dn+d+6}}\\
&\qquad\times\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta\big(\big|\nabla u^{\frac{n+2}{6}}\big|^6+u^n|\nabla\Delta u|^2\big)\ \mathrm{d}x\ \mathrm{d}t\right)^{\frac{dn+3}{dn+d+6}}\\
&\quad + C\left(\frac{R}{2^{k+1}}\right)^{-1-\frac{d(n-1)}{2}} T^{\frac{1-\beta}{2}}\left(\sup_{t\in(0,T)}\int_{B_{r_k}(x_0)} u\ \mathrm{d}x\right)^{\frac{n}{2}}\\
&\qquad\times\left(\int_0^T\int_{B_{r_k}(x_0)} t^\beta\big|\nabla u^{\frac{n+2}{6}}\big|^6\ \mathrm{d}x\ \mathrm{d}t\right)^{\frac12}\\
&=: M_1+M_2+M_3.
\end{aligned}\tag{27}$$

Step 3. Down-propagation of the degeneracy. Let us consider the following functions:

$$\begin{aligned}
G_1(k) &:= \sup_{t\in(0,T)}\fint_{B_{r_k}(x_0)} u\ \mathrm{d}x,\\
G_2(k) &:= \sup_{t\in(0,T)}\fint_{B_{r_k}(x_0)} t^\beta|\nabla u|^2\ \mathrm{d}x\\
&\qquad + \int_0^T\fint_{B_{r_k}(x_0)} t^\beta\big(\big|\nabla u^{\frac{n+2}{6}}\big|^6+u^n|\nabla\Delta u|^2\big)\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

We want to prove that for $R > 0$ chosen large enough and for $T := c\kappa^{-n}$, for every $k \in \mathbb{N}$ the bounds

$$G_1(k) \leq \varepsilon\, T^{-1/n} \left(\frac{R}{2^k}\right)^{4/n}, \tag{28a}$$

$$G_2(k) \leq \varepsilon^\delta\, T^\beta\, T^{-2/n} \left(\frac{R}{2^k}\right)^{8/n-2} \tag{28b}$$

hold. Here, $\varepsilon, \delta > 0$ are suitable constants that will be chosen below. The parameter $\beta > 0$ is arbitrary within the bounds mentioned above. Note that the estimate (28) will immediately imply our desired estimate (24).

We will prove this claim by induction. It is immediate to check that the base step $k = 1$ is verified provided that we fix $R > 0$ large enough. Indeed,

$$\sup_{t\in(0,T)} \int_{B_{r_1}(x_0)} u \,\mathrm{d}x \leq \varepsilon T^{-1/n} \left(\frac{R}{2}\right)^{\frac{4}{n}+d}$$

follows from the fact that

$$\int_{\mathbb{R}^d} u \,\mathrm{d}x = \int_{\mathbb{R}^d} u_0 \,\mathrm{d}x < \infty. \tag{29}$$

On the other hand, to prove

$$\begin{aligned}\sup_{t\in(0,T)} \int_{B_{r_1}(x_0)} t^\beta |\nabla u|^2 \,\mathrm{d}x + \int_0^T \int_{B_{r_1}(x_0)} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla\Delta u|^2\big) \,\mathrm{d}x\,\mathrm{d}t \\ \leq \varepsilon\, T^\beta\, T^{-2/n} \left(\frac{R}{2}\right)^{\frac{8}{n}+d-2}\end{aligned}$$

we argue as follows. By the property of finite speed of propagation for the solutions to the thin-film equation (see [41, Theorem 1.3]), there exists a ball $B_{\bar{R}}(x_0)$ that contains $\operatorname{supp} u(\cdot, t)$ for $t \in [0, T)$. We consider a smooth cut-off function φ such that $\operatorname{supp}\varphi \subset B_R(x_0)$, $0 \leq \varphi \leq 1$, and $\varphi \equiv 1$ in $B_{\bar{R}}(x_0)$. Then, from the weighted energy estimate it follows that

$$\begin{aligned}&\sup_{t\in(0,T)} \int_{B_{r_1}(x_0)} t^\beta |\nabla u|^2 \,\mathrm{d}x + C \int_0^T \int_{B_{r_1}(x_0)} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla\Delta u|^2\big) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leq C \int_0^T t^{\beta-\frac{3}{2}} \left(\int_{B_{\bar{R}}(x_0)} \varphi \,\mathrm{d}x\right)^{\frac{n-2}{2}} \left(\int_{B_{\bar{R}}(x_0)} u \,\mathrm{d}x\right)^{\frac{4-n}{2}} \mathrm{d}t \\ &\quad\leq C\, T^{\beta-\frac{1}{2}} \bar{R}^{d\frac{n-2}{2}} \|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{4-n}{2}}\,.\end{aligned}$$

In view of (29), this implies the claim if we choose $R \geq \bar{R}$ large enough.

Having proved the base step of the induction, we now show that the bounds are propagated down to smaller scales: Assuming

$$\sup_{t\in(0,T)} \fint_{B_{r_k}(x_0)} u \, \mathrm{d}x \leq \varepsilon \, T^{-1/n} \left(\frac{R}{2^k}\right)^{\frac{4}{n}}, \tag{30}$$

$$\begin{aligned} \sup_{t\in(0,T)} \fint_{B_{r_k}(x_0)} t^\beta |\nabla u|^2 \, \mathrm{d}x + \int_0^T \fint_{B_{r_k}(x_0)} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla \Delta u|^2\big) \, \mathrm{d}x \, \mathrm{d}t \\ \leq \varepsilon^\delta \, T^\beta \, T^{-2/n} \left(\frac{R}{2^k}\right)^{\frac{8}{n}-2}, \end{aligned} \tag{31}$$

we want to show that

$$\sup_{t\in(0,T)} \fint_{B_{r_{k+1}}(x_0)} u \, \mathrm{d}x \leq \varepsilon \, T^{-1/n} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}}, \tag{32}$$

$$\begin{aligned} \sup_{t\in(0,T)} \fint_{B_{r_{k+1}}(x_0)} t^\beta |\nabla u|^2 \, \mathrm{d}x + \int_0^T \fint_{B_{r_{k+1}}(x_0)} t^\beta \big(\big|\nabla u^{\frac{n+2}{6}}\big|^6 + u^n |\nabla \Delta u|^2\big) \, \mathrm{d}x \, \mathrm{d}t \\ \leq \varepsilon^\delta \, T^\beta \, T^{-2/n} \left(\frac{R}{2^{k+1}}\right)^{\frac{8}{n}-2}. \end{aligned} \tag{33}$$

Plugging the induction hypothesis (30)-(31) as well as the assumption (9) into the localized energy and mass estimates (26) and (27), we obtain

$$\begin{aligned} E_1 &\leq C \, \underline{\varepsilon^{\frac{6(n+2-\delta)}{dn+d+6}}} \, \varepsilon^\delta \, T^\beta \, T^{-\frac{2}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{8}{n}-2+d} \\ E_2 &\leq C \, \underline{\varepsilon^{n+2-\delta}} \, \varepsilon^\delta \, T^\beta \, T^{-\frac{2}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{8}{n}-2+d} \\ E_3 &\leq C \, \underline{\varepsilon^{\frac{4-n-2\delta}{2}}} \, \varepsilon^\delta \, T^\beta \, T^{-\frac{2}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{8}{n}-2+d} \end{aligned}$$

$$\begin{aligned} M_1 &\leq C \, \varepsilon^{-1} \kappa \, T^{\frac{1}{n}} \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d} \\ M_2 &\leq C \, \underline{\varepsilon^{\frac{3n-dn-6+\delta dn+3\delta}{dn+d+6}}} \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d} \\ M_3 &\leq C \, \underline{\varepsilon^{\frac{n-2+\delta}{2}}} \, \varepsilon \, T^{-\frac{1}{n}} \left(\frac{R}{2^{k+1}}\right)^{\frac{4}{n}+d}. \end{aligned}$$

Putting these estimates together, we conclude that (32) and (33) hold if ε and δ are chosen in a suitable way (i. e. ε small enough and δ in such a way that the exponents in the underlined factors are positive, in particular $\delta < 2 - \frac{n}{2}$ but $\delta > (dn + 6 - 3n)/(dn + 3)$) and if we suppose that T satisfies

$$C \, \varepsilon^{-1} \, T^{1/n} \, \kappa \leq 1.$$

As a consequence, for such T the estimates (28) hold. □

Appendix A. Auxiliary inequalities

A.1. **Gagliardo-Nirenberg-Sobolev's interpolation inequality.** For the sake of completeness, we recall the version of Gagliardo-Nirenberg-Sobolev's interpolation inequality that has been used throughout the paper (see e. g. [17, Proposition A.1]).

Theorem A.1 (Gagliardo-Nirenberg-Sobolev's interpolation inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with piecewise smooth boundary $\partial\Omega$. Let $0 < q < p$, $1 \leq r \leq \infty$, and $k \in \mathbb{N}$. Let $v \in L^q(\Omega)$ such that $D^k v \in L^r(\Omega)$. Then there exist constants C_1 and C_2 (depending only on Ω, k, q, and r) such that*

$$\|v\|_{L^p(\Omega)} \leq C_1 \|D^k v\|_{L^r(\Omega)}^{\vartheta} \|v\|_{L^q(\Omega)}^{1-\vartheta} + C_2 \|v\|_{L^q(\Omega)}, \tag{34}$$

where

$$\vartheta := \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{k}{d} - \frac{1}{r}} \in (0,1).$$

In addition, there exists a positive constant C (depending on r, m, q and d and independent of Ω) such that the following propositions hold true.

(1) *If Ω is either the d-dimensional cube $Q_\lambda(0)$ centered at the origin and of side-length λ or the d-dimensional ball $B_\lambda(0)$ centered at the origin and of radius λ, then (34) holds with*

$$C_1 = C \quad \text{and} \quad C_2 = C\lambda^{-d\left(\frac{1}{q}-\frac{1}{p}\right)}.$$

(2) *If $0 \leq r_1 < r_2$, with $2r_1 > r_2$ if $d > 1$, and $\Omega = B_{r_2}(0) \setminus B_{r_1}(0)$, then (34) holds with*

$$C_1 = C \quad \text{and} \quad C_2 = C\left(r_2 - r_1\right)^{-d\left(\frac{1}{q}-\frac{1}{p}\right)}.$$

(3) *If $\Omega = \mathbb{R}^d$, then (34) holds with*

$$C_1 = C \quad \text{and} \quad C_2 = 0.$$

A.2. **Bernis-Grün's weighted interpolation inequality.** We state Grün's weighted interpolation inequality (see [42, Theorem III.1 and Corollary III.2] and [40]), which was proved by Bernis in one space dimension (see [5, Theorem 1]) and plays an important role in handling the energy estimate for the thin-film equation.

Theorem A.2 (Bernis-Grün's weighted interpolation inequality)**.** *Let $\Omega \subset \mathbb{R}^d$, with $2 \leq d < 6$, be a bounded convex domain with a smooth boundary. Assume that a strictly positive function $u \in H^2(\Omega)$ satisfies*

$$\partial_\nu u|_{\partial\Omega} = 0 \qquad \text{and} \qquad \int_\Omega u^n |\nabla\Delta u|^2 \,\mathrm{d}x < \infty,$$

where $n \in \left(2-\sqrt{\frac{8}{8+d}}, 3\right)$. *Let* $\varphi \in C^1(\overline{\Omega})$ *be a nonnegative function. Then there exists a positive constant* C, *which only depends on* d *and* n, *such that*

$$\begin{aligned}\int_\Omega \varphi^6 u^{n-4}|\nabla u|^6 \,\mathrm{d}x + \int_\Omega \varphi^6 u^{n-2}|D^2u|^2|\nabla u|^2 \,\mathrm{d}x \\ + \int_{\partial\Omega} \varphi^6 u^{n-2}|\nabla u|^2 H(\nabla u, \nabla u) \,\mathrm{d}x \\ \leq C\left(\int_\Omega \varphi^6 u^n |\nabla\Delta u|^2 \,\mathrm{d}x + \int_{\{\varphi>0\}} u^{n+2}|\nabla\varphi|^6 \,\mathrm{d}x\right),\end{aligned}$$

where $H(\cdot,\cdot)$ *is the second fundamental form of* $\partial\Omega$. *In particular, we have*

$$\begin{aligned}\int_\Omega \varphi^6 |\nabla u^{\frac{n+2}{6}}|^6 \,\mathrm{d}x + \int_\Omega \varphi^6 |\nabla\Delta u^{\frac{n+2}{2}}|^2 \,\mathrm{d}x \\ \leq C\left(\int_\Omega \varphi^6 u^n |\nabla\Delta u|^2 \,\mathrm{d}x + \int_{\{\varphi>0\}} u^{n+2}|\nabla\varphi|^6 \,\mathrm{d}x\right).\end{aligned}$$

A.3. **Weighted energy estimate.** Finally, we provide a proof of the energy estimate (25) by adopting a technique that resembles the one used in the proof of [24, Lemma 1].

Lemma A.3 (Weighted energy estimate)**.** *Let* $\Omega = \mathbb{R}^d$, $n \in \left(2-\sqrt{\frac{8}{8+d}}, 3\right)$, *and* u *be an energy-dissipating weak solution to the thin-film equation* (1) *with zero contact angle in the sense of Definition 2.5. Let* $\psi \in C_c^2(\mathbb{R}^d)$ *be a nonnegative weight function. Then we have*

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2\psi \,\mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2\psi_t \,\mathrm{d}x\,\mathrm{d}t \\ = -\int_{t_1}^{t_2}\int_{\{u(\cdot,t)>0\}} u^n|\nabla\Delta u|^2\psi \,\mathrm{d}x\,\mathrm{d}t \\ - \int_{t_1}^{t_2}\int_{\{u(\cdot,t)>0\}} u^n\nabla\Delta u\cdot\left(\Delta u\nabla\psi + D^2u\cdot\nabla\psi + \nabla u\cdot D^2\psi\right)\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{35}$$

for a.e. $t_2 \geq t_1 \geq 0$ *and a.e.* $t_2 \geq 0$ *in case* $t_1 = 0$.

Remark A.4. Let $\varphi \in C^\infty(\mathbb{R}^d)$. By applying Grün's weighted inequality and Young's inequality, from the estimate above – with $\psi := t^\beta\varphi^6$ and $\beta \in (0,1)$ – we deduce (25). We observe that in [43, Corollary 2.3] inequality (25) is proved for energy-dissipating weak solutions to the thin-film equation (1) in the sense of Definition 2.5 as constructed with the approximation procedure in [42].

Proof of Lemma A.3. Let $\psi \in C_c^2(\mathbb{R}^d)$ be a nonnegative weight function and let $\rho_\delta \in C_c^\infty(\mathbb{R}^d)$ denote a standard mollifier with respect to space. Assume that $\operatorname{dist}(\operatorname{supp}\psi, \partial\Omega\times(0,T)) > \delta$. Using $-\nabla\cdot(\rho_\delta * (\psi\,(\rho_\delta * \nabla u)))$ as a test function in

the weak formulation of the thin-film equation yields

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{2}|\nabla \rho_\delta * u|^2 \psi \,\mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \frac{1}{2}|\nabla \rho_\delta * u|^2 \psi_t \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left(\rho_\delta * u^n \nabla\Delta u\right)\cdot \psi \nabla\Delta(\rho_\delta * u)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad -\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left(\rho_\delta * u^n \nabla\Delta u\right)\cdot D^2(\rho_\delta * u)\cdot \nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad -\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left(\rho_\delta * u^n \nabla\Delta u\right)\cdot \nabla\psi\,\Delta(\rho_\delta * u)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad -\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left(\rho_\delta * u^n \nabla\Delta u\right)\cdot D^2\psi\cdot \nabla(\rho_\delta * u)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{36}
$$

We intend to pass to the limit as $\delta \to 0$. Since $u \in L^\infty((0,T);H^1(\mathbb{R}^d))$, the terms on the left-hand side converge for a.e. t_1, t_2 to

$$
\int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 \psi \,\mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 \psi_t \,\mathrm{d}x\,\mathrm{d}t.
$$

By the definition of weak energy-dissipating solution, we have

$$
\nabla u^{\frac{n+2}{6}} \in L^6((0,T);L^6(\mathbb{R}^d)) \quad \text{and} \quad u^{\frac{n}{2}}\nabla\Delta u \in L^2((0,T);L^2(\mathbb{R}^d)).
$$

From Gagliardo-Nirenberg-Sobolev's embedding theorem and the property of conservation of mass, it follows that

$$
u^{\frac{n+2}{6}} \in L^6((0,T);L^6(\mathbb{R}^d)).
$$

Therefore,

$$
\nabla u = \frac{6}{n+2} u^{\frac{4-n}{6}} \nabla u^{\frac{n+2}{6}} \in L^{n+2}\left((0,T);L^{n+2}(\mathbb{R}^d)\right).
$$

Moreover,

$$
u^{\frac{n}{2}} = \left(u^{\frac{n+2}{6}}\right)^{\frac{3n}{n+2}} \in L^{\frac{2(n+2)}{n}}\left((0,T);L^{\frac{2(n+2)}{n}}(\mathbb{R}^d)\right).
$$

In addition, due to $d \le 3$, by Sobolev's embedding theorem, for a. e. time $t \in [0,T]$ the function $u^{\frac{n+2}{6}}(\cdot,t)$ (and therefore $u(\cdot,t)$) is continuous. As a consequence, we have $\nabla\Delta u(\cdot,t) \in L^2_{loc}(\{u(\cdot,t) > 0\})$ for a.e. $t \in [0,T]$. From the regularity theory for elliptic operators, it follows that $u(\cdot,t) \in H^3_{loc}(\{u(\cdot,t) > 0\})$ for a.e. $t \in [0,T]$. Thus, on $\{u > 0\}$, we immediately obtain pointwise convergence a.e. of the integrands on the right-hand side in formula (36) in the limit $\delta \to 0$. It remains to show that the integrands are dominated by integrable functions and to identify the pointwise limit on $\{u = 0\}$ to infer convergence of the integrals.

We start by studying the first integrand on the right-hand side of formula (36). Consider a smooth monotonous function g, with $0 \le g \le 1$, such that $g \equiv 0$ for $x < 1/2$ and $g \equiv 1$ for $x > 1$ and let

$$
f_\beta(v) = \int_0^v g\left(\frac{s-\beta}{\beta}\right)\,\mathrm{d}s + \int_0^{2\beta} 1 - g\left(\frac{s-\beta}{\beta}\right)\,\mathrm{d}s,
\tag{37}
$$

where $\beta > 0$. Note that this definition in particular entails $f_\beta(v) = v$ for any $v \geq 2\beta$ and $|f_\beta(v) - v| \leq 2\beta$ for any $v \geq 0$. We may then rewrite

$$\begin{aligned}\nabla\Delta(\rho_\delta * u) &= \nabla\Delta(\rho_\delta * (u - f_\beta(u))) + \nabla\Delta(\rho_\delta * f_\beta(u)) \\ &=: I_{11} + I_{12}.\end{aligned} \tag{38}$$

We start by estimating I_{11} as follows.

$$\begin{aligned}&\big|\nabla\Delta(\rho_\delta * (u - f_\beta(u)))\big|(x_0) \leq C\delta^{-3} \fint_{B_\delta(x_0)} |u - f_\beta(u)| \, \mathrm{d}x \\ &\quad \leq C\delta^{-3}\beta \fint_{B_\delta(x_0)} \chi_{\{u<2\beta\}} \, \mathrm{d}x \leq C\delta^{-3}\beta^{-\frac{n}{2}} \fint_{B_\delta(x_0)} \big((3\beta)^{\frac{n+2}{6}} - u^{\frac{n+2}{6}}\big)_+^3 \, \mathrm{d}x \\ &\quad \leq C\delta^{-3}\beta^{-\frac{n}{2}} \Bigg(\fint_{B_\delta(x_0)} \left| u^{\frac{n+2}{6}} - \fint_{B_\delta(x_0)} u^{\frac{n+2}{6}}(y) \, dy \right|^3 \, \mathrm{d}x \\ &\qquad\qquad + \left((3\beta)^{\frac{n+2}{6}} - \fint_{B_\delta(x_0)} u^{\frac{n+2}{6}}(y) \, dy \right)_+^3 \Bigg) .\end{aligned}$$

Choose

$$\beta(x_0) := \nu \left(\fint_{B_\delta(x_0)} u^n \, \mathrm{d}x \right)^{\frac{1}{n}} \tag{39}$$

with some $\nu > 0$ to be fixed. Then, by the Poincaré inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned}&\big|\nabla\Delta(\rho_\delta * (u - f_\beta(u)))\big|(x_0) \\ &\quad \leq C\beta^{-\frac{n}{2}} \fint_{B_\delta(x_0)} |\nabla u^{\frac{n+2}{6}}|^3 \, \mathrm{d}x \\ &\qquad + C\delta^{-3}\beta^{-\frac{n}{2}} \Bigg(C\nu^{\frac{n+2}{6}} \fint_{B_\delta(x_0)} u^{\frac{n+2}{6}} \, \mathrm{d}x \\ &\qquad\qquad + C\nu^{\frac{n+2}{6}}\delta \left(\fint_{B_\delta(x_0)} |\nabla u^{\frac{n+2}{6}}|^3 \, \mathrm{d}x \right)^{\frac{1}{3}} - \fint_{B_\delta(x_0)} u^{\frac{n+2}{6}} \, \mathrm{d}x \Bigg)_+^3 .\end{aligned}$$

Choosing $\nu > 0$ small enough depending only on n and d, we infer

$$\big|\nabla\Delta(\rho_\delta * (u - f_\beta(u)))\big|(x_0) \leq C\beta^{-\frac{n}{2}} \fint_{B_\delta(x_0)} \big|\nabla u^{\frac{n+2}{6}}\big|^3 \, \mathrm{d}x \, . \tag{40}$$

Secondly, we analyze I_{12}. We have

$$\begin{aligned}&\nabla\Delta(\rho_\delta * f_\beta(u)) \\ &\quad = \rho_\delta * \big(f'_\beta(u)\nabla\Delta u + 2f''_\beta(u)\nabla u \cdot D^2 u + f''_\beta(u)\nabla u \Delta u + f'''_\beta(u)|\nabla u|^2 \nabla u \big),\end{aligned}$$

which implies (using the fact that $f''_\beta(v) = 0$ and $f'''_\beta(v) = 0$ for $v \notin [\beta, 2\beta]$ as well as the fact that $f'_\beta(v) = 0$ for $v < \beta$ and the bounds $|f'_\beta| \leq C$, $|f''_\beta| \leq C\beta^{-1}$, and $|f'''_\beta| \leq C\beta^{-2}$)

$$\begin{aligned}&\big|\nabla\Delta(\rho_\delta * f_\beta(u))\big| \\ &\quad \leq \rho_\delta * \big(\beta^{-\frac{n}{2}} |u^{\frac{n}{2}} \nabla\Delta u| + C\beta^{-\frac{n}{2}} |u^{\frac{n-2}{2}} \nabla u \otimes D^2 u| + \beta^{-\frac{n}{2}} u^{\frac{n-4}{2}} |\nabla u|^3 \big).\end{aligned} \tag{41}$$

Summing up, we obtain

$$|\nabla\Delta(\rho_\delta * u)(x_0)| \leq C\beta^{-\frac{n}{2}} \fint_{B_\delta(x_0)} \left(u^{\frac{n}{2}} |\nabla\Delta u| + u^{\frac{n-2}{2}} |\nabla u \otimes D^2 u| + \left|\nabla u^{\frac{n+2}{6}}\right|^3 \right) \, \mathrm{d}x \ .$$

For a.e. $t \in (0,T)$, we have $\nabla u^{\frac{n+2}{6}} \in L^6(\mathbb{R}^d)$, $u^{\frac{n}{2}} \nabla\Delta u \in L^2(\mathbb{R}^d)$, and $u^{\frac{n-2}{2}} \nabla u \otimes D^2 u \in L^2(\mathbb{R}^d)$. Taking into account the estimate

$$|\rho_\delta * (u^n \nabla\Delta u)(x_0)| \leq C \left(\fint_{B_\delta(x_0)} u^n |\nabla\Delta u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\fint_{B_\delta(x_0)} u^n \, \mathrm{d}x \right)^{\frac{1}{2}} \ ,$$

we infer by (38), (39), (40), and (41)

$$\begin{aligned} &\left|(\rho_\delta * (u^n \nabla\Delta u))(x_0) \ \cdot \ \psi(x_0) \ \ \nabla\Delta(\rho_\delta * u)(x_0)\right| \\ &\quad \leq C \left(\fint_{B_\delta(x_0)} u^n |\nabla\Delta u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \psi(x_0) \\ &\qquad \times \fint_{B_\delta(x_0)} \left(u^{\frac{n}{2}} |\nabla\Delta u| + u^{\frac{n-2}{2}} |\nabla u \otimes D^2 u| + \left|\nabla u^{\frac{n+2}{6}}\right|^3 \right) \, \mathrm{d}x \end{aligned}$$

This shows that the first integrand on the right-hand side of (36) is dominated by a (space-time) integrable function and also implies that the pointwise limit of the integrand vanishes on $\{u(\cdot,t) = 0\}$ for a. e. $t \in [0,T]$.

For the other integrands on the right-hand side of (36), we use analogous arguments. Let us sketch the estimates for the second one. Consider as before a smooth monotonous function g, with $0 \leq g \leq 1$, such that $g \equiv 0$ for $x < 1/2$ and $g \equiv 1$ for $x > 1$, and let f_β be as in (37). We then write

$$\begin{aligned} D^2(\rho_\delta * u) &= D^2(\rho_\delta * (u - f_\beta(u))) + D^2(\rho_\delta * f_\beta(u)) \\ &=: I_{21} + I_{22}. \end{aligned}$$

We start by estimating I_{21} as

$$\begin{aligned} \left|D^2(\rho_\delta * (u - f_\beta(u)))\right|(x_0) &\leq C\delta^{-2} \fint_{B_\delta(x_0)} |u - f_\beta(u)| \, \mathrm{d}x \\ &\leq C\delta^{-2}\beta \fint_{B_\delta(x_0)} \chi_{\{u<2\beta\}} \, \mathrm{d}x \leq C\delta^{-2}\beta^{-\frac{n-1}{3}} \fint_{B_\delta(x_0)} \left((3\beta)^{\frac{n+2}{6}} - u^{\frac{n+2}{6}}\right)_+^2 \, \mathrm{d}x \\ &\leq C\delta^{-2}\beta^{-\frac{n-1}{3}} \left(\fint_{B_\delta(x_0)} \left| u^{\frac{n+2}{6}} - \fint_{B_\delta(x_0)} u^{\frac{n+2}{6}}(y) \, dy \right|^2 \, \mathrm{d}x \right. \\ &\qquad \left. + \left((3\beta)^{\frac{n+2}{6}} - \fint_{B_\delta(x_0)} u^{\frac{n+2}{6}}(y) \, dy \right)_+^2 \right) \ . \end{aligned}$$

Choosing β as in (39), we obtain by the Poincaré inequality and the Sobolev embedding

$$\begin{aligned}
&\big|D^2(\rho_\delta * (u - f_\beta(u)))\big|(x_0) \\
&\quad \le C\beta^{-\frac{n-1}{3}} \fint_{B_\delta(x_0)} |\nabla u^{\frac{n+2}{6}}|^2 \, \mathrm{d}x \\
&\qquad + C\delta^{-2}\beta^{-\frac{n-1}{3}} \Bigg(C\nu^{\frac{n+2}{6}} \fint_{B_\delta(0)} u^{\frac{n+2}{6}} \, \mathrm{d}x \\
&\qquad\qquad + C\nu^{\frac{n+2}{6}}\delta \bigg(\fint_{B_\delta(x_0)} |\nabla u^{\frac{n+2}{6}}|^2 \, \mathrm{d}x \bigg)^{\frac{1}{2}} - \fint_{B_\delta(x_0)} u^{\frac{n+2}{6}} \, \mathrm{d}x \Bigg)_+^2 .
\end{aligned}$$

Choosing $\nu > 0$ small enough (depending only on n and d), we infer

$$\big|D^2(\rho_\delta * (u - f_\beta(u)))\big|(x_0) \le C\beta^{-\frac{n-1}{3}} \fint_{B_\delta(0)} \big|\nabla u^{\frac{n+2}{6}}\big|^2 \, \mathrm{d}x \, .$$

We next estimate I_{22}. To this aim, we may rewrite

$$D^2(\rho_\delta * f_\beta(u)) = \rho_\delta * \big(f'_\beta(u) D^2 u + f''_\beta(u) \nabla u \otimes \nabla u\big),$$

which implies

$$\begin{aligned}
&\big|D^2(\rho_\delta * f_\beta(u))\big| \\
&\qquad \le \rho_\delta * \big(\beta^{-\frac{n-1}{3}} u^{\frac{n-1}{3}} |D^2 u| + C\beta^{-\frac{n-1}{3}} u^{\frac{n-4}{3}} |\nabla u|^2\big) \, .
\end{aligned}$$

For a.e. $t \in (0,T)$, we have $\nabla u^{\frac{n+2}{6}} \in L^6(\mathbb{R}^d \times [0,T])$ and $u^{\frac{n}{2}} \nabla \Delta u \in L^2(\mathbb{R}^d \times [0,T])$. Using these facts and the computation

$$\begin{aligned}
&\sum_{i,j} \int u^{n-1} |\partial_i \partial_j u|^3 \, dx \\
&= -2 \sum_{i,j} \int u^{n-1} |\partial_i \partial_j u| \partial_j u \, \partial_i^2 \partial_j u \, dx - (n-1) \sum_{i,j} \int u^{n-2} |\partial_i \partial_j u| \partial_i \partial_j u \partial_j u \, \partial_i u \, dx
\end{aligned}$$

as well as Hölder's inequality, we can show that $u^{\frac{n-1}{3}} D^2 u \in L^3(\mathbb{R}^d \times [0,T])$. Then we can establish convergence of the second integral on the right-hand side of (36) arguing as we have done for the first integral.

The convergence of the other integrals on the right-hand side of (36) in the limit $\delta \to 0$ may be shown analogously, thereby establishing Lemma A.3. □

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